Applications of algebraic microlocal analysis in symplectic geometry and representation theory

by

James Mracek

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Abstract

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This thesis investigates applications of microlocal geometry in both representation theory and symplectic geometry. Accordingly, there are two bodies of work contained herein.

The first part of this thesis investigates a conjectural geometrization of local Arthur packets. These packets of representations of a p-adic group were invented by Arthur for the purpose of classifying the automorphic discrete spectrum of special orthogonal and symplectic groups. While their existence has been established, an explicit construction of local Arthur packets remains difficult. In the case of real groups, Adams, Barbasch, and Vogan showed how one can use a geometrization of the local Langlands correspondence to construct packets of equivariant D-modules that satisfy similar endoscopic transfer properties as the ones defining Arthur packets. We classify the contents of these "microlocal" packets in the analogue of these varieties for p-adic groups, under certain restrictions, for a plethora of split classical groups.

The goal of the second part of this thesis is to find a way to make sense of the Duistermaat-Heckman function for a Hamiltonian action of a compact torus on an infinite dimensional symplectic manifold. We show that the Duistermaat-Heckman theorem can be understood in the language of hyperfunction theory, then apply this generalization to study the Hamiltonian $T \times S^1$ action on $\Omega SU(2)$. The essential reason for introducing hyperfunction theory is that the local contribution to the Duistermaat-Heckman polynomial near the image of a fixed point is a Green's function for an infinite order differential equation. Since infinite order differential operators do not act on Schwarz distributions, we are forced to use this more general theory.

To my family, who have provided me with opportunity

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Contents

Chapter 1

Introduction

1.1 Microlocal geometry

This thesis studies applications of algebraic microlocal geometry in the representation theory of p-adic groups and symplectic geometry. Microlocal geometry, in a rough sense, provides us with analytic objects that arise as solutions to a very broad class of differential equations. Our perspective on the theory originates from the work of Sato [Sat59] and his descendents. For the reader uninitiated with hyperfunctions and microfunctions, one can imagine that the level of generalization is akin to using distributions to study solutions to differential equations instead of analytic functions. For example, it is well known that the equation $xf(x) = 0$ has no non-trivial solutions in the space of analytic functions, but it has $\delta(x)$ as a distributional solution.

Hyperfunctions and microfunctions are analytic objects which admit solutions to equations of the form $Pf = u$ which are broader than the theory of distributions can accomodate. For example, the differential operator:

$$P = \sum_{j=0}^{\infty} \frac{2\pi i}{j!(j+1)!} \frac{d^j}{dx^j}$$

cannot act on distributions. For example, if we tried to take $P\delta(x)$, the result would have to be supported at zero, and it is well known that any such distribution is a finite sum of δ and its derivatives. Hyperfunctions were originally invented to provide a space which contains Schwarz distributions, but allows for the action of infinite order differential operators (satisfying certain convergence criteria). The insight of Sato was to realize that distributions can be thought of as boundary values of holomorphic functions. For example, the distribution $\delta(x)$ is defined by the property $\langle \delta, \varphi \rangle = \varphi(0)$ for any compactly supported real valued smooth function φ. Sato noticed the similarity between this expression and the Cauchy integral formula:

$$\varphi(0) = \frac{1}{2\pi i} \int_C \frac{\varphi(z)}{z}\, dz$$

and so he set out to realize distributions as being boundary values of holomorphic functions. He defined a hyperfunction on $\Omega \subseteq \mathbb{R}$ to be a holomorphic function on $V \backslash \Omega$, where $V \subseteq \mathbb{C}$ is an open set such that $V \cap \mathbb{R} = \Omega$, but modulo the relation that any function which extends analytically across Ω is zero. If $\phi(z)$ is a holomorphic function on $\mathbb{C} \backslash \mathbb{R}$, we will denote its equivalence class as a hyperfunction by $[\phi(z)]$.

With the idea that $\delta(x) = [1/z]$, we can see that as a hyperfunction:

$$P\delta(x) = [\exp(-1/z)]$$

so we have accomodated the action of an infinite order differential operator. Chapter five of this thesis adapts the Duistermaat-Heckman theorem of symplectic geometry to the language of hyperfunctions, to the end that certain generalizations of those theorems to infinite dimensional manifolds necessitate the study of solutions to infinite order differential operators.

While functions (and even hyperfunctions) are objects that are sections of sheaves on X, microfunctions are sections of a sheaf over the cotangent bundle T^*X. Microfunctions are obtained from the sheaf of hyperfunctions by modding out by those hyperfunctions which extend analytically across the real axis when we approach it along a fixed direction. One can see this relocation as a necessary, in light of the method of characteristics for solving linear systems of partial differential equations; we know that this method only applies to Cauchy problems where the initial surface's conormal bundle avoids the zero set of the principal symbol of a partial differential operator. In fact, the fundamental theorem of Sato (and its extension by Kawai) tells us that differential operators P (which comprise a special class of microlocal operators) have inverses on any open set in T^*X which does not intersect the principal symbol of P.

In his M.Sc. thesis, Kashiwara demonstrated how one can turn differential equations into algebraic objects of study [Kas95], and in doing so, opened the field to the application of a breadth of techniques from homological algebra and category theory. Instead of directly studying differential equations of the form $Pu = 0$, with P a differential operator, one instead introduces the ring of all differential operators D, and then constructs a sheaf of D-modules (the introduction of sheaf theory can be seen as essential, and arises from the local nature of solutions to differential equations):

$$\mathcal{M} := \frac{D}{\operatorname{Ann}_D u}$$

The differential operator P is a generator for the left ideal $\operatorname{Ann}_D u$. Various invariants and homological operations one can perform on this module turn out to have relevance to the solutions of the differential equation $Pu = 0$. For example, its solutions (in any sheaf $\mathcal{C}$, such as algebraic functions, holomorphic functions, distributions, or hyperfunctions) are represented by the set of module homomorphims $\mathcal{M} \to \mathcal{C}$.

$$\operatorname{Sol}(P) = \operatorname{Hom}_{\text{D-mod}}(\mathcal{M}, \mathcal{C}) = \{u \in \mathcal{C} \mid Pu = 0\}$$

One can then consider the higher derived functors as a sort of "higher solution" to $Pu = 0$, and this notion was eventually connected to the theory of perverse sheaves (which of course are by now ubiquitous in geometric representation theory). For the class of *regular holonomic* D-modules on X, $\operatorname{Sol}(P)$ restricts to a local system on an open subset of X.

In light of the existence of microfunctions, the algebraization above was taken one step further by Sato, Kashiwara, and Kawai [KKS], who introduced a ring of "microlocal operators" which acts on the sheaf of microfunctions. In a similar manner as above, one may consider modules over the ring of microlocal operators as algebraic replacements for systems of microdifferential equations, and morphisms in this module category as representing solutions to those equations. The sheaf of microfunction solutions to a *regular holonomic* microlocal operator equation restricts to a local system on an open subset of a Lagrangian subvariety of T^*X. These local systems, or rather, their ranks, are a central object of study in chapters three and four of this thesis.

1.2 The local Langlands correspondence

Let G be a quasi-split reductive group over a $\mathfrak{p}$-adic field, and let the cardinality of its residue field be q. The local Langlands correspondence asserts that there is a surjection from the set of equivalence classes of irreducible admissible representations of $G(F)$ to the set of equivalence classes of L-parameters, which are essentially group homomorphisms from the Galois group of F into the Langlands dual group that satisfy a list of properties (see Chapter three for more details). The fibers of this surjection are called L-packets, and their contents are conjecturally parameterized by the irreducible representations of a finite group associated to each parameter.

L-packets have a number of other properties, but there are two which are important to contextualize the historical discussion which follows. The first property is that when the image of an L-parameter is relatively compact (i.e. the parameter is *tempered*), it allows us to build a "stable distribution" on G. The second property is that maps between dual groups induce maps on Langlands parameters by pullback, and one would hope that there is some relationship between the representations in the original packet and the transfered one. The theory of endoscopy (and its generalization twisted endoscopy), developed by a number of people [LS87, KS99b], yields character relations between stable distributions built from tempered parameters of "endoscopic" subgroups and invariant distributions on G.

The analogue of these relationships in the non-tempered setting was examined by Arthur in his work on the classification of the automorphic discrete spectrum of groups over global fields. In an early paper [Art89], Arthur conjectured the existence of a new class of local parameters (now known as A-parameters), as well as a new type of packet (now known as an A-packet), that would satisfy a list of properties that enable a similar type of endoscopic transfer of distributions in the non-tempered setting. He later proved these conjectures for quasi-split special orthogonal and symplectic groups [Art13].

There is a history of geometric methods being used to prove and study the local Langlands correspondence. The earliest of which is probably the work of Kazhdan and Lusztig, who constructed the simple modules of the affine Hecke algebra associated to $G(F)$ using the module structure of the equivariant K-theory of Springer fibers [KL87]. They showed that the classification of representations is given by triples (u, s, ρ), where u is a unipotent element in the Lie algebra of the dual group, s is a semisimple element such that $sus^{-1} = qu$, and ρ is a representation of the component group of the centralizer of u and s that appears in the Springer correspondence.

The work of Kazhdan and Lusztig motivated the study of the varieties of unipotent elements such that $sus^{-1} = qu$, for some fixed semisimple element s. Very soon thereafter it was noticed that Kazhdan and Lusztig's classification theorem harkens one to the classification theorem of simple equivariant perverse sheaves by a pair of an orbit and a local system on that orbit. Vogan investigated the categories of simple, equivariant perverse sheaves on the variety of unipotent elements satisfying the previous condition (for fixed s), and made a number of conjectures about how quantities of representation theoretic interest are expected to be reflected in the geometric properties of these perverse sheaves [Vog93]. In particular, Vogan conjectured that the composition factors of standard modules associated to an irreducible admissible representation should be related to the Euler characteristics of perverse sheaves on the various strata contained in its support. He also realized that one can build "microlocal" packets of perverse sheaves on these varieties by grouping together those perverse sheaves whose characteristic cycles contain a fixed Lagrangian cycle, and that these packets share certain features in common with A-packets; for instance, the L-packets include into these "microlocal" packets, but may contain more perverse sheaves. What is important about these observations is that the perverse sheaf is giving some interesting representation theoretic information beyond which its underlying local system might be able to tell us about. The geometry and singularities of orbits in these varieties would then seem to have interesting and important connections to representation theory.

Around the same time as Vogan's work, Lusztig studied the gradings that arise on Lie algebras from the adjoint action of the torus generated by a semisimple element s, and also the categories of equivariant perverse sheaves that arise from the action of the centralizer of s on the degree d part of the grading [Lus95b]. Lusztig's focus in this paper was on showing how to decompose the categories of perverse sheaves on these varieties using a geometric analogue of cuspidal support. To that end, he introduced a parabolic induction functor for the categories of perverse sheaves on these varieties. These parabolic induction functors factor heavily into the discussion in chapter four.

The first body of work in this thesis is primarily concerned with investigating the conjectures of Vogan in the $\mathfrak{p}$-adic setting through a classification of the microlocal packets associated to a restricted class of varieties. In this sense, the work contained in this thesis can be understood as progress towards a fuller understanding of the geometric side of Vogan's conjectures.

1.3 The Duistermaat-Heckman theorem

Let (X, ω) be a symplectic manifold with the Hamiltonian action of a compact torus T. Denote the moment map by $\mu : X \to \mathfrak{t}^*$. From this data, one can take the measure on M arising from the Liouville form $\omega^n/n!$ and push it forward by μ to get a measure on $\mathfrak{t}^*$. The Duistermaat-Heckman theorem tells us about the properties of the pushforward measure $\mu_*(\omega^n/n!)$ [DH82]:

1. Its Radon-Nikodym derivative (called the Duistermaat-Heckman function, which we will denote $\eta(x)$) is described by a polynomial on each connected component of the regular values of μ - i.e. it exhibits piecewise behaviour on the images of fixed point sets of rank one subtori.

2. The value of the Duistermaat-Heckman function at $\xi \in \mathfrak{t}^*$ is equal to the volume of the symplectic reduced space at ξ.

3. The inverse Fourier transform of this measure has an exact expression as a sum over the fixed points of the group action, where the summands are described using properties of the local geometry of the action near the fixed points.

Later on, it was discovered by Jeffrey and Kirwan (who were mathematically formalizing observations of Witten from physics) that η serves as a generating function for the ring structure on the cohomology ring of the symplectic reductions [JK95b, Wit92]. The cohomology ring structure of the symplectic reductions is determined entirely by the Duistermaat-Heckman function.

Our attention was redrawn to this observation after having studied hyperfunctions and microfunctions while carrying out the first body of work in this thesis. One of our original goals was to realize the Duistermaat-Heckman localization formula as a consequence of adjunction formulas between the D-module pushforward and pullback functors. Given the Hamiltonian action of T on M, we obtain a Lie algebra homomorphism $\mathfrak{t} \to \Gamma(TM)$ by taking Hamiltonian vector fields. The condition that the action is symplectic can be succinctly expressed by the differential equation:

$$\mathcal{L}_{\xi^\#}(\omega^n) = 0 \quad \text{for every Hamiltonian vector field } \xi^\#$$

One might then be led to study the corresponding (right) D-module on X obtained as the quotient:

$$\mathcal{M} = \frac{D_X}{(\xi^\# : \xi \in \mathfrak{t}^*)D_X}$$

By construction we have that $\mathrm{Hom}_{\mathrm{mod}-D_X}(\mathcal{M}, \Omega_X)$ contains the element ω^n. On the other hand, the Duistermaat-Heckman measure gives us a distribution on $\mathfrak{t}^*$, and therefore a $D_{\mathfrak{t}^*}$-module by taking the cyclic submodule of the sheaf of hyperfunctions generated by η; we let this module be denoted by $D_{\mathfrak{t}^*} \cdot \eta$. Recall also that we have a notion of Fourier transform of D_X-modules. In local coordinates, the functor $\mathscr{F} : D_X - \mathrm{mod} \to D_X - \mathrm{mod}$ is obtained as a twisting of D_X by the automorphism:

$$D_X \to D_X$$

$$x \mapsto -\partial$$

$$\partial \mapsto x$$

We are then led to the following conjecture:

Conjecture 1.3.1. *Let (X, ω) be a symplectic manifold with a Hamiltonian action of a compact torus T, and let the moment map be $\mu : X \to \mathfrak{t}^*$. If $\mu : X \to \mathfrak{t}^*$ is proper, then there is a quasi-isomorphism:*

$$\int_\mu \mathscr{F}^{-1}(\mathcal{M}) \to \mathscr{F}^{-1}(D_{\mathfrak{t}^*} \cdot \eta)$$

We had hoped to realize the Duistermaat-Heckman localization formula as arising from a proof of this conjecture, together with the adjunction formula of [HTT08, Corollary 2.7.3]. Furthermore, we expected the relations in the cohomology ring structure to somehow be encoded in the coordinate ring of the irreducible components of the characteristic variety of these D-modules. Unfortunately, we have not made progress on either of these (admittedly abstract) conjectures.

In order to provide some concrete understanding to provide a stable base for attacking these conjectures, we were motivated to transplant the theory of hyperfunctions onto the Duistermaat-Heckman theorem. The reader unaccustomed to hyperfunctions might argue that the introduction of this theory is unecessarily complicated, and that the theory of Schwarz distributions suffices for studying this theorem. To that suspicious reader, we would offer the following justifications.

Firstly, there has been recent interest in application of microlocal geometry to studying other problems in symplectic geometry such as Lagrangian non-displaceability, Legendrian knots, and Fukaya categories [STZ16, NZ09, Tam08], but we were not aware of any such application to Hamiltonian group actions. The second reason is arguably more compelling. The Duistermaat-Heckman function can be expressed as a sum of local contributions from each vertex of the moment map image $\eta(x) = \sum_{v_i} \eta_{v_i}(x)$. Each of these functions is itself a Green's function; that is, it solves a differential equation:

$$D_{\lambda_1} D_{\lambda_2} \dots D_{\lambda_k}(\eta_{v_i}) = \delta(x)$$

Where the D_{λ_j} are constant coefficient differential operators obtained from the isotropy representation of T on the tangent space to the fixed point corresponding to v_i. If we were to allow X to become an infinite dimensional symplectic manifold, then the tangent spaces become infinite dimensional, and we would instead be forced to consider a differential operator of infinite order. As we have previously discussed, such operators act more naturally on hyperfunctions (and not on distributions). Therefore any analogue of the Duistermaat-Heckman theorem in the setting of infinite dimensional Hamiltonian group actions would be forced to use hyperfunction theory in place of the theory of distributions.

The work in chapter five outlines the constructions that would be involved in a hyperfunction version of the Duistermaat-Heckman formula, and then applies them to study a Hamiltonian group action on an infinite dimensional manifold. We associate to any Hamiltonian group action with proper moment map, a hyperfunction on $\mathfrak{t}$ which we call the *Picken hyperfunction*, named after R. Picken who also studied applications of the localization formula for infinite dimensional symplectic manifolds [Pic89]. By example, we show that in the finite dimensional setting one can recover a hyperfunction analogue of the Duistermaat-Heckman distribution by Fourier transforming the Picken hyperfunction. We then construct the Picken hyperfunction of $\Omega SU(2)$ with its standard Hamiltonian group action [PS86].

Chapter 2

Foundational results on D-modules

2.1 Introduction

In this chapter we review some of the properties of equivariant D-modules that will be needed for this thesis. In section two we recall the basic definitions of the principle objects of study of chapter four. Section three provides a modern viewpoint on equivariant D-modules, following [BD91]; the main aim is to describe a pushforward functor for D-modules along equivariant maps. Finally, in section four we review the notion of a characteristic cycle of a D-module. The characteristic cycle of an equivariant D-module is an invariant valued in Lagrangian chains in T^*X. It will factor heavily into chapters three and four. We assume that all D-modules are quasi-coherent.

2.2 A classical description of equivariant D-modules

In this chapter we suppose that we have an action of a complex linear algebraic group G on a smooth variety X. Let us denote the action morphism by $a_G : G \times X \to X$. We will let $\mu : G \times G \to G$ denote the morphism of group multiplication, and $p_{G\times X} : G \times G \times X \to G \times X$ the projection $(g_1, g_2, x) \mapsto (g_2, x)$. The action of G on X yields an action of G on the sheaf D_X of differential operators on X. If P is a differential operator on X, then for any $g \in G$ the action is $(g \cdot P)(f) = g \cdot (P(g^{-1} \cdot f))$ for all $f \in \mathcal{O}_X$.

Definition 2.2.1. *A G-**equivariant** D_X**-module** is a pair $(\mathcal{M}, \alpha)$ where M is a D_X-module and $\alpha : p_X^*\mathcal{M} \to a_G^*\mathcal{M}$ is a chosen isomorphism of $D_{G\times X}$-modules satisfying the cocycle condition:*

$$pr_{G\times X}^*(\alpha) \circ (\mathrm{id}_G \times a_G)^*(\alpha) = (\mu \times \mathrm{id}_X)^*(\alpha)$$

A morphism $g : (\mathcal{M}, \alpha_M) \to (\mathcal{N}, \alpha_N)$ of G-equivariant D_X-modules is a morphism $g : \mathcal{M} \to \mathcal{N}$ of D_X-modules such that the following diagram commutes:

$$\begin{tikzcd} p_X^*\mathcal{M} \arrow[r, "\alpha_M"] \arrow[d, "p_X^*(g)"'] & a_G^*\mathcal{M} \arrow[d, "a_G^*(g)"] \\ p_X^*\mathcal{N} \arrow[r, "\alpha_N"] & a_G^*\mathcal{N} \end{tikzcd}$$

The above definition, while succinct and useful for formal manipulations in proofs, obscures the intuition of what we would like an equivariant D-module to be. The chosen isomorphism α can be identified with, for any invariant open set U, a map $G \times M(U) \to M(U)$. Tracing the cocycle condition through the necessary identifications yields the usual associativity condition for G-modules. Further, the condition that α be an isomorphism of D_X-modules translates to the compatibility of the G-module structure on $M(U)$ with the action of differential operators on U; for any $P \in D_X(U)$, $g \in G$, and $m \in M(U)$ we have:

$$g \cdot (Pm) = (g \cdot P)(g \cdot m) \tag{2.1}$$

A morphism of equivariant D_X-modules is then simply a morphism of sheaves of D_X-modules which respects this G-module structure.

Example 2.2.1. *Equivariant D-modules arising from local systems on orbits*

The following construction comprises the main source of examples of equivariant D-modules considered herein. Let G be an algebraic group acting on a complex algebraic variety X. We let $\mathcal{O}(x)$ denote the orbit of G through the point x, $S_x = \mathrm{Stab}_G(x)$ the G stabilizer of x, and S_x° the connected component of the identity. Then there exists a covering map:

$$\tilde{\mathcal{O}}(x) = G/S_x^\circ \to \mathcal{O}(x)$$

$$[g] \mapsto g \cdot x$$

with fibers the component group $\mathcal{S}_x = S_x/S_x^\circ$. Every irreducible representation $\chi : \mathcal{S}_x \to GL(V)$ yields an equivariant vector bundle with flat connection on $\mathcal{O}(x)$ because as a group of deck transformations, $\mathcal{S}_x$ is a quotient of the fundamental group of $\mathcal{O}(x)$. The total space of the vector bundle is $\tilde{\mathcal{O}}(x) \times_\chi V$; G acts on the total space by its action on $\tilde{\mathcal{O}}(x)$, which induces the required action on sections. By minimal extension we get an equivariant D_X-module [HTT08, Section 11.6].

Definition 2.2.2. *Let X be a G-variety and Y an H-variety, and fix a homomorphism $\varphi : G \to H$. We say that $f : X \to Y$ is φ-equivariant if and only if the following diagram commutes:*

$$\begin{tikzcd} G \times X \arrow[r, "a_G"] \arrow[d, "\varphi \times f"'] & X \arrow[d, "f"] \\ H \times Y \arrow[r, "a_H"] & Y \end{tikzcd}$$

Throughout, when we say that a morphism $f : X \to Y$ is φ-equivariant we are presupposing the existence of all the necessary structure.

2.3 The Equivariant Derived Category of D-modules on X

The bulk of the information in this section seems to be well known to experts, yet is not written down in the context we require. Some of the necessary theory appears in [BL06], and otherwise in [BD91].

Recall that a (semi)simplicial object of a category $\mathcal{C}$ is simply a contravariant functor $[X]_\bullet : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$, where Δ_{inj} is the category whose objects are the sets $\{0, \dots, n\}$ and whose morphisms are increasing injections. A morphism of simplicial objects is a natural transformation of such functors. If $\mu : [n] \to [m]$ is a morphism in Δ, we will also denote $[X]_\bullet(\mu) : [X]_m \to [X]_n$ simply by μ. We will henceforth assume that when $[X]_\bullet$ is a simplicial scheme that all the morphisms μ are smooth.

Definition 2.3.1. *A D-module $\mathcal{M}_\bullet$ on the simplicial space $[X]_\bullet$ is the data of:*

1. *For every $j \in \mathbb{N}$, a D-module $\mathcal{M}_j \in \mathrm{Mod}[X]_j$*

2. *For every morphism $\mu : [X]_j \to [X]_k$, a choice of morphism $\alpha_\mu : \mu^*\mathcal{M}_k \to \mathcal{M}_j$*

Subject to the axiom that if $\nu : [X]_i \to [X]_j$ and $\mu : [X]_j \to [X]_k$, then $\alpha_{\mu\circ\nu} = \alpha_\nu \circ \nu^*(\alpha_\mu)$.

A morphism $\kappa : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ of sheaves on a simplicial space $[X]_\bullet$ is a collection of morphisms $\kappa_j : \mathcal{M}_j \to \mathcal{N}_j$ such that for any morphism $\mu : [X]_j \to [X]_k$ the following diagram commutes:

$$\begin{tikzcd} \mu^*\mathcal{M}_k \arrow[r, "\alpha_\mu"] \arrow[d, "\mu^*(\kappa_k)"'] & \mathcal{M}_j \arrow[d, "\kappa_j"] \\ \mu^*\mathcal{N}_k \arrow[r, "\beta_\mu"] & \mathcal{N}_j \end{tikzcd}$$

Strictly speaking, the above definition doesn't make sense until one has chosen an isomorphism of functors $(\mu\circ\nu)^* \to \nu^*\circ\mu^*$. For categories of D-modules this will simply come from the usual isomorphism of transfer modules [HTT08, Proposition 1.5.11].

Let $\mathrm{Mod}[X_\bullet]$ denote the category of sheaves of D-modules on the simplicial space $[X]_\bullet$, and let $\mathrm{Mod}^\circ[X_\bullet]$ denote the full subcategory of $\mathrm{Mod}[X_\bullet]$ consisting of sheaves $(\mathcal{M}_\bullet, \alpha_\mu)$ such that every α_μ is an isomorphism.

2.3.1 An important example of a simplicial space

Let X be a smooth complex variety with an action of a complex algebraic group G. For every $n \in \mathbb{N}$, define the complex variety $[X/G]_n = G^n \times X$. Define maps $d_i : [X/G]_n \to [X/G]_{n-1}$ by:

$$\begin{aligned} d_0(g_1, \dots, g_n, x) &= (g_2, \dots, g_n, x) \\ d_i(g_1, \dots, g_n, x) &= (g_1, \dots, g_i g_{i+1}, \dots, g_n, x) \quad 0 < i < n \\ d_n(g_1, \dots, g_n, x) &= (g_1, \dots, g_{n-1}, g_n \cdot x) \end{aligned}$$

In the other direction, define maps $s_i : [X/G]_{n-1} \to [X/G]_n$ by:

$$s_i(g_1, \dots, g_{n-1}, x) = (g_1, \dots, g_i, e, g_{i+1}, \dots, g_{n-1}, x)$$

Proposition 2.3.1. *The datum $([X/G]_\bullet, s_i, d_i)$ is a simplicial space*

Proof. This is a routine check that that the simplicial identities hold. □

In what follows, we will forget the degeneracy maps, s_i, and think of $[X/G]_\bullet$ as a semisimplicial space.

Lemma 2.3.1. *If $f : X \to Y$ is a φ-equivariant map, then there is a corresponding morphism of simplicial spaces $\tilde{f} : [X/G]_\bullet \to [Y/H]_\bullet$*

Proof. It suffices to prove that for any $n \in \mathbb{N}$ and for any $j \in \{0, \dots, n\}$, the following diagrams are commutative:

$$\begin{tikzcd} G^n \times X \arrow[r, "\varphi^n \times f"] \arrow[d, "d_j"'] & H^n \times Y \arrow[d, "d_j"] \\ G^{n-1} \times X \arrow[r, "\varphi^{n-1} \times f"'] & H^{n-1} \times Y \end{tikzcd}$$

For $j = n$, this follows from the condition that f is φ-equivariant, for $0 < j < n$ it follows from the fact that φ is a group homomorphism, and for $j = 0$ it is trivial. □

The following theorem of Deligne identifies the category $\mathrm{Mod}^\circ[X/G]$ with the classical category of equivariant D-modules on X.

Theorem 2.3.1. *[Del74] There is an equivalence of categories $Mod^\circ[X/G]_\bullet \to Mod_G(D_X)$*

2.3.2 Pushforwards and pullbacks in the equivariant derived category

Definition 2.3.2. *The bounded equivariant derived category of D-modules on X is the full subcategory of $D^b(\mathrm{Mod}[X/G]_\bullet)$ consisting of bounded complexes of sheaves on $[X/G]_\bullet$ such that for all i, $H^i(\mathcal{M}_\bullet) \in \mathrm{Mod}^\circ[X/G]_\bullet$. We denote this category by $D^b_G(X)$.*

By construction, there is a canonical functor:

$$\mathrm{For}_G : D^b_G(X) \to D^b(X)$$

$$\mathcal{M}^\cdot_\bullet \mapsto \mathcal{M}^\cdot_0$$

These categories are equipped with the usual t-structure consiting of complexes concentrated in either positive or negative degree. The heart of these t-structures can be identified as the category of (regular holonomic) G-equivariant D-modules on X, by Theorem 2.3.1. Alternatively, one can obtain the t-structure on $D^b_G(X)$ by pulling back the usual t-structure on $D^b(X)$ via the functor For_G.

We will define a pushforward functor in the special case that the map $\varphi : G \to H$ is an isomorphism. A more general construction exists in several other forms. For instance, instead of using the category of D-modules on the simplicial space, we could have equivalently chosen to work in the category of quasi-coherent sheaves on the deRham space of the functor of points corresponding to the quotient stack $[X/G]_\bullet$ (see [GR14] for details on this approach). While powerful in its generality, this approach would take us much too far afield, and the breadth of its applicability is not required for the work herein.

The following proposition will be what allows us to push forward equivariant sheaves.

Proposition 2.3.2. *If $\varphi : G \to H$ is an isomorphism of groups, and $f : X \to Y$ is a φ-equivariant map, then for every $i \in \mathbb{N}$, and for every $0 \leq j \leq n$, the following is a pull-back diagram:*

$$\begin{tikzcd} G^i \times X \arrow[r, "\varphi^i \times f"] \arrow[d, "d_j"'] & H^i \times Y \arrow[d, "d_j"] \\ G^{i-1} \times X \arrow[r, "\varphi^{i-1} \times f"'] & H^{i-1} \times Y \end{tikzcd}$$

Proof. For any i, and for any of the degeneracy maps d_j, the fiber product $(G^{i-1} \times X) \times_{H^{i-1} \times Y} (H^i \times Y)$ is isomorphic to $H \times G^{i-1} \times X$. Under this identification, the universal morphism $\psi : G^i \times X \to H \times G^{i-1} \times X$

is one of the following three maps:

$$\psi(g_1, \ldots, g_i, x) = (\varphi(g_i), g_2, \ldots, g_i, x) \quad \text{when } j = 0$$

$$\psi(g_1, \ldots, g_i, x) = (\varphi(g_i), g_1, \ldots, g_j g_{j+1}, \ldots, g_i, x) \quad \text{when } j < i$$

$$\psi(g_1, \ldots, g_i, x) = (\varphi(g_i), g_1, \ldots, a_G(g_i, x)) \quad \text{when } j = i$$

As φ is invertible, all of the above are invertible maps. □

We may now define a pushforward functor between the equivariant derived categories, but we must restrict to the case where $\varphi : G \to H$ is an isomorphism, as in Proposition 2.3.2. This happens, for instance, if $f : X \to Y$ is an equivariant map between two G-varieties.

Proposition 2.3.3. *If $f : X \to Y$ is a φ-equivariant morphism and $\varphi : G \to H$ is an isomorphism, then there exists a functor:*

$$\int_f^\bullet : D_G(X) \to D_H(Y)$$

which is right adjoint to $f^!_\bullet$. Furthermore, the following diagram commutes:

$$\begin{tikzcd} D^b_G(X) \arrow[r, "\int_f^\bullet"] \arrow[d, "For_G"'] & D^b_H(Y) \arrow[d, "For_H"] \\ D^b(X) \arrow[r, "\int_f"'] & D^b(Y) \end{tikzcd}$$

Proof. The proof is outlined in [MV88]; we add only a few more details. For brevity of notation, we denote $f_j = \varphi_j \times f$. Given a complex $\mathcal{M}^\cdot_\bullet$, we need to construct a complex of sheaves of modules $\displaystyle\int_f^\bullet \mathcal{M}^\cdot_\bullet$ on $[Y/H]_\bullet$, which is the data of a chain complex of D-modules on the $[Y/H]_j = H^j \times Y$ for every j, together with isomorphisms:

$$\alpha_{\tilde{\mu}} : \tilde{\mu}^* \left(\int_f^\bullet \mathcal{M} \right)_j \to \left(\int_f^\bullet \mathcal{M} \right)_i$$

for every $\tilde{\mu} : Y_i \to Y_j$, which satisfy the required compatibility conditions. We pushforward "level-wise"; for every $j \in \mathbb{N}$, we let:

$$\left(\int_f^\bullet \mathcal{M}^\cdot_\bullet \right)_j = \int_{f_j} \mathcal{M}^\cdot_j$$

where $\displaystyle\int_{f_j}$ is the usual pushforward of D-modules from X_j to Y_j.

To $\mu : X_i \to X_j$ (this is intended to be the same composition of face maps as $\tilde{\mu}$; the lack of a tilde is meant only to distinguish between being a map of X_i and not the Y_i), we have an isomorphism $\alpha_\mu : \mu^* \mathcal{M}^\cdot_j \to \mathcal{M}^\cdot_i$. Consider the following map:

$$\int_{f_i} (\alpha_\mu) : \int_{f_i} \circ \, \mu^* \mathcal{M}^\cdot_j \to \int_{f_i} \mathcal{M}^\cdot_i$$

By 2.3.2, we have a pullback diagram:

$$\begin{tikzcd} G^i \times X \arrow[r, "\varphi^i \times f"] \arrow[d, "\mu"'] & H^i \times Y \arrow[d, "\tilde{\mu}"] \\ G^j \times X \arrow[r, "\varphi^j \times f"'] & H^j \times Y \end{tikzcd}$$

so by an application of base change [HTT08, Theorem 1.7.3], we deduce there is a canonical isomorphism:

$$\int_{f_i} \circ \mu^* \mathcal{M}_j^{\cdot} \quad \simeq \quad \tilde{\mu}^* \circ \int_{f_j} \mathcal{M}_j$$

so by composition of the aforedescribed isomorphisms we have produced, for any $\tilde{\mu}$, an isomorphism

$$\alpha_{\tilde{\mu}} : \tilde{\mu}^* \int_{f_j} \mathcal{M}_j \to \int_{f_i} \mathcal{M}_i$$

That these isomorphisms satisfy the cocycle identity $\alpha_{\tilde{\mu} \circ \tilde{\nu}} = \alpha_{\tilde{\nu}} \circ \tilde{\nu}^* \alpha_{\tilde{\mu}}$ follows from the corresponding fact for the isomorphisms α_μ and that f is a morphism of simplicial spaces.

The fact that $\int_f^\bullet$ is an adjoint to $f_\bullet^!$ follows from the corresponding fact about classical D-modules, applied level wise. Finally, the statement about commuting with the forgetful functor is clear from the construction.

□

2.4 Characteristic cycles of D-modules

In this section we introduce an invariant of D-modules called the characteristic cycle. It is a $\mathbb{Z}$-valued cycle of irreducible closed subvarieties in T^*X. Roughly speaking, the characteristic cycle of $\mathcal{M}$ tells us about the directions in our manifold in which local solutions the of the system of differential equations underlying $\mathcal{M}$ cannot be extended. One of the main theorems of this thesis groups together those equivariant D-modules on a fixed variety whose characteristic cycles contain a common irreducible cycle.

2.4.1 Filtrations on the ring of differential operators and D-modules

The ring of differential operators on a variety X is filtered. The order zero differential operators are simply functions on X. The sheaf of differential operators of order k is defined inductively as the subsheaf of D_X consisting of differential operators such that for any $f \in \mathcal{O}_X$, $[P, f]$ is an order $k-1$ differential operator. In a local coordinate system, the differential operators of order less than or equal to k are given by those differential operators that can be expressed as a sum:

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

Let $F_k D_X$ denote the sheaf of differential operators of order less than or equal to k. The associated graded of D_X with respect to this filtration is a sheaf of commutative rings on X, and there is a canonical isomorphism:

$$\operatorname{gr} D_X = \bigoplus_{k=0}^{\infty} F_{k+1}D_X/F_kD_X \simeq \pi_*\mathcal{O}_{T^*X}$$

where $\pi : T^*X \to X$ is the bundle projection.

To each level of this filtration, we define a symbol map:

$$\sigma_k : F_kD_X \to F_kD_X/F_{k-1}D_X$$

If P is a differential operator defined in a local corrdinate system as above, then its symbol $\sigma_k(P) \in \pi_*\mathcal{O}_{T^*X}$ is given by:

$$\sigma_k(P) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

Here, $(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ are local coordinates on T^*X.

Similarly, if $\mathcal{M}$ is a D-module, then for fixed filtration

$$\dots \subseteq F_{l-1}\mathcal{M} \subseteq F_l\mathcal{M} \subseteq F_{l+1}\mathcal{M} \subseteq \dots$$

there is a corresponding (surjective) symbol map $\mu_l : F_l\mathcal{M} \to F_l\mathcal{M}/F_{l-1}\mathcal{M}$. These symbol maps make $\operatorname{gr}\mathcal{M}$ into a sheaf of modules over $\pi_*\mathcal{O}_{T^*X}$. Explicitly, for sections $\mu_l(m) \in \operatorname{gr}\mathcal{M}$ and $\sigma_k(P) \in \operatorname{gr} D_X$, we have $\sigma_k(P) \cdot \mu_l(m) = \mu_{l+k}(P \cdot m)$. It is easy to show that this is independent of the choice of lifts m and P.

Definition 2.4.1. *Let $\mathcal{M}$ be a filtered D_X-module, as above. We say that the filtration on $\mathcal{M}$ is good if and only if $\operatorname{gr}_F \mathcal{M}$ is a coherent sheaf of $\pi_*\mathcal{O}_{T^*X}$-modules.*

Remarks:

1. The definition has other equivalent formulations [HTT08, Prop. 2.1.1], but in practice, the one we have cited is the easiest to check.

2. A D_X-module admits a good filtration if and only if it is coherent [HTT08, Theorem 2.1.3].

2.4.2 Singular support and the characteristic cycle

Suppose that we give $\mathcal{M}$ a good filtration. As $\operatorname{gr}\mathcal{M}$ is a sheaf of modules over $\pi_*\mathcal{O}_{T^*X}$, using the (π_*, π^*)-adjunction we can make $\pi^*\operatorname{gr}\mathcal{M}$ into an $\mathcal{O}_{T^*X}$-module.

Definition 2.4.2. *The singular support of a D_X-module $\mathcal{M}$, $SS(\mathcal{M}) \subseteq T^*X$ is the sheaf theoretical support of the $\mathcal{O}_{T^*X}$-module $\pi^*\operatorname{gr}\mathcal{M}$.*

Remark: $SS(\mathcal{M})$ does not depend on the choice of good filtration.

On an affine open set, the $\mathcal{O}_{T^*X}$-module $\pi^*\operatorname{gr}\mathcal{M}$ restricts to the sheaf associated to a module N over $\mathcal{O}_{T^*X}(U)$. The singular support is then locally cut out by the ideal $V(\sqrt{\operatorname{Ann} N})$. Writing $\sqrt{\operatorname{Ann} N} = \bigcap_{i=1}^m \mathfrak{p}_i$ as an intersection of prime ideals gives a decomposition of $\mathrm{SS}(\mathcal{M}) = \bigcup_{i=1}^m C_i$ into its irreducible components. The stalk of $\pi^*\operatorname{gr}\mathcal{M}$ at the generic point of C_i is an Artinian module over $\mathcal{O}_{T^*X}(U)_{\mathfrak{p}_i}$. We denote the length of this module $m_{C_i}(\mathcal{M})$ and call it the multiplicity of $\mathcal{M}$ along C_i.

Definition 2.4.3. *Let $\mathcal{M}$ be a D_X-module, and let $V(\mathfrak{p}_i) = C_i$ be the irreducible components of its singular support. The characteristic cycle of $\mathcal{M}$ is:*

$$CC(\mathcal{M}) = \sum_{C_i} m_{C_i}(\mathcal{M})\, C_i$$

Recall that a D_X-module is called holonomic if and only if its singular support is a union of Lagrangian subvarieties of T^*X. In what follows, we will assume that whenever $\mathcal{M}$ is a holonomic D_X-module, for every irreducible component of the singular support C_i, there exists a smooth subvariety $Z_i \subseteq X$ such that $C_i = \overline{T^*_{Z_i}X}$.

The following two lemmas will be used in chapter three.

Lemma 2.4.1. *If $i : X \to Y$ is a closed embedding of smooth varieties and $\mathcal{M}$ is a D_X-module such that:*

$$CC(\mathcal{M}) = \sum_j m_{Z_j}(\mathcal{M})\overline{[T^*_{Z_j}X]}$$

then:

$$CC\left(\int_i \mathcal{M}\right) = \sum_j m_{Z_j}(\mathcal{M})\,\overline{[T^*_{Z_j}Y]}$$

Proof. For details of the specific constructions surrounding characteristic cycles, see [KS13, chap. 9, sect. 3]. By [KS13, Prop. 9.3.2], the closed embedding $i : X \to Y$ induces a linear map between the global sections of the sheaves of Lagrangian cycles:

$$i_* : H^0(T^*X, \mathcal{L}_X) \to H^0(T^*Y, \mathcal{L}_Y)$$

and by [KS13, Prop. 9.4.2] we have:

$$\begin{aligned} CC\left(\int_i \mathcal{M}\right) &= i_* CC(M) \\ &= \sum_j m_{Z_j}(\mathcal{M}) i_*[T^*_{Z_j}Y] \end{aligned}$$

so we are reduced to proving that $i_*[T^*_{Z_j}Y] = [T^*_{Z_j}X]$. This follows from [KS13, ex. 9.3.4(iii)]. □

Lemma 2.4.2. *If $\pi : X \to Y$ is a smooth morphism of smooth varieties and $\mathcal{M}$ is a D_Y-module such that*

$$CC(\mathcal{M}) = \sum_j m_{Z_j}(\mathcal{M})\overline{[T^*_{Z_j}Y]}$$

then:

$$CC(\pi^*\mathcal{M}) = \sum_j m_{Z_j}(\mathcal{M})\,\overline{[T^*_{\pi^{-1}(Z_j)}X]}$$

Proof. This proof is similar to the previous one. Smooth morphisms of smooth varieties are submersions, and in particular, are vacuously non-characteristic for every sheaf. By [KS13, prop. 9.3.2], the smooth morphism $\pi : X \to Y$ induces a linear map between the global sections of the sheaves of Lagrangian cycles:

$$\pi^* : H^0(T^*Y, \mathcal{L}_Y) \to H^0(T^*X, \mathcal{L}_X)$$

and by [KS13, Prop. 9.4.3] we have:

$$\begin{aligned} CC(\pi^*\mathcal{M}) &= \pi^* CC(M) \\ &= \sum_j m_{Z_j}(\mathcal{M})\pi^*[T^*_{Z_j}Y] \end{aligned}$$

so we are reduced to proving that $\pi^*[T^*_{Z_j}Y] = [T^*_{\pi^{-1}(Z_j)}X]$. This follows from [KS13, ex. 9.3.4(iv)]. □

If $\mathcal{M}$ is a D_X-module and $\mathcal{N}$ is a D_Y-module, then we recall *external direct product* [HTT08, p.38], which is the $D_{X\times Y}$-module $\mathcal{M} \boxtimes \mathcal{N}$. Suppose that:

$$CC(\mathcal{M}) = \sum_i m_i[\Lambda_i]$$

$$CC(\mathcal{N}) = \sum_j n_j[\Lambda'_j]$$

then we define:

$$CC(\mathcal{M}) \boxtimes CC(\mathcal{N}) = \sum_{i,j} m_i n_j [\Lambda_i \times \Lambda'_j]$$

Proposition 2.4.1. *Let X and Y be smooth varieties over $\mathbb{C}$, let $\mathcal{M}$ be a regular holonomic D_X-module, and let $\mathcal{N}$ be a regular holonomic D_Y-module. Then,*

$$CC(\mathcal{M} \boxtimes \mathcal{N}) = CC(\mathcal{M}) \boxtimes CC(\mathcal{N})$$

Proof. This follows immediately from [Sai17, Theorem 2.2(2)], together with the Riemann-Hilbert correspondence. □

2.4.3 Calculating characteristic cycles

The following three examples of D-modules on $\mathbb{C}$, together with Proposition 2.4.1, will allow us to compute the general characteristic cycles of the D-modules appearing in this thesis.

Example 2.4.1. $\mathcal{M}(\mathbb{C}) = \mathbb{C}[x]$

Throughout, it suffices to consider the global sections, which we again denote by $\mathcal{M}$. Let $\mathbb{C}[x]$ denote the ring of polynomial functions in the variable x. We give this the structure of a $D_{\mathbb{C}}$-module using the usual action of differentiation and multiplication by polynomial functions. Consider the filtration:

$$F_i\mathcal{M} = \begin{cases} 0 & i < 0 \\ \mathbb{C}[x] & i \geq 0 \end{cases}$$

for which the associated graded (as a graded $\mathbb{C}$-vector space) is:

$$\operatorname{gr} \mathcal{M} \simeq \mathbb{C}[x]$$

with the factor $\mathbb{C}[x]$ concentrated in degree zero. We now describe the $\pi^*\mathcal{O}_{T^*X}(\mathbb{C}) \simeq \mathbb{C}[x,\xi]$-module structure on $\operatorname{gr}\mathcal{M}$, recalling that x has degree zero and ξ has degree one. We will denote an element of degree i in $\operatorname{gr} \mathcal{M}$ by $[m]_i$The action on the associated graded factors through the symbol maps:

$$x \cdot [f(x)]_0 = [xf(x)]_0$$

$$\xi \cdot [f(x)]_0 = [\partial_x f]_1 = 0$$

so ξ annihilates every element, while x just acts by multiplication on $\operatorname{gr} \mathcal{M}$. This shows that $\operatorname{gr} \mathcal{M}$ is generated by the element 1, so in particular, our filtration was good.

We recall that if $Y = \operatorname{Spec} R$ is an affine scheme and $\mathscr{F}$ is the quasi-coherent sheaf of $\mathcal{O}_Y$-modules associated to the R-module M, we have:

$$\operatorname{supp} \mathscr{F} = V(\operatorname{Ann} M)$$

We had just computed that $\operatorname{Ann} \mathcal{M} = (\xi)$, so:

$$SS(\mathcal{M}) = V(\xi) = T^*_{\mathbb{C}}\mathbb{C} := \Lambda$$

that is, the singular support is exactly the zero section of the conormal bundle.

We now compute the multiplicity of $\mathcal{M}$ along Λ. We need to compute the length of the $\mathbb{C}[x,\xi]_{(\xi)}$-module gotten by localizing $\mathbb{C}[x]$ at the prime (ξ). We claim that $\mathbb{C}[x]_{(\xi)}$ is a simple module over $\mathbb{C}[x,\xi]_{(\xi)}$. Notice we always have:

$$\frac{f(x)}{g(x,\xi)} \sim \frac{f(x)}{g(x,0)}$$

in $\mathbb{C}[x]_{(\xi)}$ because we may write $g(x,\xi) = \tilde{g}(x) + \xi h(x,\xi)$ for some polynomials g, h, and:

$$g(x,0) \cdot f(x) - g(x,\xi) \cdot f(x) = \tilde{g}(x) \cdot f(x) - (\tilde{g}(x) + h(x,\xi)\xi) \cdot f(x) = 0$$

since $\xi \cdot f(x) = 0$. We also notice that $1/1$ is a generator of $\mathbb{C}[x]_{(\xi)}$. Suppose that $\mathcal{N} \subseteq \mathbb{C}[x]_{(\xi)}$ is a non-zero submodule, so that there is some non-zero $f(x)/g(x) \in \mathcal{N}$. Then,

$$1 = \frac{g(x)}{f(x)} \cdot \frac{f(x)}{g(x)} \in N$$

so $N = \mathbb{C}[x]_{(\xi)}$. This completes the proof that $m_\Lambda(\mathcal{M}) = 1$, and:

$$CC(\mathbb{C}[x]) = [T^*_{\mathbb{C}}\mathbb{C}]$$

Example 2.4.2. $\mathcal{M}_\alpha = D_{\mathbb{C}} \cdot x^\alpha, \alpha \notin \mathbb{Z}$

Consider the $D_{\mathbb{C}}$-submodule of the holomorphic functions on $\mathbb{C}$ generated by the function x^α, with $\alpha \notin \mathbb{Z}$. We have an isomorphism:

$$\frac{D_{\mathbb{C}}}{D_{\mathbb{C}}(x\partial - \alpha)} \to D_{\mathbb{C}} \cdot x^\alpha$$

$$[P] \mapsto P \cdot x^\alpha$$

which is well defined since x^α solves the differential equation:

$$x\frac{df}{dx} - \alpha f = 0$$

We can give $\mathcal{M}_\alpha$ a filtration by setting $\mathcal{M}_\alpha^i = (\mathcal{F}_i D_\mathbb{C}) \cdot x^\alpha$ for all i. With respect to this filtration, the associated graded is:

$$\operatorname{gr} \mathcal{M}_\alpha = \mathbb{C}[x]x^\alpha \oplus \mathbb{C}x^{\alpha-1} \oplus \mathbb{C}x^{\alpha-2} \oplus \dots$$

with the first summand in degree zero, the second summand in degree one, and so forth. The action of $\mathbb{C}[x,\xi]$ on $\operatorname{gr}\mathcal{M}_\alpha$ is as follows. In degree zero, we have:

$$\xi \cdot [a_0 x^\alpha + a_1 x^{\alpha+1} + \dots]_0 = \left[a_0 \alpha x^{\alpha-1} + a_1(\alpha-1)x^\alpha + \dots\right]_1 = [a_0\alpha x^{\alpha-1}]_1$$

$$x \cdot [a_0 x^\alpha + a_1 x^{\alpha+1} + \dots]_0 = [a_0 x^{\alpha+1} + a_1 x^{\alpha+2} + \dots]_0$$

while in degree $k > 0$ we have:

$$\xi \cdot [a_k x^{\alpha-k}]_k = [a_k(\alpha-k)x^{\alpha-(k+1)}]_{k+1}$$

$$x \cdot [a_k x^{\alpha-k}]_k = [a_k x^{\alpha-k+1}]_k = 0$$

From the above computations, we can see that $(x\xi) \subseteq \operatorname{Ann}\operatorname{gr}\mathcal{M}_\alpha$. It is clear to see that this is actually an equality, because any polynomial not in $(x\xi)$ must have a non-zero x term or a non-zero ξ term, but no element containing these terms can be in the annihilator. We have computed that the singular support has two irreducible components. One of them is the zero section in $T^*\mathbb{C}$, and the other is the fiber of $T^*\mathbb{C}$ above $x = 0$:

$$SS(\mathcal{M}_\alpha) = V(x\xi) = V(x) \cup V(\xi) = T^*_\mathbb{C}\mathbb{C} \cup T^*_{\{0\}}\mathbb{C}$$

A similar computation to Example 2.4.1 shows that $\mathcal{M}_\alpha$ has multiplicity one along each of these irreducible components, so:

$$CC(\mathcal{M}_\alpha) = [T^*_\mathbb{C}\mathbb{C}] + [T^*_{\{0\}}\mathbb{C}]$$

Example 2.4.3. $\mathcal{M} = \mathbb{C}[\partial]$

This computation is very similar to Example 2.4.1, so we omit most of the details. We list the result of the computation here for future reference:

$$CC(\mathcal{M}) = [T^*_{\{0\}}\mathbb{C}]$$

In other words, the characteristic cycle of $\mathcal{M}$ is the cotangent fiber above $x = 0$. One could alternatively observe that $\mathbb{C}[\partial]$ is the D-module Fourier transform of $\mathbb{C}[x]$ and apply [KS13, Theorem 5.5.5].

Chapter 3

The microlocal conjecture on A-packets

3.1 Introduction

The purpose of this chapter is to give an outline of the microlocal conjecture on the structure of local Arthur packets. These packets consist of collections of irreducible representations of a p-adic group. The microlocal conjecture will assert that, through a geometrization of the local Langlands correspondence, we can explicitly construct these packets using microlocal invariants associated to D-modules on a smooth algebraic variety. For real reductive groups, the microlocal conjecture (although not spoken of by this name) was examined in the book of Adams, Barbasch and Vogan [ABV12]. We study the analogous problem for reductive groups over $\mathfrak{p}$-adic fields, extending the work of Vogan in [Vog93].

In section two we review the main properties of the local Langlands correspondence. In order to accomplish this, we quickly survey the relevant structures involved in the correspondence (the Wei-Deligne group, L-groups, pure inner forms, etc.). Section three outlines the geometrization of the local Langlands correspondence, as envisioned by Vogan and Lusztig [Vog93, Lus95b]. In this geometrization, the space of L-parameters is identified with a the variety of SL_2-triples in the Langlands dual group, with the notion of equivalence of L-parameters being represented by an orbit equivalence for a group action. In section four, we review Arthur's work on endoscopic character formulas for non-tempered L-parameters. In this work, Arthur introduces a more general class of parameters called A-parameters. From these A-parameters, he builds a new packet of representations, which we call an A-packet. In section five, we motivate how one can realize some of the natural properties Arthur packets satisfy using purely geometric constructions. One is then led to the notion of a *micropacket*, and we conjecture that these correspond to A-packets through the local Langlands conjecture. Finally, in section six we introduce a wide range of issues that arose during our work which are directions of future study.

3.2 The local Langlands correspondence

3.2.1 L-groups

Let G be a quasi-split reductive algebraic group over a non-archimedean local field F of characteristic zero. The ring of integers of F will be denoted $\mathfrak{O}_F$ and its unique maximal ideal will be denoted by $\mathfrak{p}$. The residue field $k_F = \mathfrak{O}_F/\mathfrak{p}$ is a finite field; we will denote its cardinality by q. The Galois group of F in its algebraic closure $\overline{F}$ will be denoted $\Gamma = \mathrm{Gal}(F/\overline{F})$. There is an exact sequence:

$$1 \to I_F \to \Gamma \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

The subgroup I_F is called the inertia subgroup; we give it the subspace topology inherited from Γ. The Weil group of F, W_F, is the preimage of $\mathbb{Z}$ under the map $\Gamma \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F})$. We do not give W_F the subspace topology. Instead, we give it a topology by insisting that I_F is a compact open neighbourhood of the identity. We fix $\mathrm{Fr} \in W_F$, a lift of the Frobenius automorphism.

If G is a reductive group defined over an algebraically closed field $\overline{F}$, then G is uniquely characterized by its root datum $\Psi = (X^*, R, X_*, R^\vee)$. Here,

- X^* and X_* are the weight and coweight lattices, respectively
- $R \subseteq X^*$ and $R^\vee \subseteq X_*$ are the set of roots and coroots, respectively

More generally, if G is defined over F (maybe not algebraically closed), then G is uniquely characterized by the root datum Ψ of the base change to the algebraic closure, together with a group homomorphism:

$$\mu_G : \Gamma \to \mathrm{Aut}\,\Psi$$

One recovers $G(F)$ by first building $G(\overline{F})$ from the root datum, then taking the fixed points of the action of Γ on $G(\overline{F})$ induced by the homomorphism μ_G.

To every reductive algebraic group G defined over F we associate its *L-group* ${}^LG$ as follows [Lan79, Bor79]. From the root datum Ψ we construct a new root datum $\hat{\Psi} = (X_*, R^\vee, X^*, R)$ by swapping the role of the roots and coroots. The root datum $\hat{\Psi}$ yields a reductive group $\hat{G}/\mathbb{C}$ which we call the *Langlands dual group*. We get a homomorphism $W_F \to \mathrm{Aut}\,\hat{G}$ by composing the natural maps:

$$W_F \hookrightarrow \Gamma \to \mathrm{Aut}\,\Psi \to \mathrm{Aut}\,\hat{\Psi} \to \mathrm{Aut}\,\hat{G}$$

The L-group of G is then defined to be the semidirect product:

$${}^LG = \hat{G} \rtimes W_F$$

We will say that an element $x \in {}^LG$ is *semisimple* if and only if its image in any representation of ${}^LG$ is semisimple.

3.2.2 Galois cohomology and the zoo of forms

Let G be a reductive group defined over F. We say that G' is a *rational form* of G if and only if after base change to $\overline{F}$, G and G' become isomorphic as algebraic groups over $\overline{F}$. Rational forms of G are classified by elements of a Galois cohomology group. A good reference for Galois cohomology is Serre's book [Ser13].

Suppose that we have two groups Γ and A, together with a homomorphism $\Gamma \to \mathrm{Aut}(A)$. This data allows us to build a semidirect product group $A \rtimes \Gamma$. We define the 1-cocycles to be:

$$Z^1(\Gamma, A) = \{\varphi : \Gamma \to A \rtimes \Gamma \,|\, \varphi \text{ is a group homomorphism}, \varphi(g) = (\alpha(g), g)\}$$

Stated slightly differently, 1-cocycles are simply the possible splittings of the exact sequence:

$$1 \to A \to A \rtimes \Gamma \to \Gamma \to 1$$

We will refer to α as the cocycle. Two 1-cocycles α and β are called cohomologous when there exists a $b \in A$ such that $\beta(g) = (g \cdot b)\alpha(g)b^{-1}$ for all $g \in \Gamma$. We then denote $H^1(\Gamma, A) = Z^1(\Gamma, A)/\sim$.

We now suppose that Γ is the Galois group of F in its algebraic closure. We will describe three different notions of isomorphism classes of forms of G. When $A = \mathrm{Aut}\, G$ the set $H^1(\Gamma, A)$ parameterizes the rational forms of G. When $A = G_{\mathrm{ad}}$, elements of $H^1(\Gamma, G_{\mathrm{ad}})$ are called *inner forms*. Finally, when $A = G$ we call the elements $\delta \in H^1(\Gamma, G)$ *pure inner forms*. The maps $G \to G_{\mathrm{ad}}$ and $G_{\mathrm{ad}} = \mathrm{Inn}(G) \hookrightarrow \mathrm{Aut}\, G$ descend to maps on cohomology:

$$H^1(\Gamma, G) \to H^1(\Gamma, G_{\mathrm{ad}}) \to H^1(\Gamma, \mathrm{Aut}\, G)$$

Using these maps we can associate a rational form to any pure inner form, although this association is in general neither injective or surjective. If δ is a pure inner form of G, then we will denote $G(F, \delta)$ the rational form of G so obtained.

Kottwitz has given an alternative characterization of pure inner forms which will turn out to be more convenient for our exposition of Vogan's work [Kot84]. There exists a canonical commutative square:

$$\begin{tikzcd}
H^1(\Gamma, G) \arrow[r] \arrow[d, "\sim"] & H^1(\Gamma, G_{\mathrm{ad}}) \arrow[d, "\sim"] \\
\mathrm{Irr}\, \pi_0(Z(\hat{G})^\Gamma) \arrow[r] & \mathrm{Irr}\, \pi_0(Z(\hat{G}_{\mathrm{sc}})^\Gamma)
\end{tikzcd}$$

In the above diagram, the vertical maps are bijections, $\hat{G}_{\mathrm{sc}}$ is the simply connected covering group of $\hat{G}_{\mathrm{ad}}$, and $Z(\hat{G})^\Gamma$ denotes the invariants for the Galois action of Γ on the center of $\hat{G}$. The Kottwitz isomorphism allows us to think of pure inner forms as being parameterized by characters. If δ is a pure inner form, then we will denote its image under the Kottwitz isomorphism by $\chi_\delta : \pi_0(Z(\hat{G})^\Gamma) \to \mathbb{C}^*$.

A *representation of a pure inner form of* G is a pair (π, δ) where δ is a pure inner form and π is an admissible representation of $G(F, \delta)$. We will denote the set of equivalence classes of representations of pure inner forms of G by $\Pi(G/F)$.

3.2.3 The Langlands correspondence for pure inner forms

For brevity of notation, we let $L_F = W_F \times SL_2(\mathbb{C})$.

Definition 3.2.1. *Let G be a reductive algebraic group over F. A Langlands parameter for G (or simply L-parameter) is a continuous homomorphism $\phi : L_F \to {}^L G$ satisfying:*

1. *If $p : {}^LG \to W_F$ and $q : W_F \times SL_2(\mathbb{C})$ are the projection morphisms, then $p \circ \phi = q$*

2. *The restriction of ϕ to $SL_2(\mathbb{C})$ is a morphism of algebraic varieties*

3. *The image of $\phi|_{W_F}$ consists of semisimple elements*

The group $\hat{G}$ acts on the set of all Langlands parameters by conjugation; we denote the set of Langlands parameters for G by $\Phi(G/F)$.

For any $w \in W_F$, we denote by $d_w \in \mathrm{SL}_2(\mathbb{C})$ the element:

$$d_w = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \tag{3.1}$$

where $|\cdot| : W_F \to \mathbb{R}$ is a fixed norm homomorphism, trivial on I_F and sending Fr to q. When $\phi \in \Phi(G/F)$ is an L-parameter, we can associate its *infinitesimal parameter*, which is the homomorphism $\lambda : W_F \to {}^LG$ defined by the conditin $\lambda(w) = \phi(w, d_w)$ for all $w \in W_F$.

If G' is any quasi split pure inner form of G, then there is an identification ${}^LG \simeq {}^LG'$, and an inclusion $\Phi(G/F) \hookrightarrow \Phi(G'/F)$ [Bor79]. To any Langlands parameter for G, we may associate its *L-component group*:

$$A_\phi = \pi_0(Z_{\hat{G}}(\phi(L_F))) = Z_{\hat{G}}(\phi(L_F))/Z_{\hat{G}}(\phi(L_F))^\circ$$

Every central element $z \in Z({}^LG)^\Gamma$ gives a way to twist Langlands parameters. Write $\phi(w, x) = \varphi(w, x) \rtimes w$ for a cocycle $\varphi \in H^1(L_F, \hat{G})$, then $z \cdot \phi(w, x) = (z\varphi(w, x)) \rtimes w$. Furthermore, every such central element naturally yields an unramified character of $G(F, \delta)$ [SZ14]. For example, when $G(F) = \mathrm{GL}_n(F)$ and $z = \mathrm{diag}(q^a, \ldots, q^a)$, $a \in \mathbb{C}$, we have $\chi_z : \mathrm{GL}_n(F) \to \mathbb{C}^*$ is the character $\chi_z(g) = |\det(g)|^a$.

Every equivalence class of Langlands parameters for G is conjectured to have an associated packet of representations of pure inner forms, satisying a list of properties [Vog93, Bor79]:

Conjecture 3.2.1. *(Local Langlands conjecture for pure inner forms) Let G be a connected reductive group over F, and let ${}^LG$ be an L-group for G. For every $\phi \in \Phi(G/F)$, there exists an L-packet $\Pi_\phi \subseteq \Pi(G/F)$ satisfying the following list of properties:*

1. $\Pi(G/F) = \coprod_{\phi \in \Phi(G/F)} \Pi_\phi$

2. *There exists a bijection $\Pi_\phi \to \mathit{Irr}\, A_\phi$*

3. *If δ is a pure inner form of G, then then $\Pi_\phi(\delta) = \{\pi \in \Pi(G(F, \delta)) \,|\, (\pi, \delta) \in \Pi_\phi\}$ is finite. If δ corresponds to the quasi-split rational form of G, then $\Pi_\phi(\delta)$ is non-empty.*

4. *For every $\alpha \in H^1(L_F, Z({}^LG))$, if $\phi' = \alpha \cdot \phi$ then $\Pi_{\phi'} = \{\pi_\alpha \otimes \phi \,|\, \pi \in \Pi_\phi\}$*

5. *The following conditions on Π_ϕ are equivalent:*

   (a) Π_ϕ *contains a discrete series representation*

   (b) Every representation in Π_ϕ is a discrete series representation

   (c) $\phi(L_F)$ *is a discrete set*

6. *The following conditions on Π_ϕ are equivalent:*

(a) Π_ϕ *contains a tempered representation*

(b) Every representation in Π_ϕ *is tempered*

(c) $\phi(L_F)$ *is bounded*

7. *Other properties that will not be relevant for this thesis. See [Bor79] for the complete list.*

The bijection $\Pi_\phi \to \operatorname{Irr} A_\phi$ and the Kottwitz isomorphism give a decomposition of the packets Π_ϕ by pure inner forms:

$$\Pi_\phi = \coprod_{\delta \in H^1(\Gamma, G)} \Pi_\phi(\delta)$$

Indeed, the injection $Z(\hat{G})^\Gamma \hookrightarrow Z_{\hat{G}}(\phi(L_F))$ induces a map $\zeta : \pi_0(Z(\hat{G})^\Gamma) \to Z(A_\phi)$. If $\psi \in \operatorname{Irr} A_\phi$ has central character $z_\psi : Z(A_\phi) \to \mathbb{C}^*$, then $\zeta^*(z_\psi) : \pi_0(Z(\hat{G})^\Gamma) \to \mathbb{C}^*$ corresponds through the Kottwitz isomorphism to a pure inner form δ.

3.3 Geometrization of the local Langlands correspondence

3.3.1 Vogan varieties

It was soon thereafter noticed that the characterization of objects in an L-packet by irreducible representations of the finite group A_ϕ closely resembles the characterization of simple, equivariant perverse sheaves on an algebraic variety [Vog93, Lus95b]. When a linear algebraic group acts on a variety, the simple objects in the category of equivariant D-modules are classified by pairs $(\mathcal{O}, \mathcal{L})$, where $\mathcal{O}$ is an orbit and $\mathcal{L}$ is an equivariant, irreducible local system on $\mathcal{O}$. The category of equivariant local systems on $\mathcal{O}$ is equivalent to the category of representations of the deck transformations of a covering associated to $\mathcal{O}$ using the group action. To each pair $(\mathcal{O}, \mathcal{L})$, there is a corresponding simple equivariant perverse sheaf, and further, all such simple equivariant perverse sheaves are obtained from some $(\mathcal{O}, \mathcal{L})$.

Vogan set out to construct a complex variety X_λ for which many of the properties of Conjecture 3.2.1 will arise from purely geometric considerations. The points in X_λ are to be in bijection with Langlands parameters, and the notion of equivalence of Langlands parameters should arise as an orbit equivalence for an action of $\hat{G}$ on X_λ. Therefore, the representations in an L-packet Π_ϕ must correspond to the representations of a finite group attached to the orbit corresponding to x_ϕ. If $x_\phi \in S$ is a representative basepoint of some orbit, then we may define:

$$A_\phi = \pi_0(\operatorname{Stab}_{G_\lambda}(x_\phi))$$

By our previous remarks, the category of irreducible representations of A_ϕ (modulo conjugacy) is equivalent to the category of $\hat{G}$-equivariant local systems on S. By minimal extension, these local systems give $\hat{G}$-equivariant D-modules on X_λ.

With these geometric constructions in place, packets of irreducible representations of pure inner forms can now be thought of as packets of simple $\hat{G}$-equivariant D-modules. The representations of rational forms corresponding to each δ - i.e. the elements of $\Pi_\phi(\delta)$ - are to be distinguished by the central character of the equivariant D-module.

Vogan's construction works as follows. Since every Langlands parameter has a unique infinitesimal character, it suffices to construct, for every $\lambda : W_F \to {}^LG$, a complex variety V_λ whose points correspond to Langlands parameters with infinitesimal character $\lambda_\phi = \lambda$. We restrict our attention to the set of ϕ whose corresponding infinitesimal character is some fixed morphism $\lambda : W_F \to {}^L G$. Since λ is continuous and I_F has the profinite topology, λ must factor through a finite quotient of the inertia subgroup. As such, the image $\lambda(I_F)$ is a discrete set; thus, $\hat{G}^{I_F} = Z_{\hat{G}}(\lambda(I_F))$ is a reductive subgroup of $\hat{G}$. Furthermore, since I_F is normal in W_F, $\lambda(\mathrm{Fr})$ must normalize $\hat{G}^{I_F}$, and therefore acts as a semisimple automorphism on the Lie algebra $\hat{\mathfrak{g}}^{I_F}$.

The restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is a morphism of algebraic groups, so we get a corresponding map of Lie algebras $d\phi : \mathfrak{sl}_2 \to \hat{\mathfrak{g}}$. In other words, every Langlands parameter yields a sl_2-triple for $\hat{\mathfrak{g}}$ in a natural way. Since $\lambda_\phi(w) = \phi(w, d_w)$ for all $w \in W_F$, we require:

$$d\phi(h) = d\phi \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \frac{\log \lambda(\mathrm{Fr})}{\log q} \in \hat{\mathfrak{g}}^{I_F}$$

recalling that q is the cardinality of the residue field of F. With this restriction, ϕ is determined by the value of $d\phi(e)$, where e is the standard generator of the positive root space of $\mathfrak{sl}_2$. It is not difficult to show that $[d\phi(h), d\phi(e)] = 2\, d\phi(e)$ if and only if $\mathrm{Ad}_{\lambda(\mathrm{Fr})}(d\phi(e)) = q\, d\phi(e)$. Then, points in V_λ -i.e. homomorphisms $\phi : L_F \to {}^L G$ with infinitesimal character λ - correspond bijectively with points in the q-eigenspace for the action of $\mathrm{Ad}(\lambda(\mathrm{Fr}))$ on $\hat{\mathfrak{g}}^{I_F}$. We define:

$$V_\lambda = \left\{ X \in \hat{\mathfrak{g}}^{I_F} \mid \mathrm{Ad}_{\lambda(\mathrm{Fr})}(X) = qX \right\}$$

The group $\hat{G}_\lambda = Z_{\hat{G}}(\lambda(W_F))$ acts on V_λ through the restriction of the adjoint action.

We now let $\mathcal{O} \subseteq \hat{G}$ denote a fixed semisimple conjugacy class. Define the set:

$$P(\mathcal{O}, {}^L G) = \left\{ \phi : L_F \to {}^L G \mid \lambda_\phi(\mathrm{Fr}) \in \mathcal{O} \right\}$$

Clearly, $\hat{G}$ acts on $P(\mathcal{O}, {}^L G)$ by conjugation. Vogan's construction yields the following theorem:

Theorem 3.3.1. *[Vog93] There is a $\hat{G}$-equivariant bijection:*

$$T : P(\mathcal{O}, {}^L G) \to \hat{G} \times_{\hat{G}^\lambda} V_\lambda := X_\lambda$$

The orbits of $\hat{G}$ on $P(\mathcal{O}, \hat{G})$ are in bijection with the orbits of $\hat{G}_\lambda$ on V_λ. Furthermore, there is an isomorphism:

$$A_\phi \simeq \pi_0(\mathrm{Stab}_{\hat{G}}(T(\phi))$$

Consequently, if $\phi \in P(\mathcal{O}, \hat{G})$, then the simple $\hat{G}$-equivariant perverse sheaves on X_λ are in bijection with the irreducible admissible representations in Π_ϕ.

By induction equivalence [BL06], the category of $\hat{G}$-equivariant D-modules on X_λ is equivalent to the category of $\hat{G}_\lambda$-equivariant D-modules on V_λ. Since the variety V_λ and its orbit stratification are much simpler, we will always study the latter category of D-modules. Throughout this thesis, we will refer to V_λ as a *Vogan variety.*

3.3.2 A selection of examples

It will be helpful to keep a number of examples in mind. In what follows, we say that V_λ is *unramified* if $\lambda : W_F \to {}^LG$ is trivial on I_F; otherwise, we say that V_λ is *ramified*.

Example 3.3.1. *Unramified, regular Vogan varieties for split classical groups*

The following class of examples will constitute the main objects of study of chapter three. When $G(F)$ is split, its corresponding L-group is simply the direct product $\hat{G} \times W_F$. This means that L-parameters are simply group homomorphisms $\phi : L_F \to \hat{G}$. We say that an infinitesimal parameter $\lambda : W_F \to \hat{G}$ is *regular and unramified* when λ is trivial on inertia and $\lambda(\mathrm{Fr})$ is a regular semisimple element. By conjugating λ, we may as well assume that $\lambda(\mathrm{Fr})$ is contained in a fixed maximal torus $\hat{T}$. By the restriction that $\lambda(\mathrm{Fr})$ is regular semisimple we have that $\hat{G}_\lambda = \hat{T}$, and by the restriction that λ is unramified we have that $V_\lambda \subseteq \hat{\mathfrak{g}}$ is the q-eigenspace for the adjoint action of $\lambda(\mathrm{Fr})$.

Since $\lambda(\mathrm{Fr}) \in \hat{T}$, V_λ can be written as a product of root spaces. Letting $R_\lambda = \{\alpha \in R \,|\, \alpha(\lambda(\mathrm{Fr})) = q\}$, we have:

$$V_\lambda = \prod_{\alpha \in R_\lambda} E_\alpha$$

The maximal torus $\hat{T}$ acts on V_λ by its action on the corresponding root spaces, and the orbits of this action are clearly in bijection with subsets $J \subseteq R_\lambda$ (J simply determines which coordinates can be non-zero). Stated slightly differently, the orbit S_J is determined by the unipotent class in $\hat{\mathfrak{g}}$ determined by $d\phi(e)$. The study of equivariant D-modules on these varieties will be taken up in details in chapter 3, so at the moment we will say no more about it other than it is possible to explicitly present these modules by generators and relations. We will make a comment about which representations should be in the packets Π_ϕ corresponding to these orbits (c.f. section 10.4, [Bor79] and Example 4.9, [Vog93]).

Let δ be the pure inner form whose corresponding rational form is G. The element $\lambda(\mathrm{Fr}) \in \hat{T}$ corresponds to an unramified character of $T(F, \delta)$ in a natural way. When $\hat{G}$ is a matrix group, then we can write $\lambda(\mathrm{Fr}) = \mathrm{diag}(q^{a_1}, \ldots, q^{a_n})$ for some complex numbers $a_1, \ldots, a_n \in \mathbb{C}$. The corresponding unramified character is obtained as follows:

$$\chi_{\mathrm{Fr}} : T(F, \delta) \to \mathbb{C}^*$$

$$\chi_{\mathrm{Fr}}(t_1, \ldots, t_n) = |t_1|^{a_1} \, |t_2|^{a_2} \ldots |t_n|^{a_n}$$

Where here $| \cdot | : F \to \mathbb{R}$ is the norm on F coming from its $\mathfrak{p}$-adic valuation. Normalized parabolic induction through any Borel subgroup containing $T(F, \delta)$ gives a representation:

$$PS(\chi_{\mathrm{Fr}}) = i_B^G(\chi_{\mathrm{Fr}})$$

The composition factors of this representation are in bijection with the subsets of R_λ [Cas80]. For an orbit $S_J \subseteq V_\lambda$, there is always a trivial representation of the component group A_J, and having trivial central character, it corresponds to a representation of the split form of the group. This representation should be the composition factor corresponding to the subset $J \subseteq R_\lambda$.

Example 3.3.2. *Unramified Vogan varieties in GL_n as spaces of quiver representations*

An unramified inifinitesimal parameter $\lambda : W_F \to \mathrm{GL}_n(\mathbb{C})$ is completely determined by where the homomorphism sends a lift of the Frobenius element. The element $\lambda(\mathrm{Frob})$ is semisimple, so we may as well assume that it lies in the subgroup of diagonal matrices:

$$\lambda(\mathrm{Frob}) = \mathrm{diag}(q^{a_1}, \ldots, q^{a_n})$$

For some choice of complex numbers $a_1, \ldots, a_n$. We use this data to construct a quiver as follows: Build a graph whose vertices v_{a_i} correspond to the distinct a_i. There is an edge joining $v_{a_i} \to v_{a_j}$ if and only if $a_i - a_j = 1$. The quiver Q is obtained by associating V_{a_i}, the q^{a_i}-eigenspace of $\lambda(\mathrm{Frob})$, to the vertex v_{a_i}. The variety of quiver representations $\mathrm{Rep}\, Q$ is acted on by $\prod_i \mathrm{GL}(V_{a_i}) = Z_{\mathrm{GL}_n}(\lambda(\mathrm{Frob}))$

Example 3.3.3. *A ramified example for $PGL_4(F)$*

The simplest class of ramified Vogan varieties are those which are "tamely ramified" - i.e. their infinitesimal parameter factors through the tame inertia subgroup P_F. Let i_F denote a lift of the generator of I_F/P_F. The infinitesimal parameter is determined by a choice of two semisimple elements, $f = \lambda(\mathrm{Frob})$ and $\tau = \lambda(i_F)$, satisfying $f\tau f^{-1} = \tau^q$. For $\mathrm{PGL}_4(F)$, an infinitesimal parameter is a map $\lambda : W_F \to \mathrm{SL}_4(\mathbb{C})$. We make the choices:

$$\tau = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^q & 0 \\ 0 & 0 & 0 & \zeta^q \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \\ q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \end{pmatrix}$$

Where ζ is a both a $q^2 - 1$'th root of unity and a $2q + 2$'th root of unity. The former condition ensures that $f\tau f^{-1} = \tau^q$, while the latter ensures $\tau \in \mathrm{SL}_4(\mathbb{C})$.

The Vogan variety is the q-eigenspace of Frobenius acting on $\mathrm{Lie}\, Z_{\mathrm{SL}_4}(\tau)$. It is easy to see that this consists of all matrices of the form:

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

The group ${}^\vee G_\lambda = Z_{\mathrm{SL}_4}(f)$ acts on V_λ. Notice that f can be decomposed as a commuting product:

$$f = f_e f_h = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$$

and an element commutes with f if and only if it commutes with both of these matrices. So $\hat{G}_\lambda$ is the disconnected group consisting of diagonal matrices $t = \mathrm{diag}(t_1, t_2, t_1, t_2)$ subject to the condition

$(t_1t_2)^2 = 1$, and it acts on V_λ by the restriction of the adjoint action. For this example, the Vogan variety can be identified $V_\lambda \simeq \mathbb{C}$ with the action $(t_1, t_2) \cdot x = t_1 t_2^{-1} x$.

Example 3.3.4. *Two examples in $SO_7(F)$*

When $G(F) = \mathrm{SO}_7(F)$ we obtain a Langlands dual group $\hat{G} = \mathrm{Sp}_6(\mathbb{C})$. We use the symplectic form:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and fix a maximal torus consisting of matrices $t = \mathrm{diag}(t_1, t_2, t_3, t_3^{-1}, t_2^{-1}, t_1^{-1})$. Consider the following two unramified infinitesimal parameters:

$$\lambda_1(\mathrm{Fr}) = \mathrm{diag}(q^{1/2}, q^{1/2}, q^{1/2}, q^{-1/2}, q^{-1/2}, q^{1/2})$$

$$\lambda_2(\mathrm{Fr}) = \mathrm{diag}(q^{3/2}, q^{1/2}, q^{1/2}, q^{-1/2}, q^{-1/2}, q^{-3/2})$$

For λ_1, we get an action of the group $\hat{G}_{\lambda_1} = \mathrm{GL}_3(\mathbb{C})$ on the variety which is the product of the following root spaces in $\mathfrak{sp}_6$:

$$\begin{pmatrix} 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries in the above matrix are not all independent; if X is the 3×3 matrix corresponding to the top right block in the above matrix, then it must give a symmetric bilinear form if the corresponding point of V_λ is to be an element of $\mathfrak{sp}_6$. Raicu has classified the GL_3-equivariant D-modules on the space of 3×3 symmetric matrices and computed their characteristic cycles [Rai16]; actually, his result is far more general, in that he characterizes the equivariant D-modules on any space of symmetric matrices. These spaces will arise as Vogan varieties for infinitesimal parameters in higher rank Sp_{2n}.

For λ_2, we get an action of the group $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on the Vogan variety $V_{\lambda_2} \simeq \mathbb{C}^5$. A point in V_λ can be written:

$$x_\phi = \begin{pmatrix} 0 & u & v & 0 & 0 & 0 \\ 0 & 0 & 0 & z & y & 0 \\ 0 & 0 & 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can use different coordinates to write $x_\phi = (\omega, X)$, where $\omega = (u, v)$ and:

$$X = \begin{pmatrix} z & y \\ x & -z \end{pmatrix}$$

The action of $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ is given by:

$$(t, g) \cdot (\omega, X) = (t \det(g) \omega g^{-1}, \det(g) g X g^{-1})$$

There are seven orbits for this group action. They are:

1. $(0, 0)$
2. $(\omega, 0)$, with $\omega \neq 0$
3. $(0, X)$ with $X \neq 0$ and $\det X = 0$
4. (ω, X) with $X \neq 0$, $\det X = 0$ and $\omega X = 0$
5. $(0, X)$ with $X \neq 0$ and $\det X \neq 0$
6. (ω, X) with $X, \omega \neq 0$ and ω is an eigenvector of X
7. (ω, X) with $X, \omega \neq 0$ and ω is not an eigenvector of X

The point of this example is to show that the geometry of the singularities of these orbits can become quite complicated. The equivariant D-modules on V_λ arise from minimal extension of local systems on these orbits, and should be presentable as modules over $\mathbb{C}\langle u, v, x, y, z, \partial_u, \partial_v, \partial_x, \partial_y, \partial_z\rangle$. It is unclear to the author how to present the minimal extensions by generators and relations, or how to compute the characteristic cycles directly using D-module theory. It is for this reason that in this thesis we have focused our attention on the case that $\lambda(\mathrm{Fr})$ is a regular semisimple element. Regardless, these kinds of examples can be investigated using other methods. For details see a forthcoming paper [CFM$^+$17b].

3.4 Character relations coming from the theory of endoscopy

In this section we briefly review Arthur's work on endoscopic character formulas for non-tempered L-packets. This section is not intended to be a comprehensive review, but is meant to capture the essential structural elements of Arthur's work that will allow us to motivate some geometric constructions. Let G be a quasi-split classical group over F. When $\phi \in \Phi(G)$ is a tempered parameter (i.e. its image is bounded), there is a bijection between pairs $(G', \phi') \leftrightarrow (\phi, s)$, where:

- G' denotes an endoscopic datum for G [Art06]
- $\phi' \in \Phi(G')$
- $s \in Z_{\hat{G}}(\phi)$

This bijection can be used to relate linear combinations of the Harish-Chandra characters of the representations in Π_ϕ to certain stable distributions on G'. The study of invariant distributions on G in terms of these so called stable distributions on endoscopic subgroups is called endoscopy, which we will now describe in more detail, following [Art06].

Recall that an element $\gamma \in G(F)$ is called *strongly regular* if the stabilizer of γ under conjugacy is a torus. Denote the set of strongly regular elements by $\Gamma(G(F))$. A *stable conjugacy class* is a disjoint union of conjugacy classes in $G(F)$ which become identified upon passing to the algebraic closure. The set of all stable, strongly regular conjugacy classes is denoted $S\Gamma(G(F))$. We will denote orbital integrals by $O(\gamma, f)$, for $f \in C_c^\infty(G(F))$ and $\gamma \in \Gamma(G(F))$ a strongly regular conjugacy class. An orbital integral is an expression of the form

$$O(\gamma, f) = \int_{G/G_\gamma} f(x\gamma x^{-1})\, d\mu(x)$$

for a chosen invariant measure μ on G/G_γ. Recall that a function on $G(F)$ is called *invariant* if it is in the closed linear span of the orbital integrals, and we denote the subspace of invariant functions by $I(G)$. The dual space $\hat{I}(G)$ is the space of invariant distributions. Similarly, we recall that stable orbital integrals $SO(\delta, f)$ are given by expressions:

$$SO(\delta, f) = \sum_{\gamma \to \delta} O(f, \gamma)$$

where $f \in C_c^\infty(G(F))$, $\delta \in S\Gamma(G(F))$ is a stable, strongly regular conjugacy class, and the sum above is taken over those conjugacy classes γ which become identified over $\overline{F}$. The stably invariant functions $SI(G)$ are the closed linear span of the stable orbital integrals. A stable distribution is an element of the dual space $\hat{SI}(G)$.

The theory of endoscopy developed by Langlands, Shelstad, and Kottwitz [LS87, KS99b] allows one to express certain invariant distributions on $G(F)$ in terms of stable distributions on an endoscopic group $G'(F)$.

$$\text{Trans}_{G'}^{G} : \hat{SI}(G') \to \hat{I}(G)$$

This transfer mapping arises as the pullback on distributions of a map $I(G) \to SI(G')$, which we now describe.

The Langlands-Shelstad transfer mapping for the endoscopic group G' is a function:

$$\Delta : S\Gamma(G') \times \Gamma(G) \to \mathbb{C}$$

For a fixed endoscopic datum, Δ is canonically determined up to scalar multiplication by a complex number of norm one. This function has the property that for any fixed $\delta' \in S\Gamma(G')$, there are only finitely many strongly regular conjugacy classes in G such that $\Delta(\delta', \gamma) \neq 0$. The function Δ allows one to transfer invariant functions on G to functions on G' by using Δ as a kernel:

$$f \mapsto f'(\delta') = \sum_{\gamma} \Delta(\delta', \gamma) O(\gamma, f)$$

It is a consequence of the fundamental lemma [Ngô10, Wal97] that the image of the Langlands-Shelstad transfer is contained in $SI(G')$ - i.e. there exists a $g \in C_c^\infty(G')$ such that:

$$SO(\delta', g) = f'(\delta')$$

The transfer mapping $\mathrm{Trans}_{G'}^{G}$ arises as the dual map on distributions. The takeaway message is that stable distributions on an endoscopic group yield invariant distributions on G.

An important property of L-packets is that that to each tempered Langlands parameter $\phi' \in \Phi_{\mathrm{bdd}}(G')$ there exists a corresponding stable distribution:

$$D_{\phi'}(g) = \sum_{\pi' \in \Pi_{\phi'}} \mathrm{tr}\,\pi'(g) \tag{3.2}$$

which is a sum of the Harish-Chandra characters of the representations in the L-packet determined by ϕ'. This stable distribution on G' can be transfered to an invariant distribution on G, as described above. The resulting distribution is then a linear combination of orbital integrals, as described by the following theorem [She79, LL79]:

Theorem 3.4.1. *For any $\phi \in \Phi_{bdd}(G)$ and for any $s \in Z_{\hat{G}}(\phi)$, there exists an endoscopic group G' and a Langlands parameter $\phi' \in \Phi(G')$ such that the endoscopic transfer of the stable distribution $D_{\phi'}$ (as in Equation 3.2) on G' can be expressed by the following character formula:*

$$\mathrm{Trans}_{G'}^{G}(D_{\phi'})(f) = \sum_{\pi \in \Pi_{\phi}} \langle s, \pi \rangle\, \mathrm{tr}\,\pi(f) \tag{3.3}$$

The coefficients $\langle s, \pi\rangle$ and representations appearing in an L-packet for G are therefore implicitly determined by the transfer factors when one varies over endoscopic subgroups.

It is natural to attempt to expand the theory of endoscopy to deal with the L-parameters which correspond to non-tempered representations. Unfortunately, the naive attempts to carry out this generalization fail because the sum of the Harish-Chandra characters for the representations in a non-tempered packet fails to be stable. Arthur came across this when attempting to classify discrete series automorphic representations [Art89], as non-tempered irreducible representations arise as local constituents of such automorphic representations. To remedy this problem, he invented a new class of local parameters.

Definition 3.4.1. *Let G be a reductive group over F, and ${}^LG$ an L-group for G. An Arthur parameter (or A-parameter) for G is a homomorphism:*

$$\psi : W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to {}^LG$$

$$(w, x, y) \mapsto \psi^{\circ}(w, x, y) \rtimes w$$

such that:

1. *The restriction of $\psi|_{W_F \times SL_2}$ is a Langlands parameter for G*
2. *The restriction of ψ° to the last copy of SL_2 is a morphism of algebraic groups*
3. *$\psi^{\circ}|_{W_F}$ has bounded image in the complex topology on $\hat{G}$*

The collection of all A-parameters for G will be denoted $\Psi(G)$.

To an A-parameter ψ, we associate the Arthur component group:

$$\mathcal{S}_{\psi} = \frac{Z_{\hat{G}}(\psi)}{Z_{\hat{G}}(\psi)^{\circ}\, Z(\hat{G})^{\Gamma}}$$

and denote $\hat{\mathcal{S}}_\psi$ the collection of irreducible representations of $\mathcal{S}_\psi$.

Recall the element $d_w \in \mathrm{SL}_2(\mathbb{C})$ given by Equation 3.1. There is a natural way to associate a (non-tempered, in general) L-parameter to ψ. To every ψ, we may associate the L-parameter:

$$\psi \mapsto \phi_\psi(w,x) = \psi\,(w,x,d_w)$$

The Jacobson-Morozov theorem implies that the correspondence $\psi \mapsto \phi_\psi$ is an injection. Going one step further, we may also associate an infinitesimal parameter to ψ:

$$\psi \mapsto \lambda_\psi(w) = \psi(w,d_w,d_w)$$

We will denote the set of Arthur parameters whose corresponding infinitesimal parameter is λ by $\Psi(G/F,\lambda)$.

To develop an analogue of Theorem 3.4.1 in the non-tempered setting using Arthur parameters, certain features of the theory need to be reconsidered. First, a bijection between pairs $(\psi,s) \in \Psi(G) \times Z_{\hat{G}}(\psi)$ and endoscopic subgroups with an Arthur parameter (G',ψ') must be established. We must also suitably reinterpret the objects on both the left and the right hand sides of equation 3.3.

The left hand side of 3.3 needs to be the Langlands-Shelstad transfer of a stable distribution from the endoscopic group G'. When G' is a special orthogonal or symplectic group, Arthur has constructed this stable distribution [Art13]. We have the following three mappings:

1. Since G' is an endoscopic group, one has a map ${}^L G' \hookrightarrow \mathrm{GL}_N(\mathbb{C})$, and we may use this map to get an injection $\Psi(G') \hookrightarrow \Psi(N)$, where here $\Psi(N)$ is referring to a certain class of representations of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{GL}(N,\mathbb{C})$.

2. By the above discussion, we have an injection $\Psi(N) \hookrightarrow \Phi(N)$, where $\Phi(N)$ denotes the the collection of Langlands parameters for $\mathrm{GL}(N,F)$

3. The local Langlands correspondence, established by Henniart, Harris, Sholze, and Taylor [Sch13, Hen86, Har98, HT01] yields a bijection $\Phi(N) \to \Pi(N)$, where $\Pi(N)$ denotes the irreducible admissible representations of $\mathrm{GL}(N,F)$.

Tracing through the composition:

$$\Psi(G') \hookrightarrow \Psi(N) \hookrightarrow \Phi(N) \to \Pi(N)$$

to each Arthur parameter ψ' one may attach a representation π_ψ of $\mathrm{GL}(N,F)$.

When the endoscopic group is simple, the necessary stable distribution attached to ψ is constructed as follows. The Kottwitz-Shelstad twisted endoscopic transfer of functions from $\mathrm{GL}(N,F)$ to $G'(F)$ is surjective. Furthermore, if we lift a function $f \in G'(F)$ through the twisted endoscopic transfer to a function $\tilde{f}$, the value of $\mathrm{tr}\,\pi_\psi(\tilde{f})$ only depends on f. The distribution $D_\psi(f) = \mathrm{tr}\,(\pi_\psi(\tilde{f}))$ is stable. When the endoscopic group is not simple, but is a product of simple groups, the necessary stable distribution is obtained as a product of the aforementioned distributions over each simple factor. One of Arthur's primary results is the following theorem:

Theorem 3.4.2 (Arthur, Thm. 2.2.1)**.** *(a) Suppose that G' is a quasi-split special orthogonal or symplectic endoscopic group of $GL(N,F)$ and that $\psi' \in \Psi(G')$. There exists a unique stable linear form on*

$\mathcal{H}(G')$:

$$f \mapsto D_{\psi'}(f)$$

with the properties:

1. *$D_{\psi'}(f) = \operatorname{tr} \pi_\psi(\tilde{f})$ for all $f \in \mathcal{H}(G')$*

2. *If $G' = G_S \times G_O$, $\psi' = \psi_S \times \psi_O$, and $f = f^S \times f^O$, (i.e. G' is a product of simple endoscopic groups) then $D_{\psi'}(f) = D_{\psi_S}(f_S) D_{\psi_O}(f_O)$.*

(b) If G is a simple elliptic endoscopic group of $GL(N, F)$, then for every $\psi \in \Psi(G)$ there exists a finite set Π_ψ of unitary representations of G, together with a mapping:

$$\Pi_\psi \to \hat{\mathcal{S}}_\psi$$

$$\pi \mapsto \langle \cdot\,, \pi \rangle$$

such that if $s \in Z_{\hat{G}}(\psi)$ and (G', ψ') is the pair corresponding to (ψ, s), then:

$$\mathrm{Trans}_{G'}^{G}(D_{\psi'})(f) = \sum_{\pi \in \Pi_\psi} \langle s_\psi x, \pi \rangle \operatorname{tr} \pi(f)$$

Where x is the image of s in $\mathcal{S}_\psi$, and s_ψ is the canonical central element in $\mathcal{S}_\psi$.

We will refer to the packets Π_ψ in the statement of Theorem 3.4.2 as A-packets. Arthur's theorem gives the analogue of the endoscopic character relations for classical groups in the non-tempered setting. Supposing we are interested in character relations for G, the first part of the theorem should be understood as describing how to use twisted Kottwitz-Shelstad transfer to build stable distributions on G', an endoscopic group of G. The second part of the theorem tells us how the transfer of that stable distribution to G can be written as a sum of Harish-Chandra characters for representations in the packet Π_ψ.

A fundamental problem of interest is to explicitly determine the packets Π_ψ and the association $\pi \mapsto \langle \cdot, \pi \rangle$. When ψ is trivial on the second factor of $\mathrm{SL}_2(\mathbb{C})$, the corresponding L-parameter is tempered and one recovers the usual theory of endoscopy; in this case, the L-packet should be exactly equal to the A-packet. When ψ is non-trivial on the second factor, one only obtains an inclusion of packets $\Pi_{\phi_\psi} \hookrightarrow \Pi_\psi$. One would like to have some way of understanding which "extra" representations need to be in the packet in order to get endoscopic character formulas.

3.5 Arthur parameters and Vogan's geometrization: The $\mathfrak{p}$-adic microlocal conjecture

It is natural to ask how Arthur's work on endoscopic character formulas should fit into Vogan's geometrization of the local Langlands correspondence. In the case of real groups, the work of Adams, Barbasch, and Vogan [ABV12] demonstrates that microlocal geometry gives a natural setting in which to understand endoscopic character formulas for non-tempered Arthur parameters. In their book, they prove an analogue of part (b) of Theorem 3.4.2 using purely geometric constructions. Packets of simple perverse sheaves are built by grouping together those simple perverse sheaves whose D-modules contain a common irreducible component in their characteristic cycle. Furthermore, the characters $\langle \cdot, \pi \rangle$ are constructed using microlocalization. They do not prove that these packets are equal to Arthur packets, however, because they do not address the issues in part (a) of Theorem 3.4.2. While the structure of the varieties V_λ is somewhat different in the archimedean setting, the essence of their constructions carries over to $\mathfrak{p}$-adic groups. We expose these ideas in this section.

With a view towards incorporating representations of pure inner forms of G, instead of using the group $\mathcal{S}_\psi$, we will instead consider the group:

$$A_\psi = Z_{\hat{G}}(\psi)/Z_{\hat{G}}(\psi)^\circ$$

If $\psi \in \Psi(G/F)$ and $\phi_\psi \in \Phi(G/F)$ is the associated Langlands parameter, then $\operatorname{im}\phi_\psi \subseteq \operatorname{im}\psi$. This gives an inclusion:

$$Z_{\hat{G}}(\psi) \hookrightarrow Z_{\hat{G}}(\phi_\psi)$$

which descends to a morphism of component groups:

$$A_\psi \to A_{\phi_\psi}$$

What is required is a natural geometric construction of both:

1. The packet Π_ψ
2. The association $\pi \mapsto \langle\, , \pi \rangle \in \hat{A}_\psi$

Furthermore, we would like to maintain the interpretation of representations of A_ψ as local systems, and we would also like there to be some natural way to obtain these local systems from the existing D-modules which represent local systems on $\hat{G}_\lambda$-orbits on V_λ.

Following Ginzburg [Gin86], we will see how to accomodate all of the above requirements. Suppose that $\mathcal{M}$ is a D-module on a complex variety X. Write its characteristic cycle:

$$CC(\mathcal{M}) = \sum_{\alpha \in I} m_\alpha(\mathcal{M})\,[\Lambda_\alpha]$$

where the collection $\{\Lambda_\alpha\}$ is the set of Lagrangian subvarieties of T^*X which appear in the singular support of $\mathcal{M}$. Let:

$$\Lambda_\alpha^{\mathrm{reg}} = \Lambda_\alpha \setminus \bigcup_{\beta \neq \alpha} \Lambda_\beta$$

be the regular part of a component of the singular support. Recall from chapter one that if $\mathcal{M}$ is a D-module on a variety X, then by choosing a good filtration of $\mathcal{M}$ and taking the associated graded $\operatorname{gr}\mathcal{M}$ we obtain a sheaf of $\mathcal{O}_{T^*X}$-modules. In fact, $\operatorname{gr}\mathcal{M}|_{\Lambda_\alpha^{\mathrm{reg}}}$ is a local system [Kas83, Theorem 3.2.1, p.70]. The functor which takes D-modules on a variety X and outputs objects over T^*X (which restrict to local systems on the appropriate regular locus) is called the functor of microlocalization. We will denote this functor by:

$$Q^\alpha_{\mathrm{mic}} : \text{D-mod}_{\hat{G}_\lambda}(V_\lambda) \to \mathrm{Loc}(\Lambda_\alpha^{\mathrm{reg}})$$

Microlocalization gives the correct framework in which we may construct packets having the requisite properties. Let $\mathcal{M}(\pi, \delta)$ be a simple, $\hat{G}_\lambda$-equivariant D-module on V_λ which corresponds to a representation π of a pure inner form δ under the local Langlands correspondence. Since $\mathcal{M}(\pi, \delta)$ is equivariant, its characteristic cycle can be written as a sum of the closures of Lagrangian conormals to orbits:

$$CC(\mathcal{M}(\pi,\delta)) = \sum_S m_S(\mathcal{M}(\pi,\delta))\,[\overline{T^*_S V_\lambda}]$$

Definition 3.5.1. *Let $S \subseteq V_\lambda$ be an orbit. The micropacket associated to S is defined to be the set of simple, $\hat{G}_\lambda$-equivariant D-modules on V_λ satisfying the following condition:*

$$\Pi_S^{mic} = \left\{\mathcal{M}(\pi,\delta) : Q^S_{mic}(\mathcal{M}(\pi,\delta)) \neq 0\right\}$$

Remarks:

1. The rank of the local system $Q^S_{\text{mic}}(\mathcal{M}(\pi,\delta))$ is equal to multiplicity $m_S(\mathcal{M}(\pi,\delta))$; so, the coefficients in the characteristic cycle give us information about which D-modules microlocalize non-trivially on the regular part of $T^*_{S_i}V_\lambda$. As such, an equivalent definition of the micropacket Π_S^{mic} is as the collection of D-modules whose characteristic cycles contain $[T^*_S V_\lambda]$.

2. If $\mathcal{M}(\pi,\delta)$ is equal to the minimal extension of a local system on the orbit S, then $[T^*_S V_\lambda]$ is always a non-zero summand of $CC(\mathcal{M}(\pi,\delta))$. The other components of the characteristic cycle come from strata $S' \leq S$. As such, if the orbit S corresponds to an equivalence class of Langlands parameters represented by ϕ, then we have an inclusion $\Pi_\phi \hookrightarrow \Pi_S^{\text{mic}}$.

The second remark above says that the microlocal packet attached to an orbit contains the corresponding L-packet, but it might contain more D-modules. The extra D-modules that get added to the packet are exactly those whose microsupport contains $T^*_S V_\lambda$. This should be compared to requirement that an A-packet Π_ψ must contain the L-packet Π_{ϕ_ψ}, but can also contain additional representations. To make this comparison more concrete, the remainder of this section will be devoted to explaining how Arthur parameters can be thought of as points in a conormal variety to an orbit.

Since $\hat{G}_\lambda$ acts on V_λ (for a fixed $g \in \hat{G}_\lambda$, call the action map $a_g : V_\lambda \to V_\lambda$), we also get a corresponding action on T^*V_λ:

$$\tilde{a}_g : T^*V_\lambda \to T^*V_\lambda$$

$$g\cdot(x,\xi) = (a_g(x), a^*_{g^{-1}}(\xi))$$

Let $x \in V_\lambda$, $S = \hat{G}_\lambda \cdot x$ be the orbit through x, and $\hat{G}_\lambda(x) = \text{Stab}_{\hat{G}_\lambda}(x)$. By restriction of $\tilde{a}$ we get a representation of $\hat{G}_\lambda(x)$ on the cotangent fiber $T^*_x V_\lambda$. What is more, is that since the isotropy representation of $\hat{G}_\lambda(x)$ on $T_x V_\lambda$ preserves $T_x S$, we get a well defined representation of $\hat{G}_\lambda(x)$ on $T_x V_\lambda / T_x S$, and therefore on the conormal fiber $T^*_{S,x} V_\lambda$ by duality.

Definition 3.5.2. *We define the microlocal component group by:*

$$A_\xi^{mic} = \pi_0\left(Stab_{\hat{G}_\lambda(x)}(\xi)\right)$$

where the pair $(x,\xi) \in T^*_S V_\lambda$ are as in [ABV12, Lemma 24.3(f)].

Remark: Conjecturally, for every S, $T^*_S V_\lambda$ has an open orbit for the action of $\hat{G}_\lambda$. We will show in Proposition 3.5.2 that the conormals to strata which have associated Arthur parameters have an open orbit. It suffices to take (x,ξ) a representative point of the open orbit in $T^*_S V_\lambda$.

Supposing that $x = x_\phi$ for some Langlands parameter $\phi \in \Phi(G/F)$, the inclusion

$$\mathrm{Stab}_{\hat{G}_\lambda(x_\phi)}(\xi) \hookrightarrow \hat{G}_\lambda(x_\phi)$$

maps the identity component into the identity component, and therefore descends to a group homomorphism:

$$A_\xi^{\mathrm{mic}} \to A_\phi$$

In what follows, we will assume that $\lambda : W_F \to {}^LG$ is an unramified infinitesimal parameter. The theorems that follow will also be true for arbitrary infinitesimal parameters, but the arguments are complicated by an "unramification" procedure. The general arguments can be found in [CFM⁺17a], but they obfuscate the simplicity of the construction and are not needed for this thesis, so we omit them[1]. We choose a non-degenerate, Ad-invariant, symmetric bilinear form $(\cdot,\cdot) : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \mathbb{C}$. Such a form exists, as one can extend the Cartan-Killing form on $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ to a form on $\hat{\mathfrak{g}}$ by choosing any non-degenerate form on $\mathfrak{z}(\hat{\mathfrak{g}})$ and insisting that $\mathfrak{z}(\hat{\mathfrak{g}})$ and $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ are orthogonal. Let $\lambda : W_F \to {}^LG$ be an infinitesimal parameter and consider the grading induced by λ:

$$\hat{\mathfrak{g}} \simeq \bigoplus_{\mu \in \mathbb{C}} \hat{\mathfrak{g}}_\mu$$

where $\mathrm{Ad}_{\lambda(\mathrm{Fr})}(x) = q^\mu x$ for all $x \in \hat{\mathfrak{g}}_\mu$.

Lemma 3.5.1. *For every eigenvalue q^μ of $\mathrm{Ad}_{\lambda(Fr)}$, the non-degenerate, Ad-invariant, symmetric bilinear form $(\cdot,\cdot) : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \mathbb{C}$ induces a non-degenerate, Ad-invariant, bilinear pairing:*

$$(\cdot,\cdot)_\mu : \hat{\mathfrak{g}}_\mu \times \hat{\mathfrak{g}}_{-\mu} \to \mathbb{C}$$

Proof. If $x \in \hat{\mathfrak{g}}_\mu$, $y \in \hat{\mathfrak{g}}_\nu$, then

$$(x, y) = \left(\mathrm{Ad}_{\lambda(\mathrm{Fr})}(x), \mathrm{Ad}_{\lambda(\mathrm{Fr})(y)}\right) = q^{\mu+\nu}\,(x, y)$$

so by invariance of the pairing we have $(x, y) \neq 0$ if and only if $\mu + \nu = 0$. If the restricted pairing were degenerate, then so too would be the original pairing on $\hat{\mathfrak{g}}$. The invariance of the restricted pairing is trivial. □

In light of the previous lemma, we define the *dual Vogan variety*:

$$V_\lambda^\dagger = \left\{x \in \hat{\mathfrak{g}} \,|\, \mathrm{Ad}_{\lambda(\mathrm{Fr})}(x) = q^{-1}x\right\}$$

The next proposition will allow us to view the cotangent bundle to a Vogan variety as also being a subvariety of $\hat{\mathfrak{g}}$.

Proposition 3.5.1. *There exists a canonical $\hat{G}_\lambda$-equivariant isomorphism of varieties $V_\lambda \times V_\lambda^\dagger \to T^*V_\lambda$*

Proof. Being an affine space, there exist canonical isomorphisms $TV_\lambda \simeq V_\lambda \times V_\lambda$ and $T^*V_\lambda \simeq V_\lambda \times V_\lambda^*$,

[1] The unramification procedure works by considering the decomposition of $\lambda(\mathrm{Fr})$ as a product of commuting elliptic and hyperbolic semisimple elements. Setting J_λ^0 to be the connected component of the centralizer of the hyperbolic part of $\lambda(\mathrm{Fr})$, one then defines an unramified parameter $\lambda_{\mathrm{nr}} : W_F \to J_\lambda^0 \times W_F$ by sending a lift of Frobenius to the elliptic part of $\lambda(\mathrm{Fr})$. One may then relate the categories of equivariant perverse sheaves on V_λ and $V_{\lambda_{\mathrm{nr}}}$

with V_λ^* the dual vector space to V_λ. Define the map

$$V_\lambda \times V_\lambda^\dagger \to T^*V_\lambda \simeq V_\lambda \times V_\lambda^*$$

$$(x,\xi) \mapsto (x,(\xi,\cdot)_1)$$

This map is an isomorphism because of the non-degeneracy of the restricted pairing, as in Lemma 3.5.1. Equivariance of the map follows from invariance of the pairing. The isomorphism is canonical because $V_\lambda \subseteq [\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$, so the pairing above only depends on the Cartan-Killing form of $[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$. □

Let $\psi \in \Psi(G/F)$ be an Arthur parameter, and ϕ_ψ the associated Langlands parameter. To ϕ_ψ we can also associate a point $x_\psi \in V_\lambda$, where $\lambda(w) = \phi_\psi(w,d_w) = \psi(w,d_w,d_w)$. Denoting $\psi_r = \psi|_{\mathrm{SL}_2\times\mathrm{SL}_2}$, we have that:

$$x_\psi = d\psi_r\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

We will also let:

$$\xi_\psi = d\psi_r\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

Let us show that if $\lambda(w) = \psi(w,d_w,d_w)$, then $\xi_\psi \in V_\lambda^\dagger$. We must compute:

$$\begin{aligned}
\mathrm{Ad}_{\lambda(\mathrm{Fr})}(\xi_\psi) &= \mathrm{Ad}_{\psi(\mathrm{Fr},d_{\mathrm{Fr}},d_{\mathrm{Fr}})}\left(d\psi_r\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\right) \\
&= d\psi_r\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}\right) \\
&= d\psi_r\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{-1} \\ 0 & 0 \end{pmatrix}\right) \\
&= q^{-1}\, d\psi_r\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)
\end{aligned}$$

This computation has shown that to any Arthur parameter $\psi \in \Psi(G/F)$, we may associate a point $(x_\psi,\xi_\psi) \in T^*V_\lambda \simeq V_\lambda \times V_\lambda^\dagger$. Something more is actually true. The image of $d\psi_r$ must consist of a pair of *commuting* $\mathfrak{sl}_2$-triples. This implies that we also have $[x_\psi,\xi_\psi]=0$. Let S_ψ denote the $\hat{G}_\lambda$-orbit of x_ψ in V_λ. By Proposition 22.2 of [Lus95b], we have

$$T^*_{S_\psi}V_\lambda = \left\{(x,\xi) \in V_\lambda \times V_\lambda^\dagger \mid [x,\xi]=0\right\}$$

This means that not only is the pair $(x_\psi,\xi_\psi) \in T^*V_\lambda$, but it is actually a point in $T^*_{S_\psi}V_\lambda$.

Proposition 3.5.2. *The $\hat{G}_\lambda(x_\psi)$ orbit of ξ_ψ is open and dense in $T^*_{S_\psi,x_\psi}V_\lambda$*

Proof. The author is thankful to Bin Xu for explaining the proof. To show that the orbit is open, it suffices to prove that the tangent space to the $\hat{G}_\lambda(x_\psi)$-orbit of ξ_ψ in $T^*_{S_\psi,x_\psi}V_\lambda$ is equal to $T^*_{S_\psi,x_\psi}V_\lambda$ (denseness follows, since we are considering an algebraic action of an algebraic group). Recall from Proposition 3.5.1 that the identification $T^*V_\lambda \to V_\lambda \times V_\lambda^\dagger$ is equivariant, so we can identify the action of $\hat{G}_\lambda(x_\psi)$ on $T^*_{S_\psi,x_\psi}V_\lambda$ with the restriction of the adjoint action on $V_\lambda^\dagger$. The tangent space to the orbit of

ξ_ψ is:

$$T_{\xi_\psi}\left(\hat{G}_\lambda(x_\psi)\cdot\xi_\psi\right) = \left\{[X,\xi_\psi]\in V_\lambda^\dagger \mid X\in \operatorname{Lie}\hat{G}_\lambda(x_\psi)\right\}$$

and we are required to prove that this set is equal to $T^*_{S_\psi,x_\psi}V_\lambda$.

The morphism $\psi_r : \mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\hat{G}$, combined with the adjoint action, gives us a pair of commuting $\mathfrak{sl}_2$-representations on $\hat{\mathfrak{g}}$. This commuting pair gives us a bigrading of $\hat{\mathfrak{g}}$ by weight spaces. We write

$$\hat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}}\hat{\mathfrak{g}}_n$$

where we have denoted

$$\hat{\mathfrak{g}}_n = \bigoplus_{n=r+s}\hat{\mathfrak{g}}_{r,s}$$

for $r,s\in\mathbb{Z}$. Notice that $\hat{\mathfrak{g}}_0 = \operatorname{Lie}\hat{G}_\lambda$, and $V_\lambda^\dagger = \hat{\mathfrak{g}}_{-2}$. We are done if we can show that:

$$[\hat{\mathfrak{g}}_0\cap\operatorname{Lie}Z_{\hat{G}}(x_\psi),\xi_\psi] = \hat{\mathfrak{g}}_{-2}\cap\operatorname{Lie}Z_{\hat{G}}(x_\psi)$$

Let $r+s=0$ and consider the following diagram:

$$\begin{tikzcd}
\hat{\mathfrak{g}}_{r,s} \arrow[r, "\mathrm{ad}(x_\psi)"] \arrow[d, "\mathrm{ad}(\xi_\psi)"'] & \hat{\mathfrak{g}}_{r+2,s} \arrow[d, "\mathrm{ad}(\xi_\psi)"] \\
\hat{\mathfrak{g}}_{r,s-2} \arrow[r, "\mathrm{ad}(x_\psi)"'] & \hat{\mathfrak{g}}_{r+2,s-2}
\end{tikzcd}$$

The Jacobi identity and $[x_\psi,\xi_\psi]=0$ implies that the diagram is commutative. We have:

$$[\hat{\mathfrak{g}}_0\cap\operatorname{Lie}Z_{\hat{G}}(x_\psi),\xi_\psi] = \bigoplus_{r+s=0}[\ker\operatorname{ad}(x_\psi)|_{\hat{\mathfrak{g}}_{r,s}},\xi_\psi]$$

$$\hat{\mathfrak{g}}_{-2}\cap\operatorname{Lie}Z_{\hat{G}}(x_\psi) = \bigoplus_{r+s=0}\ker\operatorname{ad}(x_\psi)|_{\hat{\mathfrak{g}}_{r,s-2}}$$

We need to check that these two sets are equal. By $\mathfrak{sl}_2$-representation theory, $\operatorname{ad}(x_\psi)$ is injective when $r<0$, in which case the claim is trivial. We may assume then that $r\geq 0$ so that $s\leq 0$ and $\operatorname{ad}(\xi_\psi)$ is surjective.

The first containment is easy. Suppose that $y=[X,\xi_\psi]$ for $X\in\ker\operatorname{ad}(x_\psi)|_{\hat{\mathfrak{g}}_{r,s}}$, then

$$[x_\psi,y] = [x_\psi,[X,\xi_\psi]] = [[x_\psi,X],\xi_\psi] + [X,[x_\psi,\xi_\psi]] = 0$$

So $y\in\ker\operatorname{ad}(x_\psi)|_{\hat{\mathfrak{g}}_{r,s-2}}$.

Now suppose that we have $y\in\hat{\mathfrak{g}}_{r,s-2}$ such that $[x_\psi,y]=0$, so y is a highest weight vector for an irreducible representation V of the first SL_2. Since $\operatorname{ad}(\xi_\psi)$ is surjective, choose a lift $\tilde{y}\in\hat{\mathfrak{g}}_{r,s}$ and consider W the SL_2-representation generated by $\tilde{y}$. We have a map of SL_2 representations:

$$\operatorname{ad}(\xi_\psi) : W\to V$$

which we can split, since V is irreducible and the category of finite dimesional SL_2 representations is semisimple. Let $T : V \to W$ denote the splitting, then $T(y) \in \ker \mathrm{ad}(x_\psi)$ and $y \in \mathrm{im}\,\mathrm{ad}(\xi_\psi)$. This completes the proof. $\square$

As a consequence of these computations and Proposition 3.5.2, we have arrived at the following theorem:

Theorem 3.5.1. *For every $\psi \in \Psi(G/F, \lambda)$, the $\hat{G}_\lambda(x_\psi)$-orbit of $(x_\psi, \xi_\psi) \in T^*_{S_\psi} V_\lambda$ is open*

The theorem above allows us to define a microlocal Arthur component group. To any Arthur parameter $\psi \in \Psi(G/F)$, we associate a point in the conormal fiber $T^*_{S_\psi, x_\psi} V_\lambda$ whose orbit is open. Using Definition 3.5.2 we have a corresponding microlocal component group A^{mic}_ψ, together with a map:

$$A^{\mathrm{mic}}_\psi \to A_{\phi_\psi}$$

We have successfuly completed the required geometrization. Arthur parameters give rise in a natural way to conormal varieties of orbits. Given any Arthur parameter $\psi \in \Psi(G/F, \lambda)$, we can make a packet of representations from ψ in two ways. The first packet is Arthur's A-packet Π_ψ. The second packet is gotten by considering the representations that correspond to the D-modules in the micropacket $\Pi^{\mathrm{mic}}_{S_\psi}$. Both packets contain the L-packet Π_{ϕ_ψ}, but might contain more representations. The exception, of course, is that the open orbit in V_λ will always satisfy $\Pi_\psi = \Pi_{\phi_\psi} = \Pi^{\mathrm{mic}}_{S_\psi}$; the first equality follows from the fact that the corresponding A-parameter is tempered, while the second equality follows from Remark 2 following Definition 3.5.1. Furthermore, the condition that A^{mic}_ψ is the component group of the open orbit means that the image of the microlocalization functor:

$$Q^\psi_{\mathrm{mic}} : \text{D-mod}_{\hat{G}_\lambda}(V_\lambda) \to \mathrm{Loc}((T^*_{S_\psi} V_\lambda)^{\mathrm{reg}})$$

gives us a collection of local systems on $(T^*_{S_\psi} V_\lambda)^{\mathrm{reg}}$, and therefore representations of its fundamental group. These representations of π_1 descend to representations of A^{mic}_ψ. In summary, we have a chain of correspondences:

$$\begin{array}{ccccccc}
\Pi(G/F) & \longrightarrow & \text{D-mod}_{\hat{G}_\lambda}(V_\lambda) & \longrightarrow & \mathrm{Loc}((T^*_{S_\psi} V_\lambda)^{\mathrm{reg}}) & \longrightarrow & \hat{A}^{\mathrm{mic}}_\psi \\
 & & & & & & \\
(\pi, \delta) & \longmapsto & \mathcal{M}(\pi, \delta) & \longmapsto & \mathcal{L}_{(\pi,\delta)} & \longmapsto & \langle \cdot, \pi \rangle
\end{array}$$

where the restriction of $\langle \cdot, \pi \rangle$ to $Z(\hat{G})^\Gamma$ is the character corresponding to the pure inner form δ under the Kottwitz isomorphism. We are then led to the following conjecture:

Conjecture 3.5.1. *(The microlocal conjecture) Let G be a $\mathfrak{p}$-adic group. If $\psi \in \Psi(G/F)$ is any A-parameter, then:*

$$\Pi_\psi = \Pi^{mic}_{S_\psi}$$

and the association $\Pi_\psi \to \hat{A}_\psi$ agrees with the geometric construction coming from microlocalization.

The main goal of the third chapter of this thesis is to take a step towards understanding this conjecture. In chapter three we will characterize the micropackets $\Pi^{\mathrm{mic}}_{S_\psi}$ in the case that the infinitesimal parameter associated to ψ is unramified, and the image of a lift of Frobenius is a regular semisimple element. In fact, for a such a V_λ, we will use the characteristic cycle condition to associate a micropacket to any orbit. In the case of real groups, Adams, Barbasch, and Vogan use the regular semisimple case in a reduction step using a so called, "translation datum" [ABV12]. It's possible that a similar reduction is possible in the $\mathfrak{p}$-adic setting.

3.6 Wild speculation

The intent of this section is to draw attention to several peculiarities about the constructions we have made, and what the underlying cause of these peculiarities might be.

3.6.1 Geometric endoscopy

Understanding the D-modules on a Vogan variety whose corresponding infinitesimal parameter is unramified seems to be very difficult, in general, and computing the micropackets is even more difficult. We will see in chapter 3 that for a regular, unramified Vogan variety the micropackets are given by an explicit combinatorial formula. Since we can solve the problem in the latter case, but not the former, we might ask whether or not we can use information about packets for regular Vogan varieties to obtain information about packets for the non-regular ones.

The element $\lambda(\mathrm{Fr})$ may not be a regular semisimple element in $\hat{G}$, but it can still be a regular semisimple element for endoscopic subgroups. Consider again the second part of Example 3.3.4. We had an unramified infinitesimal parameter such that $\lambda(\mathrm{Fr}) = \mathrm{diag}(q^{3/2}, q^{1/2}, q^{1/2}, q^{-1/2}, q^{-1/2}, q^{-3/2})$. While this is not a regular semisimple element in Sp_6, it is a regular semisimple element in the endoscopic subgroup $\mathrm{Sp}_2 \times \mathrm{Sp}_4$:

$$\mathrm{Sp}_2 \times \mathrm{Sp}_4 = \left\{ \begin{pmatrix} * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \end{pmatrix} \in \mathrm{Sp}_6 \right\}$$

We can solve the problem of computing the micropackets easily in the latter case. It is forthcoming work to determine how the micropackets for the endoscopic subgroup with a regular parameter can be used to calculate the micropackets of a non-regular parameter [CFM$^+$17b]

Another problem of interest is whether these geometric constructions can be used to say anything about the transfer factors. The microlocal constructions give conjectural explicit constructions for both the packet Π_ψ, as well as the map $\Pi_\psi \to A_\psi$. This data completely determines the right hand side of the formula in part (b) of Theorem 3.4.2. Then the left hand side, which gives information about the transfer factors for an endoscopic subgroup, can be determined in terms of the right hand side, which is determined purely geometrically. This seems to provide further substance to some remarks of Arthur [Art06, Remark 8, p. 211].

3.6.2 Twisting by unramified characters

As we have seen, when $z \in Z(\hat{G})^\Gamma$ and $\phi : L_F \to {}^LG$ is a Langlands parameter, we can obtain a new Langlands parameter $\phi' \in \Phi(G/F)$ by twisting our old one with respect to this central element:

$$\phi'(w,x) = z^{\mathrm{ord}(w)}\phi(w,x)$$

where $\mathrm{ord}(w)$ is the integer defined by $[w] = [\mathrm{Fr}^{\mathrm{ord}(w)}] \in W_F/I_F$. To every such element $z \in Z(\hat{G})^\Gamma$, we can also associate an unramified character χ_z, as in Example 3.3.1 where we identified elements of the dual torus with unramified characters. The Langlands correspondence posits that the packets Π_ϕ and $\Pi_{\phi'}$ should be related to one another in the following way:

$$\Pi_{\phi'} = \{\chi_z \otimes \pi \,|\, \pi \in \Pi_\phi\}$$

It is not clear what the analogue of twisting by unramified characters should be in Vogan's geometrization of the local Langlands correspondence. We would like a geometrization of twisting to be, for every fixed $\lambda : W_F \to {}^LG$, a collection of functors:

$$\mathscr{F}_z : \text{D-mod}_{\hat{G}_\lambda}(V_\lambda) \to \text{D-mod}_{\hat{G}_\lambda}(V_\lambda) \quad z \in Z(\hat{G})^\Gamma$$

Satisfying the property $\mathscr{F}_{zz'} \simeq \mathscr{F}_z \circ \mathscr{F}_{z'}$.

As our current constructions stand, it is impossible for such a collection of functors to exist. Consider, as a counter-example, any unramified Vogan variety. Let the corresponding infinitesimal parameter be $\lambda : W_F \to {}^LG$. In Example 3.3.1, we discussed which representations should correspond to the simple equivariant D-modules on V_λ. These were the irreducible components of the principal series representation corresponding to the element $\lambda(\mathrm{Fr})$. If $\{\phi_i\} \subseteq \Phi(G/F)$ are the L-parameters such that $\phi_i(w, d_w) = \lambda(w)$ for all i, then we can see that the infinitesimal parameter associated to $z \cdot \phi_i$ is given by $z \cdot \lambda$. The key observation here is that twisting does not preserve the infinitesimal character, yet we will still have $V_\lambda = V_{z\cdot\lambda}$. Should the D-modules on $V_\lambda = V_{z\cdot\lambda}$ be associated through our geometric correspondence to the composition factors of χ_λ, or of $\chi_{z\cdot\lambda}$?

One possible remedy to this problem is to simply keep track of the central twist by considering the varieties:

$$V_{[\lambda]} = \coprod_{z \in Z(\hat{G})^\Gamma} V_{z\cdot\lambda} \simeq Z(\hat{G})^\Gamma \times V_\lambda$$

The group $\hat{G}_\lambda$ acts on $V_{[\lambda]}$ through its action on each of the $V_{z\cdot\lambda}$, and $Z(\hat{G})^\Gamma$ acts by permuting the connected components of $V_{[\lambda]}$. One could then try to implement the twisting functors on the category of D-modules on $V_{[\lambda]}$.

The origin of this problem seems to be the attempt to directly interpret the D-modules appearing Vogan's varieties as representations of the group $G(F)$; or equivalently, as representations of the affine Hecke algebra $\mathcal{H}(G,K)$, where $K \subseteq G(F)$ is the Iwahori subgroup. In [Lus88, Lus95a], Lusztig studies how the category of equivariant perverse sheaves on V_λ relates to the category of modules over $\mathcal{H}(G,K)$. In his work, the semisimple element $\lambda(\mathrm{Fr})$ gives a character of the Bernstein center by evaluation:

$$\chi_\lambda : \mathfrak{Z}(\mathcal{H}) \simeq \mathbb{C}[\hat{T} \times \mathbb{C}^*]^W \to \mathbb{C}$$

$$\chi_\lambda(f) = f(\lambda(\mathrm{Fr}), 1)$$

We have an ideal $I_\lambda = \ker \chi_\lambda \subseteq \mathfrak{Z}(\mathcal{H})$, which we can use to produce an I_λ-adic filtration on $\mathcal{H}(G, K)$:

$$\mathcal{H}(G, K) \supseteq I_\lambda \mathcal{H}(G, K) \supseteq I_\lambda^2 \mathcal{H}(G, K) \supseteq \dots$$

The associated graded algebra $\mathrm{gr}_\lambda \mathcal{H}(G, K)$ is called the graded Hecke algebra (for details on these constructions, see [Lus89]). Lusztig proves that there is an equivalence of categories:

$$\mathrm{Mod}\, \mathrm{gr}_\lambda \mathcal{H}(G, K) \to \mathrm{Mod}\, \mathcal{H}(G, K)_{I_\lambda}$$

where $\mathrm{Mod}\, \mathcal{H}(G, K)_{I_\lambda}$ denotes the subcategory of representations of $\mathcal{H}(G, K)$ on which the ideal I_λ is contained in the annihilator. Lusztig also proves an equivalence of categories:

$$\mathrm{Perv}_{\hat{G}_\lambda}(V_\lambda) \to \mathrm{Mod}\, \mathrm{gr}_\lambda \mathcal{H}(G, K)$$

In summary, we have a chain of correspondences:

$$\begin{tikzcd} \boxed{\text{Equivariant perverse sheaves on } V_\lambda} \arrow[r] & \boxed{\text{Modules over a graded Hecke algebra}} \arrow[dl] \\ \boxed{\text{Modules over the affine Hecke algebra}} \arrow[r] & \boxed{G(F) \text{ representations}} \end{tikzcd}$$

As we can see, in Lusztig's work the connection between modules on V_λ and representations of $G(F)$ is rather indirect. From a geometric perspective, the most important equivalence above is the one passing between the affine Hecke algebra and its graded version. In Lusztig's words [Lus88]:

> *The connection between an affine Hecke algebra and its graded version is analogous to the connection between a reductive group and its Lie algebra... (In fact this is more than an analogy).*

Passing from the category of $\mathcal{H}(G, K)$-representations to the category of representations of the graded Hecke algebra is to be thought of as a categorical analogue of linearization. We might then try to imitate this linearization directly on some category of D-modules.

Question: Is there an H-variety X whose D-module category is equivalent to $\mathcal{H}(G, K)$? Further, for every $\chi_\lambda : \mathfrak{Z}(\mathcal{H}) \to \mathbb{C}$, there should be a "linearization" functor:

$$L_\lambda : \text{D-mod}_H(X) \to \text{D-mod}_{\hat{G}_\lambda}(V_\lambda)$$

compatible with the equivalences in Lusztig's work.

3.6.3 Refining the microlocal conjecture

Our version of the microlocal conjecture states that the A-packet Π_ψ is equal to the packet of simple, equivariant D-modules whose characteristic cycles contain the the conormal to the orbit S_ψ. It is strange that the conjecture seemingly makes no use of the fact that the D-modules under consideration are equivariant (other than to regulate the number of simple objects in the category). In some examples, it can happen that the forgetful functor to the category of D-modules on V_λ (without equivariant structure) is not faithful.

In fact, this is the case in Example 3.3.3. The group $\hat{G}_\lambda$ acting on V_λ in this example was disconnected, and accordingly, there are multiple equivariant structures allowed on a fixed D-module. The category of equivariant D-modules on V_λ has six objects, but there are only three D-modules underlying these equivariant D-modules. This is happening because $\pi_0(\hat{G}_\lambda) = \mathbb{Z}_2$ in this example, so there are two distinct equivariant structures on any fixed module. They are distinguished by the character of the $\pi_0(\hat{G}_\lambda)$ action on global sections. The characteristic cycle of the equivariant D-module is not affected by the equivariant structure, and so we see that the micropackets are forced to contain multiple copies of the same D-module (with different equivariant structure), but with multiplicity corresponding to the number of characters of $\pi_0(\hat{G}_\lambda)$.

The theory of descent tells us that G-equivariant D-modules on a variety X are secretly D-modules on the quotient stack $[X/G]$. Accordingly, the characteristic cycle of an equivariant D-module should not be a Lagrangian cycle in T^*X, but a Lagrangian cycle in $T^*[X/G]$. This observation might be useful if one needs to modify the microlocal conjecture in the event that it is proven false.

3.6.4 What are the extra packets

The inclusion $\Psi(G/F) \hookrightarrow \Phi(G/F)$ is not a surjection. While some of the orbits in V_λ have Arthur parameters attached to them, many orbits do not. This is best demonstrated by example when λ is a regular, unramified infinitesimal parameter. For concreteness, let's take $\lambda : W_F \to \mathrm{SL}_4(\mathbb{C})$ given by:

$$\lambda(\mathrm{Fr}) = \begin{pmatrix} q^{3/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-3/2} \end{pmatrix}$$

Let's classify the possible Arthur parameters ψ such that $\lambda(w) = \psi(w, d_w, d_w)$. As a 4-dimensional representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, ψ must take one of the following forms:

$$\psi = \begin{cases} |\cdot|^a \otimes \nu_4 \otimes \nu_1 \\ (|\cdot|^a \otimes \nu_3 \otimes \nu_1) \oplus (|\cdot|^b \otimes \nu_1 \otimes \nu_1) \\ |\cdot|^a \otimes \nu_2 \otimes \nu_2 \\ |\cdot|^a \otimes \nu_1 \otimes \nu_4 \end{cases}$$

where a and b are complex numbers, $|\cdot|$ is the norm homomorphism $W_F \to \mathbb{C}^*$, and ν_k is the k dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. In order for $\psi(\mathrm{Fr}, d_{\mathrm{Fr}}, d_{\mathrm{Fr}})$ to have distinct eigenvalues, only the first and last cases above are possible (with $a = 0$). This happens more generally as well. For any regular, unramified Vogan variety there are at most two distrinct Arthur parameters whose corresponding infinitesimal parameter is λ. However, there are $2^{|R_\lambda|}$ orbits in a regular, unramified Vogan variety.

Never the less, it is still possible to attach microlocal packets to orbits which do not correspond to any Arthur parameter. It is a question of interest to determine how these "extra" packets, which are purely geometric, can be understood in a representation theoretic context.

Chapter 4

Computing regular microlocal A-packets for split classical groups

4.1 Notation

Throughout this chapter we will use the following notation. If F is a $\mathfrak{p}$-adic field, we let q be the cardinality of its residue field. We will always be dealing with split groups over F in this chapter, so their corresponding L-groups can be identified as the direct product ${}^LG = \hat{G} \times W_F$, with $\hat{G}$ the complex Langlands dual group. We will accordingly think of infinitesimal parameters as homomorphisms $\lambda : W_F \to \hat{G}$. If G is a reductive group, R will be its roots, $\Delta(G)$ will be its simple roots, and for an infinitesimal parameter $\lambda : W_F \to \hat{G}$, we let $R_\lambda \subseteq R$ denote the set of roots such that $\mathrm{Ad}_{\lambda(\mathrm{Fr})}(E_\alpha) = q\, E_\alpha$, where E_α is a root vector corresponding to the root α. When λ is regular and unramified, we are free to assume that $R_\lambda \subseteq \Delta(G)$. The orbits of a regular unramified Vogan variety are indexed by subsets $J \subseteq R_\lambda$, which we will find occasion to identify as subsets of $\{1, \ldots, n\}$ by having chosen a fixed labelling of the simple roots. If $\chi : \pi_1(S_J) \to \mathbb{C}^*$ is a character, we denote by $\ker \chi \subseteq R_\lambda$ the set of roots for which the monodromy of the local system corresponding to χ is trivial around the corresponding coordinates.

If $J \subseteq R_\lambda$ we will write:

$$\mathbb{C}[x_J] = \boxtimes_{j \in J} \mathbb{C}[x_j]$$

and similarly,

$$\mathbb{C}[\partial_J] = \boxtimes_{j \in J} \mathbb{C}[\partial_j]$$

If S_J denotes the orbit in V_λ corresponding to J, we will simply denote:

$$T^*_{S_J} V_\lambda = T^*_J V_\lambda$$

If $J \subseteq R_\lambda$ indexes an orbit in V_λ, then we will denote the corresponding micropacket simply by Π_J, instead of $\Pi^{\mathrm{mic}}_{S_J}$.

We will always be denoting nilpotent orbits by $\mathcal{O} \subseteq \hat{\mathfrak{g}}$, and orbits in Vogan varieties will be denoted using S (or S_J, if we wish to make a specific reference to the subset $J \subseteq R_\lambda$). If $x \in \mathcal{O}$, the component group of a nilpotent orbit is the finite group:

$$A(\mathcal{O}, x) = \pi_0 \left(\mathrm{Stab}_{\hat{G}}(x)\right)$$

while if $S \subseteq V_\lambda$ is an orbit, we will also denote the component group:

$$A(S, x) = \pi_0 \left(\mathrm{Stab}_{\hat{G}_\lambda}(x)\right)$$

The notation should cause no confusion, as S is always an orbit in V_λ, while $\mathcal{O}$ is always a nilpotent orbit. If $J \subseteq R_\lambda$ indexes the orbit $S := S_J$, then we will sometimes write the component group of S as A_J.

4.2 Introduction

We will begin this section by describing an example computation of some microlocal packets for $PGL_4(F)$, where F is a $\mathfrak{p}$-adic field. The structure of the general arguments is heavily reflected in the structure of this specific example. Consider the unramified infinitesimal parameter given by:

$$\lambda : W_F \to SL_4(\mathbb{C})$$

$$\mathrm{Frob} \mapsto \mathrm{diag}(q^{3/2}, q^{1/2}, q^{-1/2}, q^{-3/2})$$

The corresponding Vogan variety V_λ is seen to be the product of the simple root spaces.

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathfrak{sl}_4(\mathbb{C})$$

Henceforth, we use the coordinates x_1, x_2, and x_3 to identify $V_\lambda \simeq \mathbb{C}^3$. The T-action is given by the adjoint action, and may be written:

$$a : T \times \mathbb{C}^3 \to \mathbb{C}^3$$

$$(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3) \mapsto (t_1 t_2^{-1} x_1, t_2 t_3^{-1} x_2, t_3 t_4^{-1} x_3)$$

Remembering that the relation $t_1 t_2 t_3 t_4 = 1$ holds. The orbits of this group action are easily seen to correspond bijectively with subsets $J \subseteq \{1, 2, 3\}$; they are:

$$S_J = \{(x_1, x_2, x_3) : \, j \in J \Rightarrow x_j \neq 0, j \notin J \Rightarrow x_j = 0\}$$

These are locally closed subvarieties whose boundaries are normal crossing divisors in their closures.

We now give a description of the simple, T-equivariant D-modules on V_λ. All of these D-modules arise from minimal extension of an equivariant vector bundle with flat connection on some stratum S_J. To describe such an object, we only need to specify the corresponding monodromy representation. To each T-orbit in V_λ we can canonically associate a covering map whose covering group is isomorphic to the component group of the stabilizer of a point in the orbit. In order for the vector bundle so obtained to be T-equivariant, we allow only those monodromies which arise from irreducible representations of these covering groups.

Let's look at how this arises in this particular example. For the stratum $J = \{1,3\}$ we can take a representative point $x_J = (1,0,1)$. The stabilizer of this point is the subtorus:

$$\mathrm{Stab}_T\,(x_J) = \left\{(t,t,s,s) \in T:\ (ts)^2 = 1\right\}$$

This is not connected, but has the two connected components consisting of points $\left\{(t,t,t^{-1},t^{-1})\right\}$ and $\left\{(t,t,-t^{-1},-t^{-1})\right\}$, with the former set of points determining the connected component of the identity. The component group of the stratum is therefore $A_J = \mathbb{Z}_2$. Explicitly, the covering map is given by:

$$p_J : \tilde{S}_J = T/\mathrm{Stab}_T(x_J)^\circ \to S_J$$

$$t\,\mathrm{Stab}_T(x_J)^\circ \mapsto t \cdot x_J = (t_1 t_2^{-1}, 0, t_3 t_4^{-1})$$

The theory of covering spaces gives an isomorphism:

$$A_J \simeq \frac{\pi_1(S_J)}{(p_J)_*(\pi_1(\tilde{S}_J))}$$

We fix a splitting of the exact sequence (this can be done, for instance, by finding a basis of $X_*(T)$ containing $(1,1,-1,-1)$):

$$1 \to \mathrm{Stab}_T(x_J)^\circ \to T \to \tilde{S}_J \to 1$$

So that $\tilde{S}_J$ can be identified with a subtorus isomorphic to $(\mathbb{C}^*)^2$:

$$\tilde{S}_J = \left\{(1,t,st^{-1},s^{-1}) :\ s,t \in \mathbb{C}^*\right\}$$

With these identifications, the map p_J is:

$$p_J(t,s) = (t^{-1}, 0, s^2 t^{-1})$$

On fundamental groups, the induced map is:

$$(p_J)_* : \mathbb{Z}^2 \to \mathbb{Z}^2$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

The lattice $(p_J)_*(\pi_1(\tilde{S}_J))$ is spanned by these two elements, and $e_1 = (1,0)$ is a representative of the non-trivial element of $A_J = \mathbb{Z}_2$. The component group A_J has two irreducible representations; the two corresponding simple, T-equivariant D-modules on S_J are determined by their monodromies. We should say what the monodromy does on $e_2 = (0,1)$, the second generator of $\pi_1(S_J)$. Notice that if χ is an irreducible representation of $\pi_1(S_J)$ induced from an irreducible representation of A_J, then χ is trivial on $(p_J)_*(\pi_1(\tilde{S}_J))$. Applying this to the example under consideration, it must be that:

$$1 = \chi(e_1 + e_2) = \chi(e_1)\chi(e_2)$$

So $\chi(e_1) = \chi(e_2)^{-1}$. This relation completely determines the monodromy.

We have calculated that there are two simple, T-equivariant D-modules on S_J. The monodromy determines them uniquely. As $\mathcal{O}_{S_J}$-modules, they are both isomorphic to the sheaf associated to $\mathbb{C}[x_1, x_1^{-1}, x_3, x_3^{-1}]$. The difference between them is in the flat connection. For the trivial character, the ring elements ∂_i act as they usually do on functions; however, for the non-trivial character, ∂_i acts through the flat connection as the operator:

$$\partial_i = x_i \frac{\partial}{\partial x_i} - \frac{1}{2}$$

We will let $\tilde{M}_1$ denote the D_{S_J}-module corresponding to the trivial character, and $\tilde{M}_{-1}$ the D_{S_J}-module corresponding to the non-trivial character.

Recall that if $i : U \to X$ is an inclusion of a locally closed subvariety, the minimal extension of a D_U-module $\mathcal{M}$ is characterized as the unique irreducible submodule of $\int_i \mathcal{M}$, extended by zero to give a D_X-module. When the morphism i is the inclusion of an affine open set, then the pushforward $\int_i \mathcal{M}$ is simply the D_X-module obtained by restricting differential operators to the open set U.

Let $i : S_J \to \overline{S}_J$ denote the inclusion of the stratum into its closure; this is an affine, open immersion. Pushing forward gives two modules over $\mathbb{C}[x_1, \partial_1, x_3, \partial_3]$. $\int_i \tilde{M}_{-1}$ is a simple module; however, $\int_i M_1$ contains a unique irreducible submodule consisting of polynomials with positive degree. Extending by zero simply adjoins the derivatives in the x_2 direction. The minimal extensions are therefore:

$$M(S_J, \chi_1) = \mathbb{C}[x_1] \boxtimes \mathbb{C}[\partial_2] \boxtimes \mathbb{C}[x_3]$$

$$M(S_J, \chi_{-1}) = \frac{\mathbb{C}[x_1, \partial_1]}{(x_1\partial_1 - 1/2)} \boxtimes \mathbb{C}[\partial_2] \boxtimes \frac{\mathbb{C}[x_3, \partial_3]}{(x_3\partial_3 - 1/2)}$$

By Proposition 2.4.1 and Examples (2.4.1, 2.4.2, 2.4.3) the characteristic cycles are:

$$CC(M(S_J, \chi_1)) = [\overline{T^*_{S_{13}} V_\lambda}]$$

$$CC(M(S_J, \chi_{-1})) = [\overline{T^*_{S_\emptyset} V_\lambda}] + [\overline{T^*_{S_1} V_\lambda}] + [\overline{T^*_{S_3} V_\lambda}] + [\overline{T^*_{S_{13}} V_\lambda}]$$

An identical analysis can be performed on the other strata. The result of the computation would be the following table, which summarizes all of the simple T-equivariant D-modules on V_λ and their characteristic cycles:

J	A_J	M_J^χ	$CC(M(J,\chi))$
123	$\mathbb{Z}_4$	$\mathbb{C}[x_1,x_2,x_3]$	$[\overline{T^*_{S_{123}}V_\lambda}]$
		$\frac{\mathbb{C}[x_1,\partial_1]}{(x_1\partial_1+1/4)}\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_2,\partial_2]}{(x_2\partial_2+1/2)}\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_3,\partial_3]}{(x_3\partial_3+3/4)}$	$\sum_{J\subseteq\{1,2,3\}}[\overline{T^*_{S_J}V_\lambda}]$
		$\frac{\mathbb{C}[x_1,\partial_1]}{(x_1\partial_1+1/2)}\otimes_{\mathbb{C}}\mathbb{C}[x_2]\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_3,\partial_3]}{(x_3\partial_3+1/2)}$	$[\overline{T^*_{S_2}V_\lambda}]+[\overline{T^*_{S_{12}}V_\lambda}]+[\overline{T^*_{S_{23}}V_\lambda}]+[\overline{T^*_{S_{123}}V_\lambda}]$
		$\frac{\mathbb{C}[x_1,\partial_1]}{(x_1\partial_1+3/4)}\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_2,\partial_2]}{(x_2\partial_2+1/2)}\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_3,\partial_3]}{(x_3\partial_3+1/4)}$	$\sum_{J\subseteq\{1,2,3\}}[\overline{T^*_{S_J}V_\lambda}]$
13	$\mathbb{Z}_2$	$\mathbb{C}[x_1]\otimes_{\mathbb{C}}\mathbb{C}[\partial_2]\otimes_{\mathbb{C}}\mathbb{C}[x_3]$	$[\overline{T^*_{S_{13}}V_\lambda}]$
		$\frac{\mathbb{C}[x_1,\partial_1]}{(x_1\partial_1-1/2)}\otimes_{\mathbb{C}}\mathbb{C}[\partial_2]\otimes_{\mathbb{C}}\frac{\mathbb{C}[x_3,\partial_3]}{(x_3\partial_3-1/2)}$	$[\overline{T^*_{S_\emptyset}V_\lambda}]+[\overline{T^*_{S_1}V_\lambda}]+[\overline{T^*_{S_3}V_\lambda}]+[\overline{T^*_{S_{13}}V_\lambda}]$
Else	1	$\mathbb{C}[x_J]\otimes_{\mathbb{C}}\mathbb{C}[\partial_{J^c}]$	$[\overline{T^*_{S_J}V_\lambda}]$

For a stratum $I\subseteq V_\lambda$, we may simply read off the geometric A-packet Π_I for this example from the table above; however, we would like to describe the packets using a simple combinatorial formula involving J, χ, and I. One can simply notice that in the above example, the packets are given by:

$$\Pi_I=\{\mathcal{M}(J,\chi)\,|\,\ker\chi\subseteq I\subseteq J\}$$

Where $\ker\chi$ indexes the coordinates for which the local system underlying $\mathcal{M}(J,\chi)$ has trivial monodromy. In fact, this simple combinatorial formula will hold for a broad collection of classical groups.

Theorem 4.2.1. *Let $G=SL_n, PGL_n, Sp_{2n}, SO_{2n+1}$. If $S_I\subseteq V_\lambda$ is the stratum in an unramified, regular Vogan variety of G corresponding to the subset $I\subseteq R_\lambda$, then the micropacket associated to S_I is:*

$$\Pi_I=\{\mathcal{M}(J,\chi)\,|\,\ker\chi\subseteq I\subseteq J\}$$

Our efforts in this chapter will be devoted to proving this theorem, and will proceed on a case by case basis.

4.3 Stabilizer coverings of regular, unramified Vogan varieties

Let $\lambda: W_F\to\hat{G}$ be an unramified infinitesimal parameter. Such a parameter is determined by the image of the Frobenius element. We make the simplifying assumption that $\lambda(\mathrm{Fr})$ is a regular semisimple element in ${}^\vee G$, which we may as well assume lies in a fixed maximal torus T. Vogan originally defined, for any choice of infinitesimal parameter, a variety:

$$V_\lambda=\left\{X\in\hat{\mathfrak{g}}:\,\mathrm{Ad}_{\lambda(\mathrm{Fr})}(X)=qX\right\}$$

Where q is the characteristic of the residue field of F. When the image of Frobenius is a regular semisimple element, we will call V_λ a regular, unramified Vogan variety.

Under our simplifying assumption, the variety V_λ can be written using the root space decomposition of $\mathfrak{g}$. Letting $R_\lambda=\{\alpha\in R:\,\alpha(\lambda)=q\}$ and $E_\alpha\subseteq\mathfrak{g}$ be the root space corresponding to the root α, then,

$$V_\lambda=\prod_{\alpha\in R_\lambda}E_\alpha$$

Since we are only going to care about infinitesimal parameters λ up to conjugacy, we are free to assume that $R_\lambda \subseteq \Delta(G)$. The torus $T = Z_{{}^\vee G}(\lambda)$ acts on V_λ by the adjoint action on the root spaces:

$$a : T \times V_\lambda \to V_\lambda$$

$$t \cdot (x_\alpha)_{\alpha\in R_\lambda} = (\alpha(t)x_\alpha)_{\alpha\in R_\lambda}$$

The T action stratifies V_λ by its orbits, which are easily seen to be labelled by subsets $J \subseteq R_\lambda$. Let S_J be the stratum corresponding to a subset $J \subseteq R_\lambda$; a point $(x_\alpha) \in S_J$ if and only if $x_\alpha = 0$ for all $\alpha \notin J$.

Choose a basepoint $x_J \in S_J$; we may as well assume that $(x_J)_\alpha = 0$ for $\alpha \notin J$ and $(x_J)_\alpha = 1$ for $\alpha \in J$. Letting $\tilde{S}_J = T/\mathrm{Stab}^0_T(x_J)$, such a choice yields a covering of the stratum:

$$p^J : \tilde{S}_J \to S_J$$

$$t\,\mathrm{Stab}^0_T(x_J) \mapsto (\alpha(t))_{\alpha\in J}$$

Characters of the group of deck transformations yield T-equivariant vector bundles with flat connections on S_J, which then give D-modules on V_λ by minimal extension.

The D-modules so obtained are therefore prescribed by their monodromies around a generating set for $\pi_1(S_J)$; however, only monodromies induced from characters of these coverings are permitted. It is therefore crucial to determine the sublattice $p^J_*(\pi_1(\tilde{S}_J)) \subseteq \pi_1(S_J)$. Every stratum S_J is a product $(\mathbb{C}^*)^{|J|}$, whose fundamental group is $\pi_1(S_J) \simeq \mathbb{Z}^{|J|}$. We let $[\gamma_\alpha(s)] \in \pi_1(S_J)$ denote the path whose coordinates are $x_{\alpha'}(s) = 1$ for $\alpha' \neq \alpha$, $x_\alpha(s) = e^{is}$. The collection $[\gamma_\alpha]$ is a basis for $\pi_1(S_J)$.

Lemma 4.3.1. *Let $\hat{G}$ be a complex reductive group corresponding to the root datum $(X, R, X^\vee, R^\vee)$, choose an unramified, regular, infinitesimal parameter $\lambda : W_F \to \hat{G}$, and consider the associated variety V_λ equipped with its stratification into T-orbits. For any stratum $S_J \subseteq V_\lambda$,*

$$p^J_*(\pi_1(\tilde{S}_J)) = \mathrm{span}_{\mathbb{Z}}\left\langle \sum_{\alpha\in J} (\alpha, \lambda)[\gamma_\alpha] : \lambda \in X^\vee \right\rangle$$

where (α, λ) denotes the perfect pairing between X and $X^\vee$.

Proof. First we find a generating set for $\pi_1(\tilde{S}_J)$. The long exact sequence of homotopy groups for the fibration $\mathrm{Stab}^0_T(x_J) \hookrightarrow T \to \tilde{S}_J$ yields an exact sequence:

$$1 \to \pi_1(\mathrm{Stab}^0_T(x_J)) \to \pi_1(T) \to \pi_1(\tilde{S}_J) \to 1$$

So $\pi_1(T)$ surjects onto $\pi_1(\tilde{S}_J)$ and every element $[\gamma] \in \pi_1(\tilde{S}_J)$ can be written as $[\lambda]$ for some $\lambda \in X^\vee \simeq \pi_1(T)$. The statement of the lemma now follows directly from the definition of p^J_*:

$$\begin{aligned} p^J_*([\lambda(s)]) = [p^J(\lambda(s))] &= [(\alpha(\lambda(s)))_{\alpha\in J}] \\ &= [(s^{(\alpha,\lambda)})_{\alpha\in J}] \\ &= \sum_{\alpha\in J} (\alpha, \lambda)[\gamma_\alpha] \end{aligned}$$

□

Remark: In fact, the map $\pi_1(\mathrm{Stab}^0_T(x_J)) \to \pi_1(T)$ is an injection. This can be seen by choosing a complementary torus to $\mathrm{Stab}_T^0(x_J)$, then the associated projection map $T \to \mathrm{Stab}_T^0(x_J)$ is a continuous retraction. The long exact sequence of homotopy groups splits into a short exact sequence on π_1, and gives an isomorphism $\pi_1(\tilde{S}_J) \simeq X^\vee / \bigcap_{\alpha\in J} \ker\alpha$. By choosing a complement to $\bigcap_{\alpha\in J} \ker\alpha \subseteq X^\vee$ one can obtain a minimal, finite generating set for $p_*^J(\pi_1(\tilde{S}_J))$. This fact is useful when one needs to actually compute a basis for these types of sublattices.

4.4 Induction functors for D-modules on Vogan varieties

An unramified infinitesimal parameter $\lambda : W_F \to {}^LG$ is determined by a choice of semisimple element in ${}^LG$. For any parabolic subgroup $P = MN$ of ${}^LG$ containing λ, we can consider the Vogan variety associated to λ, but with λ thought of as an element of any of M, P, or ${}^LG$:

$$\begin{aligned} V_\lambda^G &= \{X \in \mathfrak{g} : \mathrm{Ad}_\lambda(X) = qX\} \\ V_\lambda^P &= \{X \in \mathfrak{p} : \mathrm{Ad}_\lambda(X) = qX\} \\ V_\lambda^M &= \{X \in \mathfrak{m} : \mathrm{Ad}_\lambda(X) = qX\} \end{aligned}$$

The groups $H_M = Z_M(\lambda)$, $H_P = Z_P(\lambda)$ and $H_G = Z_G(\lambda)$ act on V_λ^M, V_λ^P and V_λ^G, respectively. The projection $\pi : P \to P/N \simeq M$ induces a map on Lie algebras, which we will also call π, and therefore also a map $\pi : V_\lambda^P \to V_\lambda^M$. Similarly, the closed embedding $i : P \hookrightarrow G$ induces a closed embedding $i : V_\lambda^P \hookrightarrow V_\lambda^G$. We will say that a parabolic subgroup P is relevant for λ if H_G and P contain a common maximal torus in ${}^LG$; this condition is meant to ensure that if $p = mn \in P$ and $p \in Z_G(\lambda)$, then $m \in Z_G(\lambda)$. If P is relevant for λ, then there are also maps $\phi_i : Z_P(\lambda) \to Z_G(\lambda)$ and $\phi_\pi : Z_P(\lambda) \to Z_M(\lambda)$ induced by i and π, respectively.

Lemma 4.4.1. *Let P be a parabolic subgroup with is relevant for λ. There is an equivariance condition on the following maps:*

1. *$i : V_\lambda^P \to V_\lambda^G$ is a ϕ_i-equivariant map*
2. *$\pi : V_\lambda^P \to V_\lambda^M$ is a ϕ_π-equivariant map*

Proof. The equivariance of item one is the property that the following diagram commutes:

$$\begin{tikzcd} H_P \times V_\lambda^P \arrow[r, "a_P"] \arrow[d, "(\phi_i, i)"'] & V_\lambda^P \arrow[d, "i"] \\ H_G \times V_\lambda^G \arrow[r, "a_G"] & V_\lambda^G \end{tikzcd}$$

Where a_G and a_P are the respective action maps coming from the adjoint actions. The diagram commutes because H_P is a subgroup of H_G, and V_λ^P is an H_P stable subvariety of V_λ^G.

For the second item, we decompose the Lie algebra of the parabolic as $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ and write $X = X_M + X_N$ according to this decomposition. We also write elements $p = mn$ using the Levi decomposition. The map ϕ_π is equivariant if and only if $\mathrm{Ad}_m(X_M) \equiv \mathrm{Ad}_{mn}(X_M) \mod \mathfrak{n}$; or stated slightly differently, if $\mathrm{Ad}_n(X_M) \in X_M + \mathfrak{n}$. The follows from the relation between the Lie group/algebra adjoint actions:

$$\mathrm{Ad}_{\exp(X)} = \exp(\mathrm{ad}_X)$$

and the fact that $\mathfrak{n}$ is an ideal of $\mathfrak{p}$. $\square$

For the following definition we recall the functors of equivariant pushforward and pullback from chapter one.

Definition 4.4.1. *Let $P = MN$ be a parabolic subgroup of G which is relevant for λ. The geometric parabolic induction functor $i_P^G : \mathrm{Mod}_{H_M}(V_\lambda^M) \to \mathrm{Mod}_{H_G}(V_\lambda^G)$ is the composition of functors given by:*

$$i_P^G = \int_i^\bullet \circ \pi^\bullet$$

The geometric restriction functor $r_P^G : \mathrm{Mod}_{H_G}(V_\lambda^G) \to \mathrm{Mod}_{H_M}(V_\lambda^M)$ is defined by:

$$r_P^G = \int_\pi^\bullet \circ i^\bullet$$

The definitions of geometric parabolic induction and restriction are abstract enough as to be practicially uncomputable, but they simplify in our situation of interest. The main use for the following proposition is that it allows us to essentially ignore the equivariant structure for the purposes of computing the microlocal packets.

Proposition 4.4.1. *If $\lambda : W_F \to {}^LG$ is an unramified infinitesimal parameter such that the image of Frobenius is regular semisimple and $P = MN$ a parabolic subgroup which is relevant for λ, then:*

1. *$H_G = H_P = H_M = T$ is a maximal torus in ${}^LG$.*
2. *If $\mathcal{M}_\bullet$ is any T-equivariant D-module on V_λ^M, then:*

$$\mathrm{For}_T \circ i_P^G(\mathcal{M}_\bullet) = \int_i \circ \pi^*(\mathcal{M}_0)$$

Proof. The proof of the first item follows since H_G is the unique maximal torus containing λ, thus must be contained in both H_M and H_P by the assumption that P is relevant for λ. The second item follows from Proposition 2.3.3. $\square$

Theorem 4.4.1. *Suppose that $\lambda : W_F \to {}^LG$ is a regular, unramified infinitesimal parameter and $P = MN \subseteq \hat{G}$ is a parabolic compatible with λ. If $\mathcal{M}$ is an equivariant D-module on V_λ^M such that:*

$$CC(\mathcal{M}) = \sum_S m_S(\mathcal{M})[T_S^* V_\lambda^M]$$

then,

$$CC(i_P^G(\mathcal{M})) = \sum_S m_S(\mathcal{M})[T_{\pi^{-1}(S)}^* V_\lambda]$$

Proof. The map $\pi : V_\lambda^P \to V_\lambda^M$ is a submersion so it is non-characteristic for $\mathcal{M}$, and $i : V_\lambda^P \to V_\lambda$ is a closed embedding so it is proper. By Proposition 4.4.1 and Lemmas 2.4.2 and 2.4.1 we have that:

$$\begin{aligned} CC(i_P^G(\mathcal{M})) &= CC\left(\int_i \circ \pi^* \mathcal{M}\right) \\ &= i_* \circ \pi^* CC(\mathcal{M}) \\ &= \sum_S m_S(\mathcal{M})\, [T^*_{\pi^{-1}(S)} V_\lambda] \end{aligned}$$

□

The previous theorem gives a very simple way to understand the combinatorics of how induction affects the characteristic cycles of D-modules on regular unramified Vogan varieties.

4.5 Cuspidal D-modules on Vogan varieties

We are going to compute the Arthur-Vogan packets for a selection of split reductive groups over F. The reduction to split groups will simplify several aspects of the computation and allow our work to tie into the existing literature; foremost, it allows our theory to tie in to Lusztig's work on the classification of cuspidal local systems on nilpotent orbits [Lus85, Lus95b, Lus88]. The other benefit to making this simplification is that complications in the L-group arising from non-trivial Galois actions are not present when $G(F)$ is a split group. This observation allows us to replace the complex group ${}^LG$ with the simpler complex reductive Langland's dual $\hat{G}$.

The computation of the Arthur-Vogan packets will proceed according to the following observations:

1. In analogy with Lusztig's work, we can define a notion of cuspidal support for D-modules on V_λ
2. The cuspidal support indexes a decomposition of the category of D-modules on V_λ; in particular, when λ is regular then every simple equivariant D-module on V_λ is equal to the parabolic induction of a cuspidal D-module on a Levi subgroup
3. The characteristic cycles of the cuspidal objects will be easy to compute, and we can use simple formulas to observe how the characteristic cycle commutes with induction
4. The formula so obtained will allow us to deduce exactly when a given module contains a fixed conormal in its characteristic cycle.

The next few sections will follow the same logical structure to prove the observations we have listed above. The reason that a case by case analysis is required is that Lusztig's classification of cuspidal local systems varies by type (e.g. SL_n, Sp_{2n}, SO_{2n+1}, etc.), making a uniform treatment of the results difficult to express. It is for this reason that we proceed on a case by case. Before proceeding to the specific computations for each type, we will give a more specific outline of our plan of attack.

We recall the definition of a cuspidal local system from [Lus88]. Any reductive group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action. Let $\mathcal{O}$ be the orbit of an ad-nilpotent element, and let $\mathcal{E}$ be a G-equivariant local system on $\mathcal{O}$. If $\mathfrak{p} \subseteq \mathfrak{g}$ is a parabolic subalgebra, we will write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ for its splitting according to the Levi decomposition. Here, $\mathfrak{m}$ is the Lie algebra of a Levi subgroup M in G, and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$.

Definition 4.5.1. *A G-equivariant local system $\mathcal{E}$ on $\mathcal{O}$ is called cuspidal if and only if for every parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, and any $y \in \mathfrak{p}$ we have, for every i:*

$$H_c^i((y + \mathfrak{n}) \cap \mathcal{O}, \mathcal{E}) = 0$$

We call $(\mathcal{O}, \mathcal{E})$ a cuspidal pair.

If $\mathcal{E}$ is a cuspidal local system on $\mathcal{O}$, then $IC(\mathcal{O}, \mathcal{E}) = j_!\mathcal{E} = j_*\mathcal{E}$, where $j : \mathcal{O} \hookrightarrow \mathcal{N}$ is the inclusion into the nilpotent cone. This implies that the simple, cuspidal perverse sheaves on a nilpotent orbit are exactly equal to their extensions by zero.

While Lusztig's definitions apply to local systems on nilpotent orbits, we require something slightly different. We are considering the collection of simple equivariant D-modules on the varieties V_λ. These are in bijection with pairs $(A(G_\lambda, x), \chi)$ where χ is an irreducible representation of $A(G_\lambda, x)$. Of course, χ can be thought of as a G_λ-equivariant local system on the orbit $G_\lambda \cdot x$.

Notice that $\mathfrak{g}$ decomposes as a direct sum of eigenspaces for the adjoint action of the Frobenius element:

$$\mathfrak{g} \simeq \bigoplus_{\mu \in \mathbb{C}} \mathfrak{g}_\mu$$

Where $\mathrm{Ad}_{\mathrm{Fr}}(x) = q^\mu x$. By definition, we have $V_\lambda = \mathfrak{g}_1$. For any $x \in \mathfrak{g}_\mu$ and $y \in \mathfrak{g}_\nu$, we have that $[x, y] \in \mathfrak{g}_{\mu+\nu}$, so in particular, there is an inclusion of the Vogan variety into the nilpotent cone, $V_\lambda \hookrightarrow \mathcal{N}$. This inclusion is equivariant for the action of G_λ on V_λ, and G on $\mathcal{N}$, with respect to the inclusion $G_\lambda \hookrightarrow G$. As such, for every orbit $S \subseteq V_\lambda$, there exists a unique orbit $\mathcal{O} \subseteq \mathcal{N}$ such that $S \subseteq \mathcal{O} \cap V_\lambda$.

Proposition 4.5.1. *Let $x \in V_\lambda$ be a representative point of a G_λ orbit $S \subseteq V_\lambda$, and a nilpotent adjoint orbit $\mathcal{N} \subseteq \mathfrak{g}$. The inclusion $\mathrm{Stab}_{G_\lambda}(x) \hookrightarrow \mathrm{Stab}_G(x)$ induces an isomorphism $A(S, x) \to A(\mathcal{O}, x)$.*

Proof. The stabilizer of $x \in \mathcal{O}$ can be expressed as a semidirect product $\mathrm{Stab}_G(x) = U^x \rtimes G^\phi$, where U^x is a connected unipotent group whose Lie algebra is $\mathfrak{u}^x = Z_{\mathfrak{g}}(x) \cap [x, \mathfrak{g}]$, and G^ϕ is stabilizer of the unique $\mathfrak{sl}_2$-triple containing x and $\log \lambda(\mathrm{Fr})$ [CM93]. But by definition, an element of G^ϕ centralizes $\lambda(\mathrm{Fr})$ and $x \in S$, so $G^\phi = \mathrm{Stab}_{G_\lambda}(x)$. Since U^x is connected, the component group of $\mathrm{Stab}_G(x)$ is determined by G^ϕ. This completes the proof. □

The inclusion of orbits and the corresponding isomorphism of component groups will allow us to define cuspidal D-modules on the variety V_λ.

Definition 4.5.2. *Let $S \subseteq V_\lambda$ be an orbit, and let $\mathcal{O}$ be the nilpotent orbit containing S. We say that $\mathcal{M}(S, \chi)$ is cuspidal if and only if there exists a cuspidal local system $\mathcal{E}$ on $\mathcal{O}$ such that $\mathcal{L}_\chi = \mathcal{E}|_S$.*

The cuspidal local systems of the classical and exceptional groups have been completely determined by Lusztig and Spaltenstein [LS85, Spa85]. Let $A(G, u)$ denote the component group of the stabilizer of a unipotent class $u \in \mathcal{N}$. Since $Z(G) \hookrightarrow \mathrm{Stab}_G(u)$, then we get a natural map $\zeta : Z(G)/Z(G)^\circ \to A(G, u)$. Cuspidal G-equivariant local systems on nilpotent orbits correspond with representations $\rho \in A(G, u)^\vee$, which can be pulled back along ζ. We set $\chi = \zeta^*\rho$ and call it the central character of the local system. We recall the classification of cuspidal local systems from [Sho88, Lus84].

Theorem 4.5.1. *Let G be an almost simple, simply connected complex group. For every character $\chi : Z(G)/Z(G)^\circ \to \mathbb{C}^*$, there exists a unique cuspidal local system $(\mathcal{E}_\rho, \mathcal{O})$ such that $\chi = \zeta^*\rho$ only if:*

1. *(Type A_n) χ has order $n+1$*

2. *(Type B_n) If $2n+1$ is a square, then $\chi = 1$. If $2n+1$ is a triangular number then $\chi \neq 1$.*

3. *(Type C_n) If n is an even triangular number, then $\chi = 1$. If n is an odd triangular number, then $\chi \neq 1$.*

4. *(Type D_n) If $2n$ is a square and $n/2$ is even then $\chi = 1$. If $2n$ is a square and $n/2$ is odd then $\chi \neq 1$ and $\chi(\epsilon) = 1$. If $2n$ is a triangular number then $\chi \neq 1$. Here, ϵ denotes the image of the nontrivial element of $\mathbb{Z}_2$ in the sequence:*

$$1 \to \mathbb{Z}_2 \to Spin_{2n} \to SO_{2n} \to 1$$

5. *(Type E_6, E_7) Any non-trivial χ*

6. *(Type E_8, F_4, G_2) $\chi = 1$.*

In fact, this theorem also yields a classification of the cuspidal local systems associated to any Levi subgroup. Any such Levi subgroup is a complex reductive group. By passing to $L/Z(L)^\circ$ we get a bijection between cuspidal local systems on L and cuspidal local systems on a semisimple group. If $L' = L/Z(G)^\circ$ is not simply connected, then we may pass to its universal cover $\pi : \tilde{L}' \to L'$ and obtain a bijection between cuspidal local systems on $\mathcal{N}_{L'}$, and cuspidal local systems on $\mathcal{N}_{\tilde{L}'}$ whose central characters are trivial on $\ker \pi \subseteq Z(\tilde{L}')$

Lusztig and Spaltenstein's theorem on classification of cuspidal local systems, together with proposition 4.5.1, will allow us to deduce that a similar classification theorem holds for cuspidal D-modules on V_λ, where we now take ρ to be the central character of the D-module.

At this point a small, illustrative example is in order.

Example in $\mathbf{SL}_2$: Let $G = \mathrm{SL}_2(\mathbb{C})$, and write a matrix $X \in \mathfrak{sl}_2$:

$$X = \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$$

The nilpotent cone is given by $\mathcal{N} = \left\{X \in \mathfrak{sl}_2 : xy + z^2 = 0\right\}$, and consists of two orbits: The origin $(x, y, z) = 0$ and the regular orbit $\mathcal{O} = \left\{(x, y, z) : xy + z^2 \neq 0\right\}$. The covering of the regular orbit arising from the adjoint action of G on $\mathcal{N}$ can be seen to be given by:

$$\mathbb{C}^2 \setminus \{(0,0)\} \to \mathcal{O}$$

$$(u, v) \mapsto \begin{pmatrix} -uv & u^2 \\ v^2 & uv \end{pmatrix}$$

And this covering has Galois group $\mathbb{Z}_2$.

Let $\lambda : W_F \to \mathrm{SL}_2(\mathbb{C})$ be the unramified infinitesimal parameter such that $\lambda(\mathrm{Fr}) = \mathrm{diag}(q^{1/2}, q^{-1/2})$; the associated Vogan variety is exactly the $\mathbb{C}$-span of the positive root space. The group $G_\lambda = T$ acts on V_λ via the adjoint action on the positive root space $t \cdot x = t^2 x$, clearly having two orbits. For the open orbit S, we easily see there is an inclusion $S \hookrightarrow \mathcal{O}$. We may pull back the covering $\mathbb{C}^2 \to \mathcal{O}$ by this inclusion to obtain a covering of S:

$$\tilde{S} \to S$$

$$u \mapsto u^2$$

It is now easy to see that the pullback of the covering from $\mathcal{O}$ agrees with the covering of S one obtains from the component group of the stabilizer for a representative point in the orbit. This gives the isomorphism of Galois groups that allows us to restrict G-equivariant local systems from $\mathcal{O}$ to get G_λ-equivariant local systems on S.

For an infinitesimal parameter $\lambda : W_F \to {}^\vee G$, we let $S_G(\lambda)$ denote the set of triples (L, S, χ) such that:

- L is a Levi subgroup of ${}^\vee G$ which contains $\lambda(W_F)$
- $S \subseteq V_\lambda^L$ is an orbit in the L-Vogan variety associated to λ
- χ is an irreducible representation of $A(S, x)$.

To $(L, S, \chi) \in S_G(\lambda)$ we may associate $\mathcal{M}(S, \chi)$, the $Z_L(\lambda)$-equivariant cuspidal D-module on V_λ^L arising from minimal extension of the pair (S, χ). Throughout the next few sections we will characterize the set $S_G(\lambda)$ under the assumption that λ is unramified and that $\lambda(\mathrm{Fr})$ is a regular semisimple element.

4.6 Computation of micropackets

Our study of micropackets will be done on a case by case basis. We will study the groups ${}^LG = \mathrm{SL}_{n+1}, \mathrm{PGL}_{n+1}, \mathrm{Sp}_{2n}$ and SO_{2n+1}.

4.6.1 $\mathbf{SL}_{n+1}$

Let T be the maximal torus in SL_{n+1} consisting of diagonal matrices. We consider only unramified infinitesimal parameters of the form:

$$\lambda : W_F \to SL_{n+1}$$

$$\lambda(\mathrm{Fr}) = \mathrm{diag}(q^{a_1}, \ldots, q^{a_{n+1}})$$

where $a_1, \ldots, a_{n+1}$ are distinct real numbers, chosen so that $a_1 + \cdots + a_{n+1} = 0$ (i.e. $\lambda(\mathrm{Fr})$ is regular semisimple). Let R_λ denote the set of roots such that the corresponding root space belongs to V_λ.

We can freely assume that $R_\lambda \subseteq \Delta(G)$; i.e. that the Vogan variety V_λ is a product of simple root spaces. The maximal torus acts on V_λ through the root space action, and accordingly, two points are in the same orbit if and only if they have the same non-zero components for each simple root. The strata of V_λ are indexed by subsets $J \subseteq R_\lambda$, where the elements appearing in J label the non-zero coordinates. From the subset J, construct an ordered tuple of integers $(l_1, \ldots, l_k)$ by recording the Jordan block sizes of the chosen basepoint x_J; for example, when $J = \{1, 3\} \subseteq \{1, 2, 3\}$ we would get the tuple $(2, 2)$, and when $J = \{1, 2\} \subseteq \{1, 2, 3\}$ we would get the tuple $(3, 1)$. For later convenience, we set $L_m = \sum_{j=1}^m l_j$. Let A_J denote the group of deck transformations of the covering $p^J : \tilde{S}_J \to S_J$.

Lemma 4.6.1. *Let $S_J \subseteq V_\lambda$ be a stratum and $p^J : \tilde{S}_J \to S_J$ the associated covering coming from the T-action.*

$$A_J \simeq \frac{\mathbb{Z}}{\gcd(l_1, \ldots, l_k)\mathbb{Z}}$$

Furthermore, $[\gamma_1] + p^J_*(\pi_1(\tilde{S}_J))$ is a generator of A_J.

Proof. Standard theory of covering spaces gives an isomorphism $A_J \simeq \pi_1(S_J)/p_*^J(\pi_1(\tilde{S}_J))$. The desired isomorphism comes about by applying lemma 4.3.1 in order to compute the Smith normal form of the matrix of relations prescribed by $p_*^J(\pi_1(\tilde{S}_J))$. The statement about $[\gamma_1]+p_*^J(\pi_1(\tilde{S}_J))$ generating A_J also follows from a Smith normal form computation (taking a quotient of A_J by the subgroup generated by $[\gamma_1]$ gives the trivial group). $\square$

We now give a description of the simple, T-equivariant D-modules, $\mathcal{M}(J,\chi)$, which arise from minimal extension of a vector bundle with flat connection on S_J. Such vector bundles are completely determined by the conjugacy classes of their monodromy representations $\chi : \pi_1(S_J) \to \mathrm{GL}_n(\mathbb{C})$. The requirement that $\mathcal{M}(J,\chi)$ be T-equivariant is enforced by restricting attention to monodromy representations which are induced from irreducible representations of A_J.

Lemma 4.6.2. *(SL_n monodromy relations) Let $S_J \subseteq V_\lambda$ be a stratum, and construct the associated ordered tuple $(l_1,\ldots,l_k)$. If $\chi : \pi_1(S_J) \to \mathbb{C}^*$ is induced from a character of A_J, then χ satisfies the following list of relations on the generating set $[\gamma_i]$:*

1. *For every $m = 1,\ldots,k$:*
$$\chi([\gamma_{L_{m-1}-1}]) = \chi([\gamma_{L_m+1}])^{-1}$$
2. *For every $m = 1,\ldots,k$, and $0 < i < L_m - L_{m-1}$:*
$$\chi([\gamma_{L_{m-1}+1}])^i = \chi([\gamma_{L_{m-1}+i}])$$

Proof. We may assume that there is no j such that $l_j = 1$. If there were such a j, then lemma 4.6.1 would imply A_J is trivial and only induce the trivial representation of $\pi_1(S_J)$ may be induced.

As χ is induced from a character of A_J, the relations in the statement of the proposition are all derived from the condition $\chi(p_*^J(\pi_1(\tilde{S}_J))) = 1$. We use lemma 4.3.1 to express the lattice $p_*^J(\pi_1(\tilde{S}_J))$ in terms of the basis of paths $[\gamma_i]$. Let $R_{ij} = (\lambda_i, \alpha_j)$, where $\{\lambda_i\}$ is a basis for the cocharacter lattice and $j \in J$. Simply put, the matrix R_{ij} is the Cartan matrix for A_n with columns $L_1,\ldots,L_k$ removed. The sublattice $p_*^J(\pi_1(\tilde{S}_J))$ is the rowspace of R_{ij}.

The relations can now be deduced in both cases by careful examination of the matrix R_{ij}. Supposing that column L_m has been removed from the Cartan matrix, R_{ij} takes the form:

$$R_{ij} = \begin{pmatrix} & -1 & 2 & -1 & 0 & 0 & 0 & \\ & 0 & -1 & 2 & 0 & 0 & 0 & \\ \cdots & 0 & 0 & -1 & -1 & 0 & 0 & \cdots \\ & 0 & 0 & 0 & 2 & -1 & 0 & \\ & 0 & 0 & 0 & -1 & 2 & -1 & \end{pmatrix}$$

The first relation $\chi([\gamma_{L_{m-1}-1}]) = \chi([\gamma_{L_m+1}])^{-1}$ can be extracted from the middle row. The second to last row above yields $\chi([\gamma_{L_{m-1}+2}]) = \chi([\gamma_{L_{m-1}+1}])^2$. From the last row shown above, we also have the relation:

$$\chi([\gamma_{L_{m-1}+1}])^{-1}\chi([\gamma_{L_{m-1}+2}])^2\chi([\gamma_{L_{m-1}+3}])^{-1} = 1$$

Upon rearranging and substituting this yields $\chi([\gamma_{L_{m-1}+3}]) = \chi([\gamma_{L_{m-1}+1}])^3$. Continuing this process by induction gives all the relations in the statement of the proposition. $\square$

We may use these monodromy relations to characterize the set $S_{SL_n}(\lambda)$, whenever λ is a regular unramified infinitesimal parameter.

Proposition 4.6.1. *Let $\lambda : W_F \to SL_n(\mathbb{C})$ be a regular, unramified infinitesimal parameter. Then $(L, S, \chi) \in S_{SL_n}(\lambda)$ if and only if*

1. *$R_\lambda \supseteq \Delta(L)$*
2. *L is a Levi subgroup with $(n+1)/k$ blocks of size $k \times k$ (in particular, k is a divisor of $n+1$)*
3. *S is the open orbit in V_λ^L*
4. *$\chi : A_S \to \mathbb{C}^*$ has order k*

Proof. Let (L, S, χ) be a cuspidal triple. By Theorem 4.5.1, L admits a cuspidal local system if and only if L consists of blocks of size $k \times k$ with k some divisor of $n+1$. The corresponding cuspidal local systems are supported on the regular nilpotent orbit. For this orbit, the map from $Z(L)/Z(L)^\circ$ to the component group of the orbit is an isomorphism, so the cuspidal local systems on this Levi have central characters which are primitive k'th roots of unity; so χ must have order k.

If R_λ does not contain the simple roots of L, then it is not possible to find a T-orbit $S \subseteq V_\lambda^L$ whose corresponding L-orbit in $\mathcal{N}_L$ is the regular nilpotent orbit in L. We conclude in this case that there are no cuspidal triples (L, S, χ). A similar argument proves that S must be the open orbit in V_λ^L. We may henceforth assume that R_λ contains all the roots of a fixed cuspidal Levi L, and that $S \subseteq V_\lambda^L$ is the open orbit.

The preceeding arguments have shown that if $(L, S, \chi) \in S_{\mathrm{SL}_n}(\lambda)$, then L is a Levi subgroup with evenly sized blocks, $V_\lambda^L = V_\lambda \cap \mathfrak{l}$, S is the open orbit of V_λ^L, and χ is a character of a component group with order equal to the size of the blocks appearing in L. The converse direction of the proof is straightforward. $\square$

Let $(L, S, \chi) \in S_{\mathrm{SL}_n}(\lambda)$, and write $l = n/k$ so that $[L, L] = \mathrm{SL}_k \times \cdots \times \mathrm{SL}_k$ with l factors appearing in the product. We also write $\chi(1) = \xi = \exp(2\pi i m/k)$. Recall the notation from chapter one:

$$\mathcal{M}_\xi = \frac{D_\mathbb{C}}{D_\mathbb{C}(x\partial - m/k)}$$

Corollary 4.6.1. *If $\mathcal{M}_L(S, \chi)$ is the cuspidal D-module on V_λ^L corresponding to the triple $(L, S, \chi) \in S_{SL_n}(\lambda)$ (as above), then:*

$$\mathcal{M}_L(S, \chi) = \boxtimes_{i=1}^{l} \left(\mathcal{M}_\xi \boxtimes \mathcal{M}_{\xi^2} \boxtimes \cdots \boxtimes \mathcal{M}_{\xi^{k-1}} \right)$$

Furthermore,

$$CC(\mathcal{M}_L(S, \chi)) = \sum_{J \subseteq \Delta(L)} [\overline{T^*_{S_J} V_\lambda^L}]$$

Proof. By proposition 4.6.1, L is a Levi subgroup consisting of l blocks of size $k \times k$, where $kl = n$. The corresponding Vogan variety for L is a product of the simple roots in L (one can imagine l Jordan blocks of size k arranged along the diagonal; the Vogan variety is the product of the simple root spaces appearing in these Jordan blocks). From the SL_n monodromy relations of lemma 4.6.2, the character of $\pi_1(V_\lambda^L)$ induced by χ has the following monodromies:

γ_1	γ_2	γ_3	$\cdots$	γ_{k-1}	γ_k	γ_{k+1}	γ_{k+2}	$\cdots$	γ_{n-1}
ξ	ξ^2	ξ^3	$\cdots$	ξ^{k-1}	-	ξ	ξ^2	$\cdots$	ξ^{k-1}

From which the form of $\mathcal{M}(S,\chi)$ follows. The formula for the characteristic cycle follows from Proposition 2.4.1 and Example 2.4.2. □

We have now characterized the cuspidal D-modules. Our next proposition will allow us to deduce that any simple T-equivariant D-module on V_λ can be obtained by induction through some parabolic subgroup.

Proposition 4.6.2. *If $\mathcal{M}(J,\chi) \in D\text{-}mod_T(V_\lambda)$, then there exists $(L,S,\eta) \in S_{SL_n}(\lambda)$ and a parabolic $P = LU$ such that $\mathcal{M}(J,\chi) = i_P^G(\mathcal{M}_L(S,\eta))$. In particular,*

$$CC(\mathcal{M}(J,\chi)) = \sum_{I' \subseteq \Delta(L)} [\overline{T^*_{I' \coprod \ker\chi} V_\lambda}] \tag{4.1}$$

Proof. The component group of the stratum S_J is, according to lemma 4.6.1:

$$A(S_J, x) = \mathbb{Z}/\mathrm{gcd}(l_1,\dots,l_m)\mathbb{Z}$$

For integers $l_1,\dots,l_m$ which record the sizes of the Jordan blocks determined by J. If $k = \mathrm{gcd}(l_1,\dots,l_m)$, then let L be the Levi subgroup of G consisting of n/k blocks of size $k\times k$. The set of simple roots of L is indexed by $J\setminus\ker\chi$. With these choices, the component group of the open orbit $S \subseteq V_\lambda^L$ is isomorphic to $A(S_J,x)$, so we simply let $\eta = \chi$. Then by proposition 4.6.1, (L,S,η) is a cuspidal support and we let $M_L(S,\eta)$ be the corresponding cuspidal D-module on V_λ^L.

To ensure that $i_P^G(\mathcal{M}_L(S,\eta)) = \mathcal{M}(J,\chi)$, we need only choose our parabolic subgroup such that:

1. If $\alpha \in \ker\chi$ then $\alpha \in \Delta(P)$
2. If $\alpha \notin J$, then $\alpha \notin \Delta(P)$

The first condition ensures that when we pull D-modules back along the map $V_\lambda^P \to V_\lambda^L$ that the restriction of the D-module to the coordinate corresponding to α is $\mathbb{C}[x_\alpha]$. The second condition ensures that when we push forward along the morphism $V_\lambda^P \to V_\lambda^G$, that the restriction of the D-module to the coordinate corresponding to α is $\mathbb{C}[\partial_\alpha]$. We can find a parabolic with these properties by choosing an element $\lambda \in \mathbb{R}\otimes X_*(T)$ such that:

$$\begin{aligned}
\alpha(\lambda) &= 0 \quad \text{for every } \alpha \in \Delta(L)\\
\alpha(\lambda) &> 0 \quad \alpha \in \ker\chi\\
\alpha(\lambda) &< 0 \quad \alpha \notin J
\end{aligned}$$

Then choose a minimal facet of the spherical building for SL_n containing λ. The corresponding parabolic subgroup will have Levi factor L and satisfy conditions (1) and (2) above.

The formula for the characteristic cycle follows by commuting the characteristic cycle with the induction functor, as in Theorem 4.4.1, noticing that for any $I \subseteq \Delta(L)$ we have $\pi^{-1}(S_I) = S_{I\coprod\ker\chi}$ for our choice of parabolic. □

Theorem 4.6.1. *If $S_I \subseteq V_\lambda$ is the stratum in an unramified, regular Vogan variety of SL_{n+1} corresponding to the subset $I \subseteq R_\lambda$, then the micropacket associated to S_I is:*

$$\Pi_I = \{\mathcal{M}(J,\chi) \mid \ker\chi \subseteq I \subseteq J\}$$

Proof. Let $\mathcal{M}(J,\chi) \in \Pi_I$, then by definition we have that $CC(\mathcal{M}(J,\chi))$ contains $[\overline{T^*_{S_I}V_\lambda}]$. By equation 4.1 it is immediate that $\ker\chi \subseteq I$ because $I = I' \coprod \ker\chi$ for some $I' \subseteq \Delta(L)$. We can also see that if $I \not\subseteq J$, then S_I is not contained in the closure of S_J and so it is not possible for the characteristic cycle of a D-module obtained by minimal extension from a local system on S_J to have $[\overline{T^*_{S_I}V_\lambda}]$ in its characteristic cycle. This has shown that $\Pi_I \subseteq \{\mathcal{M}(J,\chi) \mid \ker\chi \subseteq I \subseteq J\}$.

The reverse inclusion follows immediately from equation 4.1, for we may write both $J = \Delta(L) \coprod \ker\chi$ for some cuspidal Levi subgroup L, and $I = (I \backslash \ker\chi) \coprod \ker\chi$. $\square$

4.6.2 Sp_{2n}

Let $G(F) = \mathrm{SO}_{2n+1}$, so that we may identify ${}^LG = \mathrm{Sp}_{2n}$. We fix a maximal torus $T \subseteq \mathrm{Sp}_{2n}$ consisting of diagonal matrices of the form $t = \mathrm{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1^{-1})$. There are n simple roots for the adjoint action of T on $\mathfrak{sp}_{2n}$; we will call them $\alpha_1,\dots,\alpha_n$. They are the homomorphisms $T \to \mathbb{C}^*$ given by:

$$\alpha_i(t_1,\dots,t_n) = \begin{cases} t_it_{i+1}^{-1} & 1 \le i < n \\ t_n^2 & i = n \end{cases}$$

The cocharacter lattice $X_*(T)$ has a basis of homomorphisms:

$$\psi_i : \mathbb{C}^* \to T$$

$$\psi_i(s) = \mathrm{diag}(1,\dots,s,\dots,1,1,\dots,s^{-1},\dots,1)$$

where the s is placed on the i'th diagonal entry.

Throughout this section we let $\lambda : W_F \to \mathrm{Sp}_{2n}$ be an unramified infinitesimal parameter, and we assume that $\lambda(\mathrm{Fr})$ is regular semisimple. Furthermore, we may assume without loss of generality (by conjugating $\lambda(\mathrm{Fr})$) that V_λ is a product of simple root spaces. We again let $R_\lambda = \{\alpha \in \Delta \mid \alpha(\mathrm{Fr}) = q\}$.

The maximal torus acts on V_λ via the adjoint action on the root spaces, and so the orbits are indexed by subsets $J \subseteq R_\lambda$. Our first proposition of this subsection characterizes the component groups of orbits.

Proposition 4.6.3. *Let $\lambda : W_F \to Sp_{2n}$ be an unramified, regular infinitesimal parameter and V_λ the associated Vogan variety. If $S_J \subseteq V_\lambda$ is an orbit, then the component group A_J is given by:*

$$A_J = \begin{cases} 1 & \text{if } \alpha_n \notin J \\ \mathbb{Z}_2 & \text{if } \alpha_n \in J \end{cases}$$

When A_J is non-trivial, we may always take $[\gamma_n] + p_*^J(\pi_1(\tilde{S}_J))$ as a generator of A_J.

Proof. The proof is similar to lemma 4.6.1. Use lemma 4.3.1 to get a generating set for the lattice $K = p_*^J(\pi_1(\tilde{S}_J))$. We can deduce the component group from the Smith normal form of the matrix $R_{ij} = (\alpha_i(\psi_j))$ where $i = 1,\dots,n$ and $j \in J$. First consider the case where $J = \{1,2,\dots,n-1\}$. In this case, R takes the form:

$$R = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{pmatrix}$$

One may compute the Smith normal form by performing rightward column replacements to see that $K = \pi_1(S_J)$. For any other J such that $\alpha_n \notin J$ one would simply delete some of the rows of R and perform an identical computation, in which case we still have $K = \pi_1(S_J)$.

In the case that $\alpha_n \in J$ (say, $J = \Delta(\mathrm{Sp}_{2n})$), the matrix R is:

$$R = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 2 \end{pmatrix}$$

For any other J which contains α_n, R would appear as above but with some of the rows deleted. In any case, one can compute the Smith normal form by rightward column replacements and see that $A_J \simeq \mathbb{Z}_2$.

From the form of R, it is clear that the final column can be taken as a generator of A_J; but $2[\gamma_n] - [\gamma_{n-1}] \sim 2[\gamma_n] \bmod p_*^J(\pi_1(\tilde{S}_J))$ so the statement about $[\gamma_n]$ being a generator of A_J follows. □

The following proposition tells us the monodromies of simple T-equivariant local systems on S_J.

Proposition 4.6.4. *(Sp_{2n} monodromy relations) Let $S_J \subseteq V_\lambda$ be a stratum. If $\alpha_n \in J$ and $\tilde{\chi} : \pi_1(S_J) \to \mathbb{C}^*$ is induced from a character $\chi : A_J \to \mathbb{C}^*$, then:*

$$\tilde{\chi}([\gamma_i]) = 1 \quad \textit{for all } i \neq n$$

$$\tilde{\chi}([\gamma_n]) = \chi(-1)$$

If $\alpha_n \notin J$ and $\tilde{\chi} : \pi_1(S_J) \to \mathbb{C}^*$ is induced from $\chi : A_J \to \mathbb{C}^*$, then $\tilde{\chi}$ is the trivial representation.

Proof. Considering again the matrix R (as in the proof of proposition 4.6.3), we see that $[\gamma_i] \in p_*^J(\pi_1(\tilde{S}_J))$ for all $i \neq n$, while $[\gamma_n]$ descends to a generator of A_J. The proposition now follows. □

The simple T-equivariant D-modules on V_λ are obtained by minimal extension of the equivariant local systems described by the previous proposition. Calling these D-modules $\mathcal{M}(J, \chi)$, we can see that if $\alpha_n \in J$ then:

$$\mathcal{M}(J, \chi) \simeq \begin{cases} \mathbb{C}[x_{J \setminus n}] \boxtimes \mathbb{C}[\partial_{J^c}] \boxtimes \frac{D_{\mathbb{C}}}{D_{\mathbb{C}}(x_n \partial_n - 1/2)} & \text{if } \chi([\gamma_n]) = -1 \\ \mathbb{C}[x_J] \boxtimes \mathbb{C}[\partial_{J^c}] & \text{if } \chi([\gamma_n]) = 1 \end{cases}$$

and if $\alpha_n \notin J$, then:

$$\mathcal{M}(J, \chi) = \mathbb{C}[x_J] \boxtimes \mathbb{C}[\partial_{J^c}]$$

As in the case of SL_{n+1}, we will present these D-modules as being induced from cuspidal ones on Levi subgroups. We first require a characterization of the cuspidal triples $S_{\mathrm{Sp}_{2n}}(\lambda)$.

Proposition 4.6.5. *Let $\lambda : W_F \to Sp_{2n}$ be an unramified, regular infinitesimal parameter. $(L, S, \chi) \in S_{Sp_{2n}}(\lambda)$ if and only if either of the following two conditions holds:*

1. *$\alpha_n \in R_\lambda$, $L = GL_1^{n-1} \times SL_2$, S is the open orbit in V_λ^L, and χ is the non-trivial character of $\mathbb{Z}_2$*

2. *$\alpha_n \notin R_\lambda$, $L = T$, $S = V_\lambda^T = \{0\}$, and χ is trivial*

Proof. Suppose that (L, S, χ) is a cuspidal triple.

The Levi subgroups of Sp_{2n} take one of two forms. Either:

$$L = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$$

where $n_1 + \cdots + n_k = n$, or:

$$L = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{Sp}_{2l}$$

where $n_1 + \cdots + n_k = n - l$.

Cuspidal local systems on a product of groups are external direct products of cuspidal local systems coming from each factor. For the first type of Levi, only $T = (\mathrm{GL}_1)^n$ supports a cuspidal local system. For the second type of Levi, only those Levi subgroups of the form $L_l = (\mathrm{GL}_1)^{n-l} \times \mathrm{Sp}_{2l}$ could possibly support cuspidal local systems; by the classification theorem 4.5.1, we get a single cuspidal local system when $l = d(d-1)/2$ is a triangular number and it is supported on the nilpotent orbit corresponding to the partition $[1^{2n-2l}, 2, 4, 6, \ldots, 2d]$.

When λ is regular and unramified, the variety V_λ can be assumed to be a product of simple root spaces. In this case, if L is a Levi subgroup and $x \in S_J \subseteq V_\lambda^L$, then the nilpotent orbit $L \cdot x$ must correspond to one of the following partitions:

$$\text{if } \alpha_n \in J, P_2 = [2, n_1^2, \ldots, n_k^2] \quad n_1 + \cdots + n_k = l - 1$$

$$\text{if } \alpha_n \notin J, P_1 = [n_1^2, \ldots, n_k^2] \quad n_1 + \cdots + n_k = l$$

In the first case, if $L \cdot x$ is to support a cuspidal local system then we must have $L = L_1$ and χ the non-trivial character of $\mathbb{Z}_2$ (and so we're in the first case of the proposition statement). In the second case, if $L \cdot x$ is to support a local system we must have $L = T$ and $\chi = 1$ (and so we're in the second case of the proposition statement).

The converse direction of the proof follows immediately from the classification theorem 4.5.1. □

The previous proposition tells us that for a fixed λ, V_λ admits at most two cuspidal supports. For $L = (\mathrm{GL}_1)^{n-1} \times \mathrm{Sp}_2$, the cuspidal D-module is:

$$\mathcal{M}_L(S, \chi) = \frac{D_{\mathbb{C}}}{D_{\mathbb{C}}(x\partial - 1/2)}$$

In this case we have:

$$CC(\mathcal{M}_L(S, \chi)) = [\overline{T^*_{\{n\}} V_\lambda^L}] + [\overline{T^*_{\emptyset} V_\lambda^L}]$$

While for the other cuspidal Levi T, we have:

$$\mathcal{M}_T(S',1) = \mathbb{C}$$

So:

$$CC(\mathcal{M}_T(S',1)) = [\overline{T^*_\emptyset V^T_\lambda}]$$

Proposition 4.6.6. *If $\mathcal{M}(J,\chi)$ is a simple T-equivariant D-module on V_λ, then there exists $(L,S,\eta) \in S_{Sp_{2n}}(\lambda)$ and a parabolic subgroup $P = LU$ such that:*

$$\mathcal{M}(J,\chi) = i_P^G(\mathcal{M}_L(S,\eta))$$

Furthemore,

$$CC(M(J,\chi)) = \begin{cases} [\overline{T^*_J V_\lambda}] + [\overline{T^*_{J\backslash n} V_\lambda}] & \alpha_n \in J, \chi \neq 1 \\ [\overline{T^*_J V_\lambda}] & \textit{else} \end{cases}$$

Remark: The formula for the characteristic cycle has an identical form to the one in Proposition 4.6.2, noticing that $\ker\chi = J\backslash n$ when $\chi \neq 1$ and $\ker\chi = J$ when $\chi = 1$.

Proof. If $\alpha_n \in J$ and $\chi([\gamma_n]) = -1$ then $\mathcal{M}(J,\chi)$ is induced from $\mathcal{M}_L(S,\chi)$. Otherwise, $\mathcal{M}(J,\chi)$ is induced from $\mathcal{M}_T(S',1)$. A suitable parabolic is selected as in the proof of Proposition 4.6.2. ☐

Theorem 4.6.2. *If $S_I \subseteq V_\lambda$ is the stratum in an unramified, regular Vogan variety of Sp_{2n} corresponding to the subset $I \subseteq R_\lambda$, then the micropacket associated to S_I is:*

$$\Pi_I = \{\mathcal{M}(J,\chi) \mid \ker\chi \subseteq I \subseteq J\}$$

Proof. The proof is identical to Theorem 4.6.1, noticing that: (1) every D-module is induced from a cuspidal one, (2) we know the characteristic cycles of the relevant cuspidal D-modules, and (3) we know how to commute the characteristic cycle functor with induction. ☐

4.6.3 $\mathbf{PGL}_{n+1}$

When $G(F) = \mathrm{SL}_{n+1}(F)$ we get a Langlands dual group PGL_{n+1}. The maximal torus of diagonal matrices in GL_{n+1} descends to a maximal torus of PGL_{n+1}. We assume as before that $\lambda : W_F \to \mathrm{PGL}_{n+1}$ is unramified, and that the image of Frobenius is a regular semisimple element of our chosen maximal torus. With these hypotheses we may also assume that V_λ is a product of simple root spaces. The simple roots are the functions $\alpha_i : T \to \mathbb{C}^*$ given by:

$$\alpha_i\left(\mathrm{diag}(t_1,\ldots,t_{n+1})\right) = t_i t_{i+1}^{-1}$$

We choose a basis of cocharacters given by the functions:

$$\psi_i : \mathbb{C}^* \to \mathrm{PGL}_{n+1}$$

$$\psi_i(s) = \mathrm{diag}(1,\ldots,s,\ldots,1)$$

with the s being placed on the i'th diagonal entry.

The following proposition will show that there are not many simple T-equivariant D-modules on V_λ.

Proposition 4.6.7. *Let $\lambda : W_F \to Sp_{2n}$ be an unramified, regular infinitesimal parameter and V_λ the associated Vogan variety. If $S_J \subseteq V_\lambda$ is an orbit, then the component group A_J is trivial.*

Proof. We again use Lemma 4.3.1 to get a generating set for the lattice $K = p_*^J(\pi_1(\tilde{S}_J))$. We can deduce the component group from the Smith normal form of the matrix $R_{ij} = (\alpha_i(\psi_j))$ where $i = 1, \ldots, n$ and $j \in J$. First consider the case where $J = \Delta_\lambda = \Delta(\mathrm{PGL}_n)$. The matrix R takes the form:

$$R = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

It is evident from the Smith normal form of R that $p_*^J(\pi_1(\tilde{S}_J)) = \pi_1(S_J)$, so A_J is trivial. The computation works identically for any other J; one need only delete some of the rows of the matrix R. □

From Propostion 4.6.7 we may immediately deduce that the simple T-equivariant D-modules on V_λ take the form:

$$\mathcal{M}(J, \chi) = \mathbb{C}[x_J] \boxtimes \mathbb{C}[\partial_{J^c}]$$

By a direct computation, the characteristic cycle of these D-modules are:

$$CC(\mathcal{M}(J, \chi)) = [\overline{T_J^* V_\lambda}]$$

Since the characteristic cycles only ever contain a single Lagrangian cycle, the microlocal packets are easily seen to be:

$$\Pi_I = \{M(I, \chi)\}$$

It would also be possible to proceed as in the previous subsections. In this example, there is only a single possible cuspidal triple, $(T, \{pt\}, \mathrm{triv})$. It is still true that every D-module on V_λ is induced from a cuspidal one through some parabolic. Never the less, we have the following:

Theorem 4.6.3. *If $S_I \subseteq V_\lambda$ is the stratum in an unramified, regular Vogan variety of PGL_{n+1} corresponding to the subset $I \subseteq R_\lambda$, then the micropacket associated to S_I is:*

$$\Pi_I = \{\mathcal{M}(J, \chi) \mid \ker \chi \subseteq I \subseteq J\}$$

Proof. The component group A_J is trivial; so too must be χ. This means that $\ker \chi = J \subseteq I \subseteq J$. □

4.6.4 $\mathbf{SO}_{2n+1}$

When $G(F) = \mathrm{Sp}_{2n}(F)$ we get a Langlands dual group $\hat{G} = \mathrm{SO}_{2n+1}$. We choose the maximal torus given by the diagonal matrices of the form $t = \mathrm{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$, which gives a basis of the cocharacter lattice consisting of the homomorphisms $\psi_i : \mathbb{C}^* \to \mathrm{SO}_{2n+1}$, where $\psi_i(s)$ is a diagonal matrix with s in the i'th diagonal position, s^{-1} in the $2n+1-i$'th diagonal position, and 1's elsewhere. We assume as before that $\lambda : W_F \to \mathrm{PGL}_{n+1}$ is unramified, and that the image of Frobenius is a regular semisimple element of our chosen maximal torus. With these hypotheses we may also assume that V_λ is a product of simple root spaces. The simple roots of SO_{2n+1} are the functions $\alpha_1, \ldots, \alpha_n$ where:

$$\alpha_i(t) = \begin{cases} t_i t_{i+1}^{-1} & i < n \\ t_n & i = n \end{cases}$$

As in the case of PGL_{n+1}, the pairing between the roots and the cocharacter lattice forces all of the component groups to be trivial.

Proposition 4.6.8. *Let $\lambda : W_F \to SO_{2n+1}$ be an unramified, regular infinitesimal parameter and V_λ the associated Vogan variety. If $S_J \subseteq V_\lambda$ is an orbit, then the component group A_J is trivial.*

Proof. The proof is identical to Proposition 4.6.7, except for that the last column of R is $(0, 0, \ldots, 0, 1)$. This does not change the result. □

The discussion at the end of the previous section applies equally in this case. The equivariant D-modules on V_λ all take the form:

$$\mathcal{M}(J, 1) = \mathbb{C}[x_J] \boxtimes \mathbb{C}[\partial_{J^c}]$$

So their characteristic cycles are:

$$CC(\mathcal{M}(J, 1)) = [T_J^* V_\lambda]$$

which trivially yields the formula for the micropackets:

$$\Pi_I = \{\mathcal{M}(I, 1)\} = \{\mathcal{M}(J, \chi) \mid \ker \chi \subseteq I \subseteq J\}$$

4.7 Remarks on the regular unramified case

In this section we make some remarks about which features of the regular unramified case have made possible an explicit computation of the microlocal packets, and then make some comments about why these issues become challenges when one considers more complicated examples.

The principal reason why the regular unramified case was within reach is that geometry of the orbits and their closures is very simple. When the image of Frobenius is a regular semisimple element, the orbits S are always isomorphic to $(\mathbb{C}^*)^k$, and the boundaries of the orbits are always normal crossing divisors in $\overline{S}$. In this situation, it is possible to explicitly present the minimal extension D-modules by generators and relations, dependent on the monodromies of the underlying local system. This feature made it possible to effectively explore the environment surrounding the problem, which ultimately led to an observation of how cuspidal local systems were playing a role. Outside of the regular setting, the geometry of the orbits can be much more complicated (c.f. Example 3.3.4) and it is unclear exactly how one should present the minimal extensions by generators and relations.

The other important feature of the regular unramified case is that the induction functors simplify substantially. In the most general case, the equivariant pushforward and pullback functors are computationally intractable. It is very difficult to even explicitly write down an object of an equivariant derived category. In the regular unramified case, however, we are pushing forward and pulling back along morphisms that are equivariant for the identity map $T \to T$. Furthermore, since T is connected, the classical equivariant category is equivalent to a full subcategory of the usual D-module category and we may ignore the equivariant structure. It also happens that the varieties involved are affine, and the induction functors involve pulling back along a smooth map then pushing forward along a closed immersion. This is essentially the only case where these functors can be easily computed, and we have wound up in this setting because of the assumption that $\lambda(\mathrm{Fr})$ is regular semisimple.

If one assumes regularity, but relaxes the assumption that λ is unramified, then one may obtain that $\hat{G}_\lambda$ is disconnected. In this case, we must be careful about considering the equivariant structure (c.f. Example 3.3.3). If one assumes that λ is unramified, but $\lambda(\mathrm{Fr})$ is not regular semisimple then $\hat{G}_\lambda$ need not be connected (although this will be true if $\hat{G}$ is simply connected). In either of these cases, a careful analysis of how the equivariant structure is affected by the induction functors would be necessary to fully understand the problem, but in the regular unramified case these complications disappear.

Chapter 5

On the Duistermaat-Heckman distribution of ΩG

5.1 Introduction

For finite-dimensional compact symplectic manifolds equipped with a Hamiltonian torus action with moment map μ, the Duistermaat-Heckman theorem gives an explicit formula for an oscillatory integral over the manifold in terms of information about the fixed point set of the torus action, and the action of the torus on the normal bundle to the fixed point set. Furthermore, the Fourier transform of this integral controls the structure of the cohomology rings of the various symplectic reductions. For Hamiltonian actions on infinite dimesional symplectic manifolds, little is known is known about the behaviour of their corresponding Duistermaat-Heckman distributions. In this paper we define the same oscillatory integral for the natural Hamiltonian torus action on the based loop group, as introduced by Atiyah and Pressley, in order to give an expression for a Duistermaat-Heckman *hyperfunction*. The essential reason for introducing hyperfunction theory is that the local contribution to the Duistermaat-Heckman polynomial near the image of a fixed point is a Green's function for an infinite order differential equation. Since infinite order differential operators do not act on Schwarz distributions, we are forced to use this more general theory. After this work had been completed, we learned of the related work of Roger Picken [Pic89].

The layout of this chapter is as follows. In Section 2 we review the theory of hyperfunctions, following [KS99a, Kan89]. In Section 3 we study hyperfunctions that arise naturally from Hamiltonian group actions via localization. Section 4 reviews the based loop group and its Hamiltonian action (introduced by Atiyah and Pressley [AP83]). Section 5 describes the fixed point set of any one parameter subgroup of this torus. In Section 6 we demonstrate the theorems of Section 5 for the based loop group of $SU(2)$. In Section 7, we compute the isotropy representations of the torus that acts on the based loop group on the tangent spaces to each of the fixed points. Finally, Section 8 applies the hyperfunction localization theorem to $\Omega SU(2)$.

5.2 Introduction to Hyperfunctions

In this section we will quickly review the elements of hyperfunction theory which are needed in order to make sense of the fixed point localization formula for a Hamiltonian action on an infinite dimensional manifold. We will assume that the reader is familiar with Hamiltonian group actions, but not necessarily with hyperfunctions. Our exposition will follow a number of sources. The bulk of the background material follows [KS99a, Kan89], while the material on the Fourier transform of hyperfunctions is covered in [Kan89] as well as the original paper of Kawai [Kaw70]. The original papers of Sato also give great insight into the motivation for introducing this theory [Sat59]. The lecture notes of Kashiwara, Kawai, and Sato also give useful insight into why hyperfunction and microfunction theory is needed to solve problems in linear partial differential equations [KKS].

We will let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^n$. Points in $\mathbb{C}^n$ will be denoted $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$, and we will write $\mathrm{Re}(z) = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathrm{Im}(z) = (y_1, \ldots, y_n) \in i\mathbb{R}^n$. An open convex cone $\gamma \subseteq \mathbb{R}^n$ is a convex open set such that, for every $c \in \mathbb{R}_{>0}$, if $x \in \gamma$ then $cx \in \gamma$. We allow $\mathbb{R}^n$ itself to be an open convex cone. If γ is an open convex cone, we will denote its polar dual cone by γ°. Let Γ denote the set of all open convex cones in $\mathbb{R}^n$. If $\gamma \in \Gamma$ and $\Omega \subseteq \mathbb{R}^n$ is an open set, then we denote $\Omega \times i\gamma = \{(x_j + iy_j) \in \mathbb{C}^n \,|\, \mathrm{Re}(x) \in \Omega, \mathrm{Im}(z) \in \gamma\}$. An *infinitesimal wedge*, denoted $\Omega \times i\gamma 0$, is a choice of an open subset $U \subseteq \Omega \times i\gamma$ which is asymptotic to the cone opening (we will not need the precise definition, so we omit it). We will denote the collection of germs of holomorphic functions on the wedge $\Omega \times i\gamma$ by $\mathcal{O}(\Omega \times i\gamma 0)$; that is, we take a direct limit of the holomorphic functions varying over the collection of all infinitesimal wedges $U = \Omega \times i\gamma 0 \subseteq \Omega \times i\gamma$:

$$\mathcal{O}(\Omega \times i\gamma 0) = \varinjlim_{U \subseteq \Omega \times i\gamma} \mathcal{O}(U)$$

We will use the notation $F(z + i\gamma 0)$ to denote an element of $\mathcal{O}(\Omega \times i\gamma 0)$.

Definition 5.2.1. *A hyperfunction on $\Omega \subseteq \mathbb{R}^n$ is an element:*

$$\sum_{i=1}^{n} F(z + i\gamma_i 0) \in \bigoplus_{\gamma \in \Gamma} \mathcal{O}(\Omega \times i\gamma 0) / \sim$$

where the equivalence relation is given as follows. If $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ are such that $\gamma_3 \subseteq \gamma_1 \cap \gamma_2$ and $F_i \in \mathcal{O}(\Omega \times i\gamma_i)$, then $F_1(z) + F_2(z) \sim F_3(z)$ if and only if $(F_1(z) + F_2(z))|_{\gamma_3} = F_3(z)$. If $\Omega \subseteq \mathbb{R}^n$, we will denote the collection of hyperfunctions on Ω by $\mathcal{B}(\Omega)$.

When we wish to keep track of the cones we will use the notation $f(x) = \sum_j F(z + i\gamma_j 0)$; we call such a sum a boundary value representation of $f(x)$. Alternatively, we will sometimes also use the notation $F(z + i\gamma 0) = b_\gamma(F(z))$ when the expression for $F(z)$ makes it notationally burdensome to include the text $+i\gamma 0$. The association $\Omega \mapsto \mathcal{B}(\Omega)$ forms a flabby sheaf on $\mathbb{R}^n$, although we will not make use of the sheaf theoretical nature of hyperfunctions in this thesis. Actually, what is more, is that this is a sheaf of D-modules on $\mathbb{C}^n$; the sheaf of differential operators acts termwise on each element of a sum $\sum_\gamma F(z + i\gamma 0)$.

The relation defining the sheaf of hyperfunctions allows us to assume that the cones appearing in the sum are disjoint. Indeed, if we have a hyperfunction $f(x) = F_1(z + i\gamma_1 0) + F_2(z + i\gamma_2 0)$ such that

$\gamma_1 \cap \gamma_2 \neq \emptyset$, then we simply observe that we have an equality of equivalence classes:

$$F_1(z + i\gamma_1 0) + F_2(z + i\gamma_2 0) = (F_1 + F_2)(z + i\gamma_1 \cap \gamma_2 0)$$

Similarly, if $\gamma_1 \subseteq \gamma_2$ and $F(z)$ is an analytic function on the wedge $\Omega \times i\gamma_1 0$ that admits an analytic extension to $\Omega \times i\gamma_2 0$, then $F(z + i\gamma_1 0) = F(z + i\gamma_2 0)$ as hyperfunctions. A particular example of this says that two hyperfunctions $f(x) = F_+(z + i0) + F_-(z - i0), g(x) = G_+(z + i0) + G_-(z - i0) \in \mathcal{B}(\mathbb{R})$ are equal when the function:

$$F(z) = \begin{cases} F_+(z) - G_+(z) & \mathrm{Im}(z) > 0 \\ F_-(z) - G_-(z) & \mathrm{Im}(z) < 0 \end{cases}$$

admits an analytic extension across the real axis.

The following definition is necessary to define the product of hyperfunctions. We say that a hyperfunction $f(x)$ is *microanalytic at* $(x, \xi) \in T^*\mathbb{R}^n$ if and only if there exists a boundary value representation:

$$f(x) = \sum_{j=1}^{n} F(z + i\gamma_j 0)$$

such that $\gamma_j \cap \{y \in \mathbb{R}^n \,|\, \xi(y) < 0\} \neq \emptyset$ for every $j \in 1, \ldots, n$. The *singular support* of a hyperfunction $f(x)$, denote $SS(f) \subseteq T^*\mathbb{R}^n$, is defined to be the set of points $(x, \xi) \in T^*\mathbb{R}^n$ such that $f(x)$ is not microanalytic at (x, ξ). If $S \subseteq T^*\mathbb{R}^n$ then we denote $S^\circ = \{(x, \xi) \in T^*\mathbb{R}^n \,:\, (x, -\xi) \in S\}$.

Definition 5.2.2. *Suppose that $f, g \in \mathcal{B}(\Omega)$ are two hyperfunctions such that $SS(f) \cap SS(g)^\circ = \emptyset$, then the product $f(x) \cdot g(x)$ is the hyperfunction defined by:*

$$f(x) \cdot g(x) = \sum_{j,k} (F_j \cdot G_k)(x + i(\gamma_j \cap \Delta_k)0)$$

where we have chosen appropriate boundary value representations:

$$f(x) = \sum_{j} F_j(z + i\gamma_j 0)$$

$$g(x) = \sum_{k} G_j(z + i\Delta_k 0)$$

such that $\gamma_j \cap \Delta_k \neq \emptyset$ for all j, k.

In the above definition, the condition on singular support is simply ensuring existence of boundary value representations of f and g such that for all pairs j, k the intersection $\gamma_j \cap \Delta_k \neq \emptyset$ [Kan89, Theorem 3.2.5].

We may define an infinite product of hyperfunctions when the singular support condition holds pairwise, and the corresponding infinite product of holomorphic functions converges to a holomorphic function. This result will be necessary to define the equivariant Euler class of the normal bundle to a fixed point in ΩG as a hyperfunction.

Lemma 5.2.1. *If $\{F_k(z + i\gamma_k 0)\}_{k=1}^{\infty}$ is a sequence of hyperfunctions on Ω such that:*

1. *For all pairs $j \neq k$, $SS(F_k(z + i\gamma_k 0)) \cap SS(F_j(z + i\gamma_k 0))^\circ = \emptyset$*

2. $\gamma = \bigcap_{k=1}^{\infty} \gamma_k$ *is open*

3. *The infinite product* $\prod_{k=1}^{\infty} F_k(z)$ *is uniformly convergent on compact subsets of* $\Omega \times i\gamma$

then there exists a hyperfunction $F(z + i\gamma 0)$ *such that:*

$$F(z + i\gamma 0) = \prod_{k=1}^{\infty} F(z + i\gamma_k 0)$$

Proof. The condition on singular support is necessary to define any product of the F_k. Since the intersection of the cones is open, the wedge $\Omega \times i\gamma$ is a well defined open set in $\mathbb{C}^n$, and the convergence condition on the infinite product ensures that the following limit is a holomorphic function on $\Omega \times i\gamma$:

$$F(z) = \lim_{N\to\infty} \prod_{k=1}^{N} F_k(z)$$

This result has shown that the infinite product of the hyperfunctions $F_k(z + i\gamma_k 0)$ is well defined and equal to $F(z + i\gamma 0)$. □

We now describe how to define the Fourier transform of a hyperfunction. The following two definitions are central to the theory of hyperfunction Fourier transforms. We will restrict our attention to the class of *Fourier hyperfunctions*, also known as *slowly increasing hyperfunctions*.

Definition 5.2.3. *A holomorphic function* $F \in \mathcal{O}(\mathbb{R}^n \times i\gamma 0)$ *is called slowly increasing if and only if for every compact subset* $K \subseteq i\gamma 0$*, and for every* $\epsilon > 0$*, there exist constants* $M, C > 0$ *such that, for all* $z \in \mathbb{R}^n \times iK$*, if* $|Re(z)| > M$ *then* $|F_j(z)| \leq C \exp(\epsilon Re(z))$.

A holomorphic function $F \in \mathcal{O}(\mathbb{R}^n \times i\gamma 0)$ *is called exponentially decreasing on the (not necessarily convex) cone* $\Delta \subseteq \mathbb{R}^n$ *if and only if there exists* $\delta > 0$*, such that for every compact* $K \subseteq i\gamma_j 0$*, and for every* $\epsilon > 0$*, there exist constants* $M, C > 0$ *such that for every* $z \in \Delta \times iK$*, if* $|Re(z)| > M$ *then* $|F_j(z)| \leq C \exp(-(\delta - \epsilon) Re(z))$.

Remarks on the definition:

1. A hyperfunction will be called slowly increasing (resp. exponentially decreasing on Δ) if and only if it admits a boundary value representation:

$$f(x) = \sum_{j=1}^{n} F_j(z + i\gamma 0)$$

such that each of the $F_j(z)$ is slowly increasing (resp. exponentially decreasing on Δ).

2. If $F(z)$ is slowly increasing and $G(z)$ is exponentially decreasing on Δ, then $F(z) \cdot G(z)$ is exponentially decreasing on Δ.

3. The class of exponentially decreasing functions is closed under the classical Fourier transform (see [Kaw70]). The Fourier transform of slowly increasing hyperfunctions will be defined to be dual to this operation via a pairing between slowly increasing hyperfunctions and exponentially decreasing holomorphic functions.

Intuitively, a hyperfunction is slowly increasing when, after fixing the imaginary part of z inside of $i\gamma_j$, its asymptotic growth along the real line is slower than every exponential function. A hyperfunction is exponentially decreasing on the cone γ when the holomorphic functions in a boundary value representation decay exponentially in the real directions which are inside of the cone γ.

As previously mentioned, there exists a pairing between slowly increasing hyperfunctions and exponentially decreasing holomorphic functions. Let $f(x) = F(z + i\gamma 0)$ be a slowly increasing hyperfunction, $G(z)$ an exponentially decreasing analytic function, and S a contour of integration chosen so that $\mathrm{Im}(z) \in i\gamma 0$ for all $z \in S$. The pairing is given by:

$$\langle f, G\rangle = \int_S F(x+iy)G(x+iy)\,dx$$

Convergence of the integral is guaranteed by the condition that $F(z)G(z)$ is exponentially decreasing. That the pairing does not depend on the choice of contour follows from the Cauchy integral formula. The pairing allows us to identify the slowly increasing hyperfunctions as the topological dual space to the space of exponentially decreasing holomorphic functions. The Fourier transform of a slowly increasing hyperfunction is then defined by a duality with respect to this pairing:

$$\langle \mathscr{F}(f), G\rangle := \langle f, \mathscr{F}(G)\rangle$$

In practice, the Fourier transform of a hyperfunction is not computed directly from the definition. Let us now introduce the practical method by which one normally computes the Fourier transform of a slowly increasing hyperfunction. Suppose that $F(z)$ is a holomorphic function which is exponentially decreasing outside of a closed convex cone Δ. Letting $z = x + iy$ and $\zeta = \sigma + i\tau$, and suppose that $x \in \Delta$. We have the following estimate:

$$|\exp(-i\zeta \cdot z)| = \exp(y\cdot\sigma + x\cdot\tau)$$

The above estimate shows that $\exp(-i\zeta z)$ will be exponentially decreasing on Δ, so long as we fix $\tau \in -\Delta^\circ$. It then follows that the product $e^{-i\zeta z}F(z)$ is exponentially decreasing on $\mathbb{R}^n$. If $f(x) = F(z+i\gamma 0)$, then its Fourier transform is the hyperfunction given by:

$$\mathscr{F}(f) = G(\zeta - i\Delta^\circ) = b_{-\Delta^\circ}\left(\int_S e^{-i\zeta z}F(z)\,dz\right)$$

This can be extended to an arbitrary boundary value expression $f(x) = \sum_j F_j(z + i\gamma_j 0)$ by linearity, assuming that each of the $F_j(z)$ decreases exponentially outside of some cone.

We must now deal with the case that $f(x) = F(z + i\gamma 0)$ is a slowly increasing hyperfunction, but that it does not decrease exponentially on any cone.

Definition 5.2.4. *Let Σ be a finite collection of closed convex cones. A holomorphic partition of unity is collection of holomorphic functions $\{\chi_\sigma(z)\}_{\sigma\in\Sigma}$ such that:*

1. $\displaystyle\sum_{\sigma\in\Sigma}\chi_\sigma(z) = 1$

2. *$\chi_\sigma(z)$ is exponentially decreasing outside of any open cone $\sigma' \supset \sigma$*

3. $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$

Example of a holomorphic partition of unity:

Let Σ denote the collection of orthants in $\mathbb{R}^n$. If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multi-index whose entries are ± 1 (clearly such objects are in bijection with the orthants), we will denote the corresponding orthant by γ_σ Consider the following two functions:

$$\chi_+(t) = \frac{1}{1+e^{-t}}$$

$$\chi_-(t) = \frac{1}{1+e^{t}}$$

where $t \in \mathbb{C}$ is a complex variable. We notice that $\chi_+(t)$ is exponentially decreasing on $\mathrm{Re}(t) < 0$ and $\chi_-(t)$ is exponentially decreasing on $\mathrm{Re}(t) > 0$. For a fixed orthant $\sigma \in \Sigma$, define the holomorphic function $\chi_\sigma(z)$ by:

$$\chi_\sigma(z) = \prod_{i=1}^{n} \frac{1}{1+e^{\sigma_i z_i}}$$

This function exponentially decreases on the complement of γ_σ. The collection $\{\chi_\sigma(z)\}_{\sigma \in \Sigma}$ is a holomorphic partition of unity.

We have introduced holomorphic partitions of unity as an abstract concept, but we will only ever use this example in our computations. The reason we have done this, as we will see later, is that the computations can be made easier or harder by a clever choice of holomorphic partition of unity (although the actual result of the computation is of course independent of any such choices). Our main result on the Duistermaat-Heckman hyperfunction of $\Omega SU(2)$ will remain in an integral form, but it is possible that the computation of the Fourier transform could be completed by redoing the computation with a judicious choice of holomorphic partition of unity.

We are now ready to explain how to compute the Fourier transform of a general slowly increasing hyperfunction. Again, by linearity of the Fourier transform, we may assume our hyperfunction takes the form $f(x) = F(z + i\gamma 0)$, and that $F(z)$ is a slowly increasing holomorphic function. Choose a holomorphic partition of unity $\{\chi_\sigma(z)\}_{\sigma \in \Sigma}$, then we observe that:

$$F(z) = \sum_{\sigma \in \Sigma} F(z)\chi_\sigma(z)$$

where now, $F(z)\chi_\sigma(z)$ is exponentially decreasing outside of σ. By our previous observations,

$$\mathscr{F}(f) = \sum_{\sigma \in \Sigma} b_{-\sigma^\circ}\left(\int_S e^{-i\zeta z} F(z)\chi_\sigma(z)\, dz\right) \tag{5.1}$$

Equation 5.1 exactly tells us how to compute the Fourier transform of a general slowly increasing hyperfunction.

5.3 Hyperfunctions arising from localization of Hamiltonian group actions

Let (M, ω) be a finite dimensionial compact symplectic manifold with a Hamiltonian action of a d-dimensional compact torus T; call the moment map $\mu : M \to \mathfrak{t}^*$. The symplectic form ω gives us the Liouville measure $\omega^n/n!$ on M, which we we may push forward to $\mathfrak{t}^*$ using the moment map μ. We let $\mathcal{F}$ denote the connected components of the fixed point set for the T action on M; furthermore, if $q \in \mathcal{F}$, we denote by e_q^T the equivariant Euler class of the normal bundle to the fixed point set. We can identify $e_q^T \in H^*(BT) \simeq \mathrm{Sym}(\mathfrak{t}^*)$ with the product of the weights appearing in the isotropy representation of T on T_qM.

Theorem 5.3.1. *[DH82] The measure $\mu_*(\omega^n/n!)$ has a piecewise polynomial density function. Furthermore, the inverse Fourier transform of $\mu_*(\omega^n/n!)$ has an exact expression:*

$$\int_M e^{i\mu(p)(X)}\omega^n/n! = \frac{1}{(2\pi i)^d}\sum_{q\in\mathcal{F}} \frac{e^{i\mu(q)(X)}}{e_q^T(X)} \tag{5.2}$$

where $X \in \mathfrak{t}$ is such that $e_q^T(X) \neq 0$ for all $q \in \mathcal{F}$.

The Duistermaat-Heckman theorem applies to the case where M is finite dimensional and compact. We are interested in finding some version of a Duistermaat-Heckman distribution in the setting where M is an infinite dimensional manifold with a Hamiltonian group action. There are some immediate technical obstructions to producing such a distribution. Most notably, the inability to take a top exterior power of ω prevents us from defining a suitable Liouville measure. There are significant analytic challenges in properly defining the left hand side of equation 5.2; a related problem is defining a rigorous measure of integration for the kinds of path integrals which appear in quantum field theory. We will not attempt to answer this question in this thesis. Nevertheless, it is possible to make sense of the right hand side of Equation 5.2.

The main goal for this section is to explain how Hamiltonian actions of compact tori yield, in a natural way, hyperfunctions on $\mathfrak{t}$. The hyperfunction one gets in this way should be a substitute for the the reciprocal of the equivariant Euler class which appears in the localization formula. We then reinterpret the sum over the fixed points in the localization formula as a hyperfunction on $\mathfrak{t}$, and define the *Duistermaat-Heckman hyperfunction* to be its Fourier transform as a hyperfunction.

We will start by considering the local picture. Suppose that T has a Hamiltonian action on a (finite dimensional, for now) complex vector space with weights λ_i. Let the weights of the action be given by $W = \{\lambda_i\}_{i\in I}$. The weights of the action are linear functionals $\mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$. For every weight $\lambda \in W$ we get a corresponding half space $H_\lambda = \{y \in \mathfrak{t} \,|\, \lambda(y) > 0\}$, as well as a hyperfunction:

$$f_\lambda(x) = \frac{1}{\lambda(z) + iH_\lambda 0}$$

The singular support of $f_\lambda(x)$ is given by:

$$SS(f_\lambda) = \{(x, \xi) \in T^*(\mathfrak{t}) \,|\, \lambda(x) = 0, \exists\, c > 0, \xi = c\, d\lambda(x)\}$$

Proposition 5.3.1. *If $\mu : V \to \mathfrak{t}^*$ is proper then for all pairs of weights λ, λ', $SS(f_\lambda) \cap SS(f_{\lambda'})^\circ = \emptyset$.*

Proof. If the moment map is proper, then all of the weights are contained in a half space [GLS88]. There exists $X \in \mathfrak{t}$ such that for any pair of weights λ, λ' we have both $\lambda(X) > 0$ and $\lambda'(X) > 0$. Suppose that $(x, \xi) \in SS(f_\lambda) \cap SS(f'_\lambda)^\circ$. This means that:

1. $\lambda(x) = \lambda'(x) = 0$
2. $\exists c, c' > 0$ such that $\xi = c\, d\lambda = -c' d\lambda'$

Rearranging the second condition implies that the function $L = \lambda' + \frac{c}{c'}\lambda$ is constant. However, we have obtained a contradiction as $L(x) = 0$, while $L(X) > 0$. $\square$

The following is immediate from the proposition.

Corollary 5.3.1. *Let* $\gamma = \bigcap_{\lambda \in W} H_\lambda$. *If* $\mu : V \to \mathfrak{t}^*$ *is proper, then the following product of hyperfunctions is well defined:*

$$\frac{1}{e^T(x)} = \prod_{\lambda \in W} f_\lambda(x) = b_\gamma \left(\prod_{\lambda \in W} \frac{1}{\lambda(z)} \right)$$

We can use the reciprocals of the equivariant Euler classes in an expression which imitates the sum over the fixed points in the Duistermaat-Heckman formula. For any $p \in M^T$, let $W_p = \{\lambda_j^p\}_{j=1}^N$ denote the set of weights of the isotropy representation of T on T_pM. As in the case of the usual localization formula we must choose a *polarization*, which is simply a choice of vector $\xi \in \mathfrak{t}$ such that for every $p \in M^T$, and for every $\lambda_j^p \in W_p$, we have $\lambda_j^p(\xi) \neq 0$. For every $\lambda_j^p \in W_p$ we define the *polarized weight* by:

$$\tilde{\lambda}_j^p = \begin{cases} \lambda_j^p & \lambda_j^p(\xi) > 0 \\ -\lambda_j^p & \lambda_j^p(\xi) < 0 \end{cases}$$

and we adopt the notation as in [GLS88] by setting $(-1)^p = \prod_{\lambda_j^p \in W_p} \operatorname{sgn} \lambda_j^p(\xi)$.

By definition, for every fixed point $p \in M^T$ we have that the polarized weights are contained in the half plane defined by ξ. We define the cone $\gamma_p = \bigcap_{\lambda_j^p \in W_p} H_{\tilde{\lambda}_j^p}$, which is simply the intersection of the half spaces defined by the polarized weights. We will call:

$$\frac{1}{e_p^T(x)} = b_{\gamma_p} \left(\prod_{\tilde{\lambda}_j^p \in W_p} \frac{1}{\tilde{\lambda}_j^p(z)} \right)$$

the reciprocal of the equivariant Euler class to the normal bundle of p.

Definition 5.3.1. *Suppose that* (M, ω) *has a Hamiltonian action of a compact, dimension* d *torus* T *such that all the fixed points are isolated; let* M^T *denote the fixed point set and* $\mu : M \to \mathfrak{t}^*$ *the moment map. We will call the following expression the Picken hyperfunction:*

$$L(x) = \frac{1}{(2\pi i)^d} \sum_{p \in M^T} (-1)^p \frac{e^{i\mu(p)(x)}}{e_p^T(x)}$$

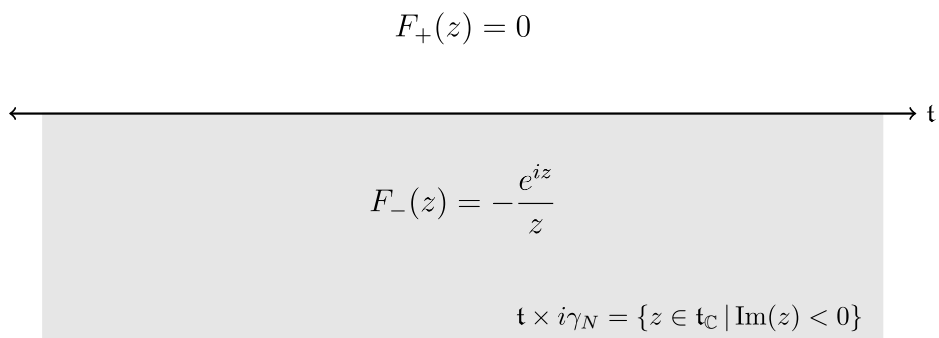

Figure 5.1: A depiction of the hyperfunction $J_N(x) \in \mathcal{B}(\mathfrak{t})$

Example: S^2 with a circle action by rotation

We will first use a simple example to demonstrate that the formalism of hyperfunctions reproduces the results one would expect from the Duistermaat-Heckman function. We choose a polarization $\xi = -1$. For the usual Hamiltonian circle action on S^2 by counterclockwise rotation about the z-axis, there are fixed points at the north and south poles, N and S, respectively. The torus acts on $T_N S^2$ with weight $+1$, while it acts on $T_S S^2$ with weight -1. Let's compute the reciprocal of the equivariant Euler class to the normal bundle of N (as a hyperfunction). There is only one weight at this fixed point. We have

$$\gamma_N = \{x \in i\mathfrak{t} \mid x < 0\}$$

$$(-1)^N = -1$$

The north pole contributes the following hyperfunction as a summand of the Picken hyperfunction, which we denote pictorially in figure 5.1:

$$J_N(x) = b_{\gamma_N}\left(-\frac{e^{iz}}{z}\right)$$

The contribution to the Picken hyperfunction coming from the south pole is computed similarly. We obtain:

$$\gamma_S = \{x \in i\mathfrak{t} \mid x < 0\}$$

$$(-1)^S = -1$$

and so

$$J_S(x) = b_{\gamma_S}\left(\frac{e^{-iz}}{z}\right)$$

Since $\gamma_N = \gamma_S$ in this example, we simply call both of these γ. The end result is that the Picken hyperfunction of this Hamiltonian group action is:

$$2\pi i\, L(x) = b_\gamma\left(-\frac{e^{iz}}{z} + \frac{e^{-iz}}{-z}\right)$$

or, thinking of hyperfunctions on $\mathbb{R}$ as pairs of holomorphic functions, this corresponds to the pair

$$2\pi i L(x) = \left[0, \frac{-e^{iz} + e^{-iz}}{z}\right]$$

Had one chosen the polarization $\tilde{\xi} = +1$, one would have alternatively obtained the presentation

$$2\pi i \tilde{L}(x) = \left[\frac{e^{iz} - e^{-iz}}{z}, 0\right]$$

however, $L(x) = \tilde{L}(x)$ as hyperfunctions because their difference extends analytically across the real axis. The observation here is that a choice of polarization is simply enabling us to write down a presentation of a hyperfunction using a specific set of cones.

The Duistermaat-Heckman hyperfunction is the Fourier transform of the Picken hyperfunction. We will now compute it according to the formula in equation 5.1. We choose the holomorphic partition of unity given by the functions:

$$\chi_+(z) = \frac{1}{1 + e^{-z}}$$

$$\chi_-(z) = \frac{1}{1 + e^{z}}$$

which gives a decomposition of the Picken hyperfunction into four parts.

$$2\pi i\, L(x) = b_\gamma\left(-\frac{e^{iz}}{z}\chi_+(z)\right) + b_\gamma\left(\frac{e^{-iz}}{z}\chi_+(z)\right) + b_\gamma\left(-\frac{e^{iz}}{z}\chi_-(z)\right) + b_\gamma\left(\frac{e^{-iz}}{z}\chi_-(z)\right)$$

The Fourier transform can now be computed termwise, noticing that the first two terms in the above expression are exponentially decreasing on $\mathrm{Re}(z) < 0$, while the third and fourth terms are exponentially decreasing on the cone $\mathrm{Re}(z) > 0$. Let $1 \gg \delta > 0$, then we may write the Fourier transform $\mathscr{F}(L(x)) = G_+(\zeta + i0) + G_-(\zeta - i0)$ where:

$$G_+(\zeta) = \int_{-\infty - i\delta}^{\infty - i\delta} -\frac{e^{-i(\zeta-1)z}}{z(1+e^z)}\,dz + \int_{-\infty - i\delta}^{\infty - i\delta} \frac{e^{-i(\zeta+1)z}}{z(1+e^z)}\,dz$$

$$G_-(\zeta) = \int_{-\infty - i\delta}^{\infty - i\delta} -\frac{e^{-i(\zeta-1)z}}{z(1+e^{-z})}\,dz + \int_{-\infty - i\delta}^{\infty - i\delta} \frac{e^{-i(\zeta+1)z}}{z(1+e^{-z})}\,dz$$

Each of these integrals can be computed by completing to a semicircular contour in the lower half plane and applying the residue theorem (noting that, as the contour is oriented clockwise, we must include an extra minus sign). The contour we use for the first integral appearing in $G_+(\zeta)$ is depicted in Figure 5.2, along with the locations of the poles.

We show how to compute the first integral in the expression for $G_+(\zeta)$; the rest are similar. The integrand of the first integral in $G_+(\zeta)$ has poles at $z_0 = 0$ and $z_k = -(2k+1)\pi i$ for $k \in \mathbb{Z}$, however, the only poles inside the contour (in the limit as the radius of the semicircle tends to infinity) are the poles at z_k for $k \geq 0$. Also, in the limit as the radius of the semicircle gets large we see that the contribution to the integral coming from the semicircular part of the contour vanishes because the integrand is exponentially

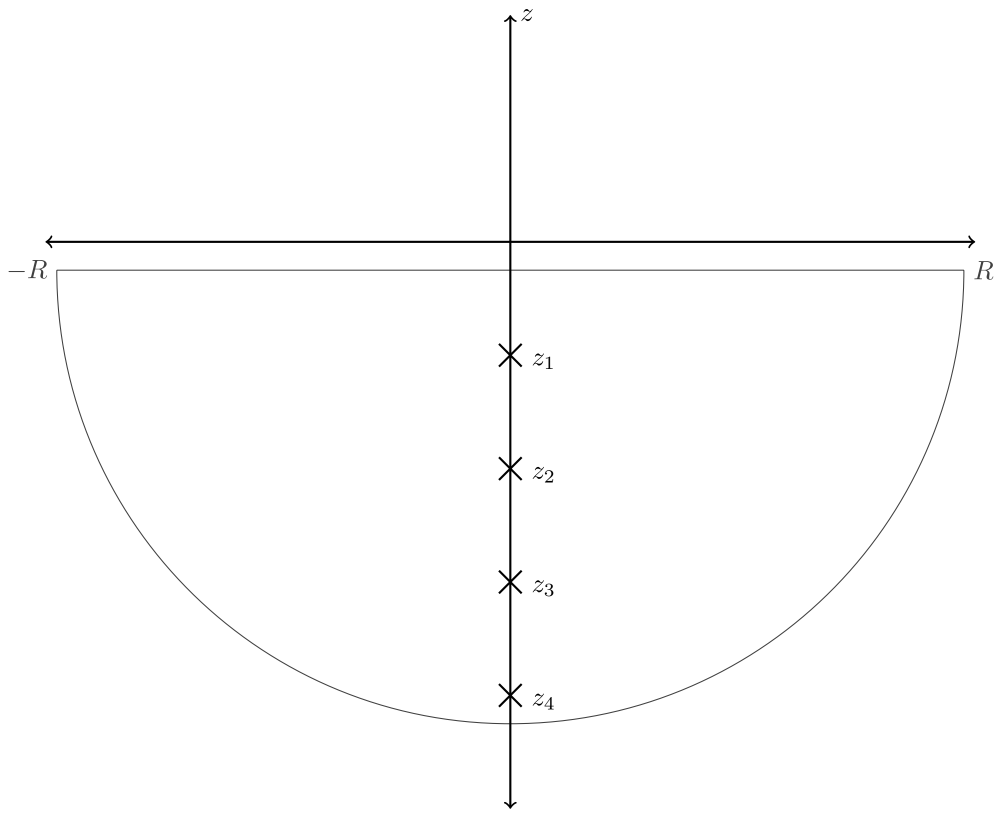

Figure 5.2: Integration contour for the first integral in $G_+(\zeta)$

decreasing in $\mathrm{Re}(z)$, and decreasing exponentially in $\mathrm{Im}(z)$ when $\mathrm{Im}(z) < 0$. By the residue theorem:

$$
\begin{aligned}
\int_{-\infty-i\delta}^{\infty-i\delta} \frac{e^{-i(\zeta-1)z}}{z(1+e^z)}\,dz &= -2\pi i \sum_{k=0}^{\infty} \mathrm{Res}\left(\frac{e^{-i(\zeta-1)z}}{z(1+e^z)}, z = z_k\right) \\
&= -2\pi i \sum_{k=0}^{\infty} \frac{e^{-(\zeta-1)(2k+1)\pi}}{-(2k+1)\pi i(-1)} \\
&= 2\pi \sum_{k=0}^{\infty} \int_c^{\zeta} e^{-(\zeta'-1)(2k+1)\pi}\,d\zeta' \\
&= 2\pi \int_c^{\zeta} \sum_{k=0}^{\infty} e^{-(\zeta'-1)(2k+1)\pi}\,d\zeta' \\
&= 2\pi \int_c^{\zeta} \frac{d\zeta'}{e^{\pi(\zeta'-1)} - e^{-\pi(\zeta'-1)}} \\
&= \pi \int_c^{\zeta} \frac{d\zeta'}{\sinh(\pi(\zeta'-1))}\,d\zeta' \\
&= \mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta-1)}{2}\right)\right) \qquad \text{valid for } \mathrm{Im}(\zeta) > 0
\end{aligned}
$$

From the second to third line, we found a primitive function for the summand. From the third to the fourth line we applied the monotone convergence theorem to interchange the order of summation and integration. A nearly identical computation yields the result:

$$\int_{-\infty-i\delta}^{\infty-i\delta} \frac{e^{-i(\zeta+1)z}}{z(1+e^z)}\,dz = \mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta+1)}{2}\right)\right) \quad \text{valid for } \mathrm{Im}(\zeta) > 0$$

Summarizing, to this point we have computed:

$$G_+(\zeta) = \mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta+1)}{2}\right)\right) - \mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta-1)}{2}\right)\right) \quad \text{valid for } \mathrm{Im}(\zeta) > 0$$

$$G_-(\zeta) = -\mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta+1)}{2}\right)\right) + \mathrm{Log}\left(\tanh\left(\frac{\pi(\zeta-1)}{2}\right)\right) \quad \text{valid for } \mathrm{Im}(\zeta) < 0$$

The above expressions can be simplified. We notice that the holomorphic function $\mathrm{Log}(\tanh(\zeta)) - \mathrm{Log}(\zeta)$ admits an analytic extension across a neighbourhood of the real axis, and is therefore zero as a hyperfunction. This means that all of the tanh factors may be ignored for the purposes of computing the hyperfunction Fourier transform. Therefore, the final result of our computation is:

$$\mathscr{F}(L(x)) = b_+\left(-\frac{1}{2\pi i}\mathrm{Log}\left(\frac{\zeta-1}{\zeta+1}\right)\right) - b_-\left(-\frac{1}{2\pi i}\mathrm{Log}\left(\frac{\zeta-1}{\zeta+1}\right)\right)$$

which we recognize as the standard defining hyperfunction of $\chi_{[-1,1]}(x)$ (see [Kan89] Example 1.3.11, p. 29). This has shown that the Fourier transform of the Picken hyperfunction gives the standard defining hyperfunction of the Duistermaat-Heckman distribution.

Jeffrey and Kirwan, building on work of Witten [Wit92], formalized the notion of a residue in symplectic geometry [JK95b]. They fruitfully applied this construction to compute relations in the cohomology ring of the moduli space of stable holomorphic bundles on a Riemann surface [JK95a]. We expect that the properties that uniquely characterize the residue (c.f. Proposition 8.11, [JK95b]) can be recovered from the usual notion of a residue [GH14] of a multivariable complex meromorphic function using our construction of the Picken hyperfunction.

5.4 ΩG and its Hamiltonian group action

Let G be a compact connected real Lie group, and call its Lie algebra $\mathfrak{g}$. In this chapter we will consider the space of smooth loops in $LG = C^\infty(S^1, G)$. LG is itself an infinite dimensional Lie group, with the group operation taken to be multiplication in G pointwise along a loop. The Lie algebra of LG is easily seen to consist of the space of smooth loops into the Lie algebra, which we denote $L\mathfrak{g}$.

We will also consider its quotient $\Omega G = LG/G$, where the quotient is taken with respect to the subgroup of constant loops. One may alternatively identify ΩG as the collection of loops, such that the identity in S^1 maps to the identity in G:

$$\Omega G = \{\gamma \in LG : \gamma(1) = e\}$$

Its Lie algebra can be identified with the subset $\Omega\mathfrak{g} = \left\{X : S^1 \to \mathfrak{g} \,|\, X(0) = 0\right\}$.

ΩG has a lot of extra structure, which essentially comes from its realization as a coadjoint orbit of a central extension of LG [KW08]. We can give ΩG a symplectic structure as follows. Since G is a compact Lie group, there exists a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$. This form induces an antisymmetric form:

$$\omega_e : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$$

$$(X, Y) \mapsto \frac{1}{2\pi} \int_0^{2\pi} \langle X(\theta), Y'(\theta) \rangle \, d\theta$$

This bilinear form is an antisymmetric, non-degenerate form when restricted to $\Omega\mathfrak{g}$, and extends to a symplectic form on ΩG using a left trivialization of the tangent bundle of ΩG. That is, for every $\gamma \in \Omega G$ we fix the isomorphism

$$T_\gamma \Omega G \simeq \Omega\mathfrak{g}$$

$$X \mapsto \big(\theta \mapsto \gamma^{-1}(\theta) X(\theta)\big)$$

This choice allows us to define a form on ΩG as:

$$\omega_\gamma : T_\gamma \Omega G \times T_\gamma \Omega G \to \mathbb{R}$$

$$(X, Y) \mapsto \omega_e(\gamma^{-1} X, \gamma^{-1} Y)$$

The form so defined is symplectic; a proof can be found in [PS86].

Consider the following group action on ΩG. Fix $T \subseteq G$ a maximal compact torus, and let $\mathfrak{t}$ be its Lie algebra. Pointwise conjugation by elements of T defines a T action on ΩG.

$$T \times \Omega G \to \Omega G$$

$$t \cdot \gamma = \big(\theta \mapsto t\gamma(\theta)t^{-1}\big)$$

There is also an auxiliary action of S^1 on ΩG, which comes about by descending the loop rotation action on LG to the quotient LG/G. Explicitly,

$$S^1 \times \Omega G \to \Omega G$$

$$\exp(i\psi) \cdot \gamma = \big(\theta \mapsto \gamma(\theta + \psi)\gamma(\psi)^{-1}\big)$$

These actions commute with one another, so define an action of $T \times S^1$ on ΩG. We will let $\mathrm{pr}_{\mathfrak{t}} : \mathfrak{g} \to \mathfrak{t}$ denote the orthogonal projection coming from the Cartan-Killing form. We now define two functions on ΩG:

$$p(\gamma) = \frac{1}{2\pi} \mathrm{pr}_{\mathfrak{t}} \left(\int_0^{2\pi} \gamma^{-1}(\theta)\gamma'(\theta) \, d\theta \right)$$

$$E(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} ||\gamma'(\theta)||^2 \, d\theta$$

Proposition 5.4.1. *[AP83] The $T \times S^1$ action on ΩG is Hamiltonian. The moment map is given by:*

$$\mu : \Omega G \to Lie(T \times S^1)$$

$$\gamma \mapsto \begin{pmatrix} p(\gamma) \\ E(\gamma) \end{pmatrix}$$

Furthermore, the Hamiltonian vector fields associated to the group action are given by:

$$(X_E)_\gamma = \gamma'(\theta) - \gamma(\theta)\gamma'(0)$$

$$(X_{p_\tau})_\gamma = \tau\gamma(\theta) - \gamma(\theta)\tau$$

where $\tau \in \mathfrak{t}$.

If $\beta \in \mathfrak{t} \oplus \mathbb{R}$ then we let $(X_\beta)_\gamma$ denote the Hamiltonian vector field evalutated at the loop γ.

5.5 Fixed Points Sets of Rank One Subtori

We will now proceed to identify the fixed point sets of dimension one subtori of $T \times S^1$ acting on ΩG. The moment map image of the fixed point submanifolds should correspond to the locus where the Duistermaat-Heckman density function is not differentiable. Using the exponential map, we identify $X_*(T \times S^1) \simeq P \times \mathbb{Z}$, where P is the coweight lattice of Lie(T). Fix an element $\beta = (\lambda, m) \in X_*(T \times S^1)$ and call the cocharacter it generates by T_β. Let $\Lambda \in X_*(T)$ be the cocharacter generated by λ,

$$\Lambda(\theta) = \exp(i\lambda\theta)$$

We will say the fixed point set of T_β is trivial when $\Omega G^{T_\beta} = \mathrm{Hom}_{\mathrm{Grp}}(S^1, T)$. In this section, we say that $L \subseteq G$ is a *Levi subgroup* if and only if there exists a parabolic subgroup $Q \subseteq G_\mathbb{C}$ such that $L_\mathbb{C}$ is a Levi factor of Q. Every Levi subgroup of G is the centralizer of a subtorus$S \subseteq T$.

If we have two groups K and N, together with a map $\varphi : K \to \mathrm{Aut}\, N$, then we can construct the semidirect product group $N \rtimes K$ whose point set is the Cartesian product $N \times K$, but the group operation is $(n, k) \cdot (n', k') = ((\phi(k') \cdot n)n', kk')$. In our specific context, if we fix any Levi subgroup $L \subseteq G$, we can construct a group homomorphism:

$$\varphi_\beta : S^1 \to \mathrm{Aut}\, L$$

$$\varphi_\beta(\psi) \cdot x = \Lambda\left(\frac{\psi}{m}\right)^{-1} x\, \Lambda\left(\frac{\psi}{m}\right)$$

Remarks:

1. Since $\varphi_\beta(1) = \mathrm{id}_L$ and S^1 is connected then we may consider $\varphi_\beta : S^1 \to \mathrm{Inn}(L)$. We identify $\mathrm{Inn}(L) \simeq L_{\mathrm{ad}}$, which may further be identified with $[L, L]/Z(L) \cap [L, L]$. Under these identifications, $\varphi_\beta \in X_*(T_{\mathrm{ad}})$ is a cocharacter of the maximal torus in L_{ad}.

2. This homomorphism is well defined if and only if $\Lambda(\frac{2\pi}{m}) \in Z(L)$. In particular, λ/m must be an element of the coweight lattice for the Levi subgroup L, mod $\mathfrak{z}(L)$.

3. $\varphi_\beta = \varphi_{\beta'}$ if and only if $\lambda/m - \lambda'/m' \in \mathfrak{z}(L)$

We will denote the resulting semidirect product group as $L \rtimes_\beta S^1$.

It can be easily seen that for any Levi subgroup L, $T \times S^1$ is a maximal torus of $L \rtimes_\beta S^1$. Any one parameter subgroup of $L \rtimes_\beta S^1$ is abelian, and is therefore contained in a maximal torus conjugate to $T \times S^1$. We can obtain all one parameter subgroups by considering one of the form $(\eta(\theta), \theta)$ for $\eta \in \mathrm{Hom}(S^1, T)$, then conjugating by an element of $L \rtimes_\beta S^1$.

$$\gamma(\theta) = \Lambda\left(\frac{\psi - \theta}{m}\right) g \Lambda\left(\frac{\theta}{m}\right) \eta(\theta) g^{-1} \Lambda\left(\frac{\psi}{m}\right)^{-1} \tag{5.3}$$

Proposition 5.5.1. *For any $\beta \in P \times \mathbb{Z}$, there exists a Levi subgroup $T \subseteq L_\beta \subseteq G$, such that $\gamma \in \Omega G^{T_\beta}$ if and only if $(\gamma(\theta), \theta)$ is a one parameter subgroup of $L_\beta \rtimes_\beta S^1$.*

Proof. Fix $\beta \in P \times \mathbb{Z}$ and set $L_\beta = Z_G(\Lambda(2\pi/m))$; that $T \subseteq L_\beta$ follows, since $\Lambda(2\pi/m) \in T$ and T is abelian.

Suppose we have a loop γ fixed by T_β. Recall how T_β acts on a loop $\gamma \in \Omega G$. For every $(\Lambda(\psi), \exp(im\psi)) \in T_\beta$, the action is:

$$(\Lambda(\psi), \exp(im\psi)) \cdot \gamma(\theta) = \Lambda(\psi)\gamma(\theta + m\psi)\Lambda^{-1}(\psi)\gamma(m\psi)^{-1} \qquad \forall\, \psi, \theta \in [0, 2\pi)$$

Let's rescale the ψ variable, then by periodicity we may write the condition to be fixed under T_β as:

$$\gamma(\theta + \psi) = \Lambda\left(\frac{\psi}{m}\right)^{-1} \gamma(\theta) \Lambda\left(\frac{\psi}{m}\right) \gamma(\psi) \qquad \forall\, \theta, \psi \in [0, 2\pi)$$

When $\psi = 2\pi$ in the above equation we get the condition $\gamma(\theta) \in L_\beta$ for all θ. That $(\gamma(\theta), \theta)$ is a one parameter subgroup of $L \rtimes_\beta S^1$ follows immediately from the multiplication rule for the semidirect product.

Now suppose conversely that $(\gamma(\theta), \theta)$ is a one parameter subgroup of $L \rtimes_\beta S^1$. There exists $\eta \in X_*(T)$, $g \in L$ and $\psi \in S^1$ such that γ can be written as in equation 5.3. To show that γ is fixed by T_β it suffices to prove that the Hamiltonian vector field corresponding to β vanishes at γ. This is a straightforward (but tedious) verification.

□

A consequence of the previous proposition is that for any such β, there exists a Levi subgroup L_β such that $\Omega G^{T_\beta} = \Omega L_\beta^{T_\beta}$. This follows, since the semidirect product formula forces any loop fixed under T_β to have its image be contained in L_β.

Proposition 5.5.2. *Every connected component of the fixed point set of T_β is a translate of an adjoint orbit in Lie$(L_\beta) \subseteq \mathfrak{g}$.*

Proof. Fix a loop γ in some connected component of the fixed point set of T_β. Using the exponential map on $L \rtimes_\beta S^1$, it can be seen that $(\gamma(\theta), \theta)$ is a one-parameter subgroup of $L_\beta \rtimes_\beta S^1$ if and only if γ is a solution to the differential equation:

$$\frac{d\gamma}{d\theta} = \left[\gamma(\theta), \frac{\lambda}{m}\right] + \gamma(\theta)\gamma'(0)$$

Compactness of G (and therefore, of L_β, since it is a closed subgroup) and the Picard-Lindelöf theorem allow us to identify the loops in the fixed point set of T_β with their initial conditions $\gamma'(0) \in \mathfrak{g}$. We can use equation 5.3 to compute $\gamma'(0)$:

$$\gamma'(0) = \mathrm{Ad}_{\Lambda(\frac{\psi}{m})g}\left[\frac{\lambda}{m} + \eta'(0)\right] - \frac{\lambda}{m}$$

Any other loop in the same connected component of the fixed point set of T_β can be obtained by varying $g \in L_\beta$ and $\psi \in [0, 2\pi)$. □

Notice that by fixing $\lambda = 0$ in the preceeding discussion, we recover the result that the fixed point set of the loop rotation action consists of the group homomorphisms $S^1 \to G$ [PS86].

The last result of this section characterizes exactly when two rank one subtori have the same fixed point sets.

Proposition 5.5.3. *Let $\beta = (\lambda, m)$ and $\beta' = (\lambda', m')$ be generators of rank one subgroups T_β, $T_{\beta'}$ of $T \times S^1$, and let $L_\beta, L_{\beta'}$ be the Levi subgroups provided by Proposition 5.5.1. Then, $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$ if and only if $\lambda/m - \lambda'/m' \in \mathfrak{z}(L_\beta)$*

Remark: If $\lambda/m - \lambda'/m' \in \mathfrak{z}(L_\beta)$ then $L_\beta = L_{\beta'}$. This is due to the fact that L_β was defined to be the G-centralizer of $\exp(2\pi i\lambda/m)$ (and similarly for $L_{\beta'}$).

Proof. Suppose that the fixed point sets of T_β and $T_{\beta'}$ are equal. Then for any γ, we have $(X_\beta)_\gamma = 0$ if and only if $(X_{\beta'})_\gamma = 0$. These conditions yield two differential equations:

$$0 = m\frac{d\gamma}{d\theta} - \gamma(\theta)\gamma'(0) + \lambda\gamma(\theta) - \gamma(\theta)\lambda$$

$$0 = m'\frac{d\gamma}{d\theta} - \gamma(\theta)\gamma'(0) + \lambda'\gamma(\theta) - \gamma(\theta)\lambda'$$

We may subtract these, and left translate back to $\mathfrak{g}$ to get the condition:

$$\forall\, \gamma \in \Omega G^{T_\beta}, \theta \in [0, 2\pi), \quad \mathrm{Ad}_{\gamma(\theta)}\left(\frac{\lambda}{m} - \frac{\lambda'}{m'}\right) = \frac{\lambda}{m} - \frac{\lambda'}{m'}$$

The derivative of this condition at the identity is

$$\left[\gamma'(0), \frac{\lambda}{m} - \frac{\lambda'}{m'}\right] = 0$$

so the statement is proved if for every element $Y \in \mathrm{Lie}([L_\beta, L_\beta])$, there exists $\gamma \in \Omega G^{T_\beta}$ and $c \in \mathbb{R}$ such that $Y = c\gamma'(0)$. By Proposition 5.5.2, we can identify the set of all such $\gamma'(0)$ with a translated adjoint orbit. This can be achieved by choosing a cocharacter $\eta(\theta)$ such that $\eta'(0) + \frac{\lambda}{m}$ is regular for the Ad_{L_β}-action and $\eta'(0)$ is sufficiently large so that the translated adjoint orbit intersects every ray through the origin.

Conversely, if $\lambda/m - \lambda'/m' \in \mathfrak{z}(L_\beta)$ then by the above remark, $L_\beta = L'_\beta$, and furthermore, β and β' yield identical automorphisms $\varphi_\beta = \varphi_{\beta'} : S^1 \to \mathrm{Aut}(L_\beta)$. Then by Proposition 5.5.1 we have $\Omega G^{T_\beta} = \Omega G^{T_{\beta'}}$. □

5.6 An explicit example: The loop space of $SU(2)$

When $G = SU(2)$, the general theory of the previous section can be understood in a very explicit way. The way to do this is to translate the condition of being fixed under the group action into a solution of a system of differential equations for the matrix parameters. Let's work through this derivation. We can describe an element $\gamma(t) \in \Omega SU(2)$ by:

$$\gamma(t) = \begin{pmatrix} \alpha(t) & -\beta(t)^* \\ \beta(t) & \alpha(t)^* \end{pmatrix}$$

Subject to the constraints $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ for all $t \in [0, 2\pi]$, $\alpha(0) = 1$, and $\beta(0) = 0$. One-parameter subgroups correspond bijectively with elements of the Lie algebra of $T \times S^1$. In that spirit, fix some element $(\theta, \psi) \in \mathfrak{t} \oplus \mathbb{R}$, exponentiate to the group, and act on our loop $\gamma(t)$

$$\begin{aligned}
\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, e^{i\psi}\right) \cdot \gamma(t) &= e^{i\psi} \cdot \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \alpha(t) & -\beta(t)^* \\ \beta(t) & \alpha(t)^* \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\right) \\
&= e^{i\psi} \cdot \begin{pmatrix} \alpha(t) & -e^{i2\theta}\beta(t)^* \\ e^{-i2\theta}\beta(t) & \alpha(t)^* \end{pmatrix} \\
&= \begin{pmatrix} \alpha(t+\psi) & -e^{i2\theta}\beta(t+\psi)^* \\ e^{-i2\theta}\beta(t+\psi) & \alpha(t+\psi)^* \end{pmatrix} \begin{pmatrix} \alpha(\psi)^* & e^{i2\theta}\beta(\psi)^* \\ -e^{-i2\theta}\beta(\psi) & \alpha(\psi) \end{pmatrix} \\
&= \begin{pmatrix} \alpha(t) & -\beta(t)^* \\ \beta(t) & \alpha(t)^* \end{pmatrix} \quad \text{when } \gamma(t) \text{ is a fixed loop}
\end{aligned}$$

so by rearranging slightly

$$\begin{aligned}
\begin{pmatrix} \alpha(t+\psi) & -e^{i2\theta}\beta(t+\psi)^* \\ e^{-i2\theta}\beta(t+\psi) & \alpha(t+\psi)^* \end{pmatrix} &= \begin{pmatrix} \alpha(t) & -\beta(t)^* \\ \beta(t) & \alpha(t)^* \end{pmatrix} \begin{pmatrix} \alpha(\psi) & -e^{i2\theta}\beta(\psi)^* \\ e^{-i2\theta}\beta(\psi) & \alpha(\psi)^* \end{pmatrix} \\
&= \begin{pmatrix} \alpha(t)\alpha(\psi) - e^{-i2\theta}\beta(t)^*\beta(\psi) & -\alpha(\psi)^*\beta(t)^* - e^{i2\theta}\alpha(t)\beta(\psi)^* \\ \alpha(\psi)\beta(t) + e^{-i2\theta}\alpha(t)^*\beta(\psi) & \alpha(t)^*\alpha(\psi)^* - e^{i2\theta}\beta(t)\beta(\psi)^* \end{pmatrix}
\end{aligned}$$

this yields the finite difference relations:

$$\alpha(t+\psi) = \alpha(t)\alpha(\psi) - e^{-i2\theta}\beta(t)^*\beta(\psi)$$

$$\beta(t+\psi) = e^{i2\theta}\alpha(\psi)\beta(t) + \alpha(t)^*\beta(\psi)$$

We use these infinitesimal form of these relations to get the necessary system of differential equations. Set $\theta = ns$ and $\psi = ms$ so that we can vary the group element along a fixed one parameter subgroup.

$$\begin{aligned}
m\frac{d\alpha}{dt} &= \lim_{s\to 0} \frac{\alpha(t+ms) - \alpha(t)}{s} \\
&= \lim_{s\to 0} \frac{\alpha(t)\alpha(ms) - e^{-i2ns}\beta(t)^*\beta(ms) - \alpha(t)}{s} \\
&= \alpha(t) \lim_{s\to 0} \frac{\alpha(ms) - 1}{s} - \beta(t)^* \lim_{s\to 0} \frac{e^{-i2ns}\beta(ms)}{s} \\
&= m\alpha(t)\alpha'(0) - m\beta(t)^*\beta'(0)
\end{aligned}$$

And similarly for $\beta(t)$,

$$\begin{aligned}
m\frac{d\beta}{dt} &= \lim_{s\to 0}\frac{\beta(t+ms)-\beta(t)}{s} \\
&= \beta(t)\lim_{s\to 0}\frac{e^{i2ns}\alpha(ms)-1}{s}+m\alpha(t)^*\beta'(0) \\
&= \beta(t)\left[i2ne^{i2ns}\alpha(ms)+me^{i2ns}\alpha'(ms)\right]\Big|_{s=0}+m\beta'(0)\alpha(t)^* \\
&= (i2n+m\alpha'(0))\beta(t)+m\beta'(0)\alpha(t)^*
\end{aligned}$$

so the system of differential equations we must solve (for $m \neq 0$, when $m = 0$ the problem is trivial) is given by:

$$\frac{d\alpha}{dt} = \alpha(t)\alpha'(0) - \beta(t)^*\beta'(0)$$

$$\frac{d\beta}{dt} = \beta'(0)\alpha(t)^* + \left(i2\frac{n}{m} + \alpha'(0)\right)\beta(t)$$

These differential equations are exactly the ones we could have gotten by searching for zeroes of the Hamiltonian vector field corresponding to $(n, m) \in \mathfrak{t} \oplus \mathbb{R}$ (c.f. the differential equation given in Proposition 5.5.2). The system we have described depends on four parameters: n, m, $\alpha'(0)$ and $\beta'(0)$. Once we fix these parameters, the solutions $\alpha(t)$ and $\beta(t)$ are uniquely determined. The parameters n and m are fixed from the start, so are only free to vary $\alpha'(0)$ and $\beta'(0)$. The choices that will turn out to yield periodic solutions will be exactly those loops whose derivatives at the identity are elements of the translated adjoint orbits of Proposition 5.5.2.

An explicit analytic solution to the system of differential equations can be found by expanding $\alpha(t)$ and $\beta(t)$ in Fourier series.

$$\alpha(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{-ikt}$$

$$\beta(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{-ikt}$$

Plugging these expressions into the system of differential equations yields a system of algebraic relations for each k:

$$0 = (\alpha'(0) + ik)\alpha_k - \beta'(0)\beta^*_{-k} \tag{5.4}$$

$$0 = \beta'(0)\alpha^*_{-k} + \left(i(2\frac{n}{m} + k) + \alpha'(0)\right)\beta_k \tag{5.5}$$

We can solve by taking $ik - i2\frac{n}{m} + \alpha'(0)^*$ times the first equation above and substituting into the conjugate of the second equation (replacing k by $-k$). For each k, this yields the expression:

$$\left(|\alpha'(0)|^2 + |\beta'(0)|^2 - k^2 + \frac{2n}{m}\alpha'(0) + \frac{2n}{m}k\right)\alpha_k = 0$$

which implies that either $\alpha_k = 0$ or (after completing the square and setting $\alpha'(0) = iA$, which is necessary for $\gamma \in \Omega SU(2)$):

$$\left(k - \frac{n}{m}\right)^2 = \left(A + \frac{n}{m}\right)^2 + |\beta'(0)|^2 \tag{5.6}$$

The purpose of equation 5.6 is to characterize the set of initial conditions for the differential equations above which yield periodic solutions; in other words, equation 5.6 exactly identifies to fixed point set of the subtorus generated by (n, m) with a disjoint union of translated adjoint orbits of $SU(2)$, as in Proposition 5.5.2. It is evident from equation 5.6 that for any loop fixed under the subgroup (n, m) at most two Fourier modes can be non-zero. These two modes correspond to precisely the values of k that satisfy $k - \frac{n}{m} = \pm C$ for some constant C, for which we require integer solutions of k. We can get two distinct solutions only if $n + Cm = ml$ and $n - Cm = ml'$, which implies that $C = (l - l')/2$ is a half integer and $n/m = (l + l')/2$ is a half integer.

We should contextualize this result in the language of Proposition 5.5.1. For $SU(2)$ only two Levi subgroups are possible: the maximal torus T, or $SU(2)$ itself. The former case arises when $n/m \notin \frac{1}{2}\mathbb{Z}$, and the latter case arises when $n/m \in \frac{1}{2}\mathbb{Z}$. Stated slightly differently, when $n/m \in P^\vee \subseteq \mathfrak{t}$ is in the coweight lattice of $SU(2)$, then $\exp(2\pi i n/m) \in Z(SU(2))$ and the Levi subgroup corresponding to (n, m) is $G = SU(2)$ (and is the maximal torus otherwise).

5.7 Isotropy Representation of $T \times S^1$

Whenever a group G acts on a manifold M and $x \in M$ is a fixed point of the action, one obtains a representation of G on $T_x M$ by taking the derivative of the action map at x. In this section, we compute this representation on the tangent space at any fixed point of the $T \times S^1$ action on ΩG. As we are considering the action of torus on a vector space, we present a splitting of the representation in terms of its weight vectors.

Proposition 5.7.1. *Let γ be fixed by $T \times S^1$ and suppose that $(t, \psi) \in T \times S^1$, then after identifying $T_\gamma \Omega G \simeq \Omega\mathfrak{g}$, the isotropy representation of $T \times S^1$ on $T_\gamma \Omega G$ is given by:*

$$(t, e^{i\psi})_* : \Omega\mathfrak{g} \to \Omega\mathfrak{g}$$

$$X(\theta) \mapsto \mathrm{Ad}_{t\gamma(\psi)} X(\theta + \psi)$$

Proof. By embedding G in $U(n)$ we may assume that G is a matrix group. Pick any variation $\delta\gamma \in T_\gamma \Omega G$ and write $\delta\gamma(\theta) = \gamma(\theta) X(\theta)$ for some $X \in \Omega\mathfrak{g}$. We compute the pushforward:

$$\begin{aligned}
(t, e^{i\psi})_*(\delta\gamma) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} [(t, \psi) \cdot (\gamma(\theta) + \epsilon\gamma(\theta)X(\theta))] \\
&= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left[t(\gamma(\theta+\psi) + \epsilon\gamma(\theta+\psi)X(\theta+\psi))(1 + \epsilon X(\psi))^{-1}\gamma(\psi)^{-1}t^{-1}\right] \\
&= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left[\sum_{j=0}^{\infty}(-1)^j \epsilon^j t(\gamma(\theta+\psi) + \epsilon\gamma(\theta+\psi)X(\theta+\psi))X(\psi)^j\gamma(\psi)^{-1}t^{-1}\right] \\
&= t\gamma(\theta+\psi)\left[X(\theta+\psi) - X(\psi)\right]\gamma(\psi)^{-1}t^{-1}
\end{aligned}$$

But now since γ is fixed under $T \times S^1$, we have $\gamma(\theta) = (t, \psi) \cdot \gamma(\theta) = t\gamma(\theta+\psi)\gamma(\psi)^{-1}t^{-1}$ which implies $\gamma(\theta)t\gamma(\psi) = t\gamma(\theta+\psi)$. Plugging in to the last line of the above yields the desired formula for the isotropy representation, noticing that the constant term is equivalent to zero in the quotient $\Omega\mathfrak{g} \simeq L\mathfrak{g}/\mathfrak{g}$. $\square$

The proposition above allows us to compute a weight basis for the isotropy representation, along with the corresponding weights.

Theorem 5.7.1. *If $\gamma(\theta) = \exp(\eta\theta) \in \Omega G$ is fixed by $T \times S^1$ (i.e. $\eta \in Q^\vee$), the $T \times S^1$ action on $T_\gamma \Omega G$ decomposes into non-trivial irreducible subrepresentations:*

$$T_\gamma \Omega G \simeq \Omega\mathfrak{g} \simeq \bigoplus_{k=1}^{\infty} \left(\bigoplus_{\alpha \in R} V_{\alpha,k} \oplus \bigoplus_{i=1}^{n} V_{i,k} \right)$$

The weight of $T \times S^1$ on $V_{\alpha,k}$ is:

$$\lambda_\alpha^k : Lie(T \times S^1)_{\mathbb{C}} \to \mathbb{C}$$

$$\lambda_\alpha^k(x_1, x_2) = \alpha(x_1 + \eta x_2) + k x_2$$

A basis of weight vectors for $V_{\alpha,k}$ is:

$$X_{\alpha,k}^{(1)} = i\sigma_y^\alpha \cos(k\theta) \pm i\sigma_x^\alpha \sin(k\theta)$$

$$X_{\alpha,k}^{(2)} = i\sigma_x^\alpha \cos(k\theta) \mp i\sigma_y^\alpha \sin(k\theta)$$

where the plus or minus sign is taken depending on whether α is a positive or negative root, respectively. The weight of $T \times S^1$ on $V_{i,k}$ is:

$$\lambda_i^k : Lie(T \times S^1)_{\mathbb{C}} \to \mathbb{C}$$

$$\lambda_i^k(x_1, x_2) = k x_2$$

A basis of weight vectors for $V_{i,k}$ is given by:

$$X_{i,k}^{(1)} = i\sigma_z^\alpha \cos(k\theta)$$

$$X_{i,k}^{(1)} = i\sigma_z^\alpha \sin(k\theta)$$

Proof. We will check that the pair $(X_{\alpha,k}^{(1)}, X_{\alpha,k}^{(2)})$ is a weight basis for $V_{\alpha,k}$ with the appropriate weight; the other cases are similar. Let $t = e^{x_1}$ for $x_1 \in \mathfrak{t}$, let $x_2 \in \mathrm{Lie}(S^1)$, and let $\Theta = \alpha(x_1 + \eta x_2)$. By Proposition 5.7.1,

$$\begin{aligned}
(t, e^{ix_2})_* X_{\alpha,k}^{(1)} &= \mathrm{Ad}_{t\gamma(x_2)} \left(i\sigma_y^\alpha \cos(k\theta) + i\sigma_x^\alpha \sin(k\theta) \right) \\
&= i \left(\sigma_y^\alpha \cos\Theta + \sigma_x^\alpha \sin\Theta \right) \cos(k(\theta + x_2)) + i \left(\sigma_x^\alpha \cos\Theta - \sigma_y^\alpha \sin\Theta \right) \sin(k(\theta + x_2)) \\
&= i \left(\sigma_y^\alpha \cos\Theta + \sigma_x^\alpha \sin\Theta \right) \left(\cos kx_2 \cos k\theta - \sin kx_2 \sin k\theta \right) \\
&\quad + i \left(\sigma_x^\alpha \cos\Theta - \sigma_y^\alpha \sin\Theta \right) \left(\sin kx_2 \cos k\theta + \cos kx_2 \sin k\theta \right) \\
&= i \cos(\Theta + kx_2)\, \sigma_y^\alpha \cos k\theta + i \sin(\Theta + kx_2)\, \sigma_x^\alpha \cos k\theta \\
&\quad - i \sin(\Theta + kx_2)\, \sigma_y^\alpha \sin k\theta + i \cos(\Theta + kx_2)\, \sigma_x^\alpha \sin k\theta \\
&= \cos(\Theta + kx_2) X_{\alpha,k}^{(1)} + \sin(\Theta + kx_2) X_{\alpha,k}^{(2)}
\end{aligned}$$

The computation for $X_{\alpha,k}^{(2)}$ is identical. This completes the proof. $\square$

5.8 An application of the hyperfunction fixed point localization formula to $\Omega SU(2)$

In this section we will present our approach to computing a regularized Duistermaat-Heckman distribution on $\mathrm{Lie}(T \times S^1)^*$ coming from the Hamiltonian action of $T \times S^1$ on ΩG. We will specialize to the case that $G = SU(2)$. This problem (and the work herein) was originally motivated by Atiyah's approach to a similar problem [Ati85]. In that paper, Atiyah showed that the Atiyah-Singer index theorem is a consequence of applying the Duistermaat-Heckman localization formula to the loop space of a Riemannian manifold. In [Ati85], Atiyah does also mention that similar methods can be applied to study ΩG, however, no further details or specific theorems are provided. Our original aim was to provide these details, as well as to study Duistermaat-Heckman distributions which come from Hamiltonian actions of compact tori on infinite dimensional manifolds.

It was discovered after completing this project that some of these issues had already been considered [Pic89]. In this paper, Picken shows that the propagator for a quantum mechanical free particle moving on G (with the invariant Riemannian metric coming from the Killing form) can be exactly expressed by applying the fixed point localization formula for ΩG. In this case, the ill defined left hand side of the localization formula for ΩG is expressed as a path integral on G, while the right hand of the localization formula tells us exactly how to express the result of this path integral in terms of solutions to the classical equations of motion. We should highlight where our approach differs from his:

1. Throughout, Picken uses a variable φ as a coordinate on $\mathfrak{t}$. We will be calling this coordinate x_1 in our work.

2. Picken is implicitly setting $x_2 = 1$ throughout (i.e. he considers the slice $\mathfrak{t} \times \{1\} \subseteq \mathrm{Lie}(T \times S^1)$. This is evident in his choice of action functional, where the kinetic energy term:

$$I_k[g] = \int \langle g^{-1}\dot{g}, g^{-1}\dot{g} \rangle \, d\theta$$

   appears without a mass coefficient.

3. We will directly apply a fixed point localization formula to ΩG with its $T \times S^1$ action, and interpret the result as a *hyperfunction* on $\mathrm{Lie}(T \times S^1)$. The advantage to this approach is that we will be able to Fourier transform this hyperfunction to obtain a closed form of a density function for what one should expect is the pushforward of the "Liouville measure" from ΩG to $\mathrm{Lie}(T \times S^1)^*$ using the moment map. Picken's formula is limited in this regard, since he does not use the localization formula to obtain a distribution on $\mathrm{Lie}(T \times S^1)$ - he only obtains its restriction to a slice through $E = 1$. He also makes no use of hyperfunctions in his paper.

Definition 5.8.1. *Let $\gamma \in \Omega G^{T \times S^1}$. The regularized equivariant Euler class of the normal bundle to γ is defined to be the holomorphic function on $\mathrm{Lie}(T \times S^1)_{\mathbb{C}}$ given by:*

$$e_\gamma^{T \times S^1}(z_1, z_2) = \prod_{k=1}^{\infty} \left(\prod_{\alpha \in \Delta} \frac{\lambda_\alpha^k(z_1, z_2)}{k z_2} \right)$$

The difference between the "usual" and the regularized equivariant Euler class of the normal bundle to γ is that we divide out by kz_2 on each weight. The regularization can be justified in a number of ways. We will see shortly that when we include the regularizing terms, the resulting infinite product will converge to a useful functional expression for $e_\gamma^{T\times S^1}$. Without the regularization, the infinite product does not converge. Picken's work provides another justification for the regularization, since the resulting regularized localization formula provides an exact determination of the quantum mechanical propagator for a free particle moving on G.

For simplicity, let's examine the example $G = SU(2)$. We always use coordinates on Lie$(T \times S^1)$ consisting of the coroot basis for $\mathfrak{t}$, and normalize the E-component of the moment map so that:

$$E\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = 1/2$$

Let $z = (z_1, z_2) \in \text{Lie}(T \times S^1)$ and let $\gamma(\theta) = \exp(i\eta\theta) \in \Omega SU(2)$ be a fixed point of the $T \times S^1$ action. When we work with $G = SU(2)$ a choice of η is really just a choice of integer, so for $\alpha \in \Delta$ the non-zero positive root we set $\alpha(\eta) = 2n$. For every k we get four weights for the isotropy representation, corresponding to the two root vectors in $\mathfrak{sl}_2$ and a the two weights cominig from a non-zero element of the Cartan subalgebra:

$$\begin{aligned}
\lambda_{h,i}^{(k)}(z_1, z_2) &= kz_2 \quad i = 1, 2 \\
\lambda_e^{(k)}(z_1, z_2) &= kz_2 + 2(z_2 n + z_1) \\
\lambda_f^{(k)}(z_1, z_2) &= kz_2 - 2(z_2 n + z_1)
\end{aligned}$$

Proposition 5.8.1. *Let $G = SU(2)$. If $\gamma_n \in \Omega G^{T\times S^1}$, then the regularized equivariant Euler class of the normal bundle to γ_n is given by:*

$$e_{\gamma_n}^T(z_1, z_2) = \frac{\sin\left(2\pi(n + z_1/z_2)\right)}{2\pi(n + z_1/z_2)} \tag{5.7}$$

Proof. Since the fixed points of the $T \times S^1$ action are isolated we have that the normal bundle to the fixed point set is simply $T_{\gamma_n}\Omega G$. We can compute the regularized equivariant Euler class of $T_{\gamma_n}\Omega G$ by taking the product over the weights appearing in the isotropy representation of $T \times S^1$ on $T_{\gamma_n}\Omega G$, according to Theorem 5.7.1:

$$\begin{aligned}
e_{\gamma_n}^T(z_1, z_2) &= \prod_{k=1}^{\infty}\prod_{\alpha\in\Delta} \lambda_\alpha^{(k)}/kz_2 \\
&= \prod_{k=1}^{\infty}\left[1 + \frac{2(z_2 n + z_1)}{z_2 k}\right]\left[1 - \frac{2(z_2\eta + z_1)}{z_2 k}\right] \\
&= \prod_{k=1}^{\infty}\left[1 - \left(\frac{2(z_2 n + z_1)}{z_2 k}\right)^2\right] \\
&= \frac{\sin\left(2\pi(n + z_1/z_2)\right)}{2\pi(n + z_1/z_2)}
\end{aligned}$$

where the last line follows from the infinite product formula for $\sin(z)$. □

Remark: In the more general case of $G = SU(n)$, each choice of positive root will give a difference of squares, which then translates to an extra $\sin(z)/z$ term in the final result. We would then take a product over all the positive roots.

In what follows we will write $e_{\gamma_n}^{T\times S^1}(z_1,z_2) = e_n(z_1,z_2)$ for notational simplicity. A formal application of the fixed point localization formula to ΩG would then yield the following expression, valid for $(x_1,x_2)\in \mathrm{Lie}(T\times S^1)$ such that $e_n(x_1,x_2)\neq 0$:

$$\int_{\Omega G} e^{\omega + i\langle \mu(\gamma), x\rangle} = \sum_{n\in\mathbb{Z}} e^{i(nx_1+\frac{n^2}{2}x_2)} \frac{2\pi(n+x_1/x_2)}{\sin\left(2\pi(n+x_1/x_2)\right)} \tag{5.8}$$

We have not addressed what types of objects that equation 5.8 asserts an equality of. In the setting of a compact symplectic manifold with a Hamiltonian action of a compact torus, one is free to understand this to be an equality of distributions, and even an equality of density functions on some open set. But for the purposes of ΩG, this perspective is insufficient. For instance, the Duistermaat-Heckman "distribution" is supposed to be obtained by taking the Fourier transform of the right hand side of equation 5.8, however, we can see that the expression obtained from the localization formula is not even integrable since it has poles, and even if we ignore the poles coming from the denominator, the numerator grows linearly in the ξ_1 variable. The terms appearing in the localization formula for ΩG also have unpleasant limiting behaviour as $x_2\to 0$. The right hand side of the localization formula should not be interpreted as a distribution (and consequently, neither should the left hand side).

There are further hints in [GLS88] which suggest that the localization formula for ΩG should be an expression positing an equality of two hyperfunctions. Suppose for a moment that we are considering a Hamiltonian action of a torus T on a finite dimensional vector space with weights $\alpha_1,\dots,\alpha_n$. To each weight we can associate a constant coefficient differential operator D_{α_i} on $\mathfrak{t}^*$. The Duistermaat-Heckman distribution is a solution to the differential equation:

$$D_{\alpha_1}\dots D_{\alpha_n}(DH(x)) = \delta(x)$$

When V is infinite dimensional and we have infinitely many weights (such as is the case for the isotropy representation of $T\times S^1$ on the tangent space to a fixed loop in ΩG), then we are forced to consider differential operators of infinite order. Infinite order differential operators do not even act on distributions. For example, any infinite order differential operator on $\mathbb{R}$ cannot act on the Dirac delta distribution because of the classical theorem which states that any distribution supported at the origin must be a finite sum of the Dirac delta distribution and its derivatives. Hyperfunctions (and the related concept of a microfunction) are a sheaf on which infinite order differential operators do have a well defined action. Furthermore, the entire classical theory of distributions is subsumed by the theory of hyperfunctions, so it makes more sense to study the Duistermaat-Heckman distribution as a hyperfunction, rather than as a distribution.

We now begin our construction of the Picken hyperfunction of $\Omega SU(2)$. Fix a polarizing vector of the form $\xi = (\delta,\delta')\in \mathrm{Lie}(T\times S^1)$, with $\delta' > 2\delta > 0$. For the chosen polarization, we must determine the structure of the polarized weights of the isotropy representation at each fixed point. Recall for $p\in \Omega G^{T\times S^1}$, we defined a cone γ_p as the intersection of the positive half spaces coming from the polarized weights. We now let p_n denote the n'th fixed point of the $T\times S^1$ action on $\Omega SU(2)$.

Proposition 5.8.2. *1. If $n>0$, then the weights of the isotropy representation at the n'th fixed point* *satisfy the following inequalities:*

$$\begin{aligned}
\lambda_\alpha^{(k)}(\xi) &> 0, \quad \alpha = +2, k \geq 1 \\
\lambda_\alpha^{(k)}(\xi) &> 0, \quad \alpha = -2, k > 2n \\
\lambda_\alpha^{(k)}(\xi) &< 0, \quad \alpha = -2, k \leq 2n
\end{aligned}$$

2. *If $n < 0$, then the weights of the isotropy representation at the n'th fixed point satisfy the following inequalities:*

$$\begin{aligned}
\lambda_\alpha^{(k)}(\xi) &> 0, \quad \alpha = -2, k \geq 1 \\
\lambda_\alpha^{(k)}(\xi) &> 0, \quad \alpha = +2, k \geq 2n \\
\lambda_\alpha^{(k)}(\xi) &< 0, \quad \alpha = +2, k < 2n
\end{aligned}$$

3. *If $n = 0$, then the weights of the isotropy representation at p_0 satisfy the following inequalities:*

$$\lambda_\alpha^{(k)}(\xi) > 0, \quad \textit{for all } \alpha = \pm 2,\ k \geq 1$$

Consequently,

$$\begin{aligned}
\gamma_{p_0} &= \left\{ (y_1, y_2) \in i\mathit{Lie}(T \times S^1) \,|\, |y_1| < y_2/2 \right\} \\
\gamma_{p_n} &= \left\{ (y_1, y_2) \in i\mathit{Lie}(T \times S^1) \,|\, |y_1| < y_2/2, y_1 > 0 \right\} \quad n \neq 0
\end{aligned}$$

Remark: Since the cones γ_{p_n} are independent of n (so long as $n \neq 0$), after the proof of this proposition will will simply denote $\gamma_{\neq 0} := \gamma_{p_n}$ and $\gamma_0 := \gamma_{p_0}$

Proof. We shall prove the result for 1, as 2 and 3 are similar. For the root $\alpha = +2$, we have that

$$\lambda_\alpha^{(k)}(\xi) = k\delta' + 2(n\delta' + \delta)$$

This is a positive number, being a sum of positive numbers. For the root $\alpha = -2$, we are interested in finding the values $k \geq 1$ such that:

$$k\delta' - 2(n\delta' + \delta) < 0$$

Dividing both sides by the positive number δ' yields

$$k - 2n - \frac{2\delta}{\delta'} < 0$$

By our choice of polarization we have $0 < 2\delta/\delta' < 1$, so the above inequality is true exactly when $1 \leq k \leq 2n$, which proves the first claim.

For a root $\alpha = \pm 2$ and $k \geq 1$, we denote $H_{k,\pm}^{(n)} = \left\{ (y_1, y_2) \in i\mathrm{Lie}(T \times S^1) \,|\, ky_2 \pm 2(ny_2 + y_1) > 0 \right\}$, which is the positive half plane corresponding to the weight $\lambda_\alpha^{(k)}$ at the n'th fixed point.

We now consider the second set of claims about the cones γ_{p_n} and γ_{p_0}. First, consider the case where

$n = 0$. By part (3) of the previous work towards this proposition, we can see that the weights of the isotropy representation at the $n = 0$ fixed point are already polarized. Letting $\eta_k = H_{k,+} \cap H_{k,-}$, we have by definition that $\gamma_{p_0} = \bigcap_{k \geq 1} \eta_k$. Notice that $k \geq k'$ implies that $\eta_k \supseteq \eta_{k'}$, and so $\gamma_{p_0} = \eta_1$. But now the proof is complete, since

$$\gamma_{p_0} = \eta_1 = \{(y_1, y_2) \,|\, y_2 + 2y_1 > 0\} \cap \{(y_1, y_2) \,|\, y_2 - 2y_1 > 0\} = \{(y_1, y_2) \,|\, y_2 > 2|y_1|\}$$

The case where $n \neq 0$ is similar; the only modification required is that the set of polarized weights of the isotropy representation at the n'th fixed point is equal to the set of weights at the $n = 0$ fixed point, with one extra weight of the form $(y_1, y_2) \mapsto 2y_1$. $\square$

Another consequence of the previous proposition is that

$$(-1)^{p_n} = \begin{cases} 1 & n = 0 \\ 1 & n > 0 \\ -1 & n < 0 \end{cases}$$

We now have all the necessary pieces to construct the Picken hyperfunction for the $T \times S^1$ action on $\Omega SU(2)$, which we expect is a hyperfunction replacement for the sum over the fixed points appearing in the Duistermaat-Heckman localization formula.

As before, we let $\tilde{\lambda}_{k,\alpha}$ denote the polarized weights of the isotropy representation at the n'th fixed point; we leave the n implicit to avoid notational clutter. For every n, we apply Lemma 5.2.1 to the set of hyperfunctions $\left\{ f^{(n)}_{\tilde{\lambda}_{k,\alpha}}(x) \right\}_{k=1}^{\infty}$ (c.f. notation of Corollary 5.3.1, making sure to use the regularized weights to guarantee uniform convergence of the infinite product. The resulting hyperfunction is the regularized equivariant Euler class to the normal bundle of the n'th fixed point:

$$\frac{1}{e_n(x_1, x_2)} = b_{\gamma_{p_n}} \left(\frac{2\pi(n + z_1/z_2)}{\sin(2\pi z_1/z_2)} \right)$$

Putting all of these results together we obtain the Picken hyperfunction for the Hamiltonian $T \times S^1$ action on $\Omega SU(2)$:

$$\begin{aligned} L_{\Omega SU(2)}(x_1, x_2) = {} & \frac{1}{(2\pi i)^2} b_{\gamma_{\neq 0}} \left(\sum_{n>0} e^{iz_1 n + iz_2 n^2/2} \frac{2\pi(n + z_1/z_2)}{\sin(2\pi z_1/z_2)} - \sum_{n<0} e^{iz_1 n + iz_2 n^2/2} \frac{2\pi(n + z_1/z_2)}{\sin(2\pi z_1/z_2)} \right) \\ & + \frac{1}{(2\pi i)^2} b_{\gamma_0} \left(\frac{2\pi z_1/z_2}{\sin(2\pi z_1/z_2)} \right) \end{aligned}$$

Ultimately, we would like to be able to take a Fourier transform of the Picken hyperfunction in order to obtain the Duistermaat-Heckman hyperfunction. The following proposition guarantees that the Picken hyperfunction of $\Omega SU(2)$ is in the class of hyperfunctions which have Fourier transforms, and so guarantees that we can find some hyperfunction analogue of the Duistermaat-Heckman distribution in this infinite dimensional example. We will do this term by term.

Proposition 5.8.3. *For every n,*

$$I_n(z_1, z_2) = \frac{n + z_1/z_2}{\sin(2\pi z_1/z_2)}$$

is a slowly increasing holomorphic function on $\mathbb{R}^2 \times i\gamma_{p_n} \subseteq Lie(T \times S^1)_{\mathbb{C}}$.

Proof. That the function in question is holomorphic on $\mathbb{R}^2 \times i\gamma_{p_n}$ follows from its expression as an infinite product of regularized weights, and that the cones γ_{p_n} are constructed to avoid the zero locus of all such weights. It remains to show that $I_n(z_1, z_2)$ is slowly increasing.

For fixed $(y_1, y_2) \in \gamma_n$, the image of the curves $x_1 = mx_2$ ($m \in \mathbb{R}$) under the mapping $(z_1, z_2) \mapsto z_1/z_2$ are the parametric curves given by:

$$\mathbb{R} \to \mathbb{C}$$
$$s \mapsto \frac{ms^2 + y_1 y_2}{s^2 + y_2^2} + i\frac{sy_1 - msy_2}{s^2 + y_2^2}$$

These are easily seen to be ellipsoidal arcs which cross the real axis at $\mathrm{Re}(z_1/z_2) = y_1/y_2$ when $s = 0$, and asymptotically approach the real axis from above (below) at $\mathrm{Re}(z_1/z_2) = m$ as $s \to \infty$ when $m > 0$ (and from below the axis if $m < 0$).

We assume $n \neq 0$, since the $n = 0$ case is similar. Fix a compact set $K \subseteq \gamma_{p_n}$ and any $\epsilon > 0$. Since $(y_1, y_2) \mapsto y_1/y_2$ is continuous on K it will achieve its maximum and minimum, so there is a $\delta > 0$ such that the estimate $\delta \leq y_1/y_2 \leq 1/2 - \delta$ holds uniformly over K. Since the numerator of $I_n(z_1, z_2)$ is slowly increasing (it is a polynomial), it suffices to prove that:

$$\left| \frac{e^{-\epsilon |\mathrm{Re}(z)|}}{\sin(2\pi z_1/z_2)} \right| \to 0$$

uniformly in K as $\mathrm{Re}(z) \to \infty$.

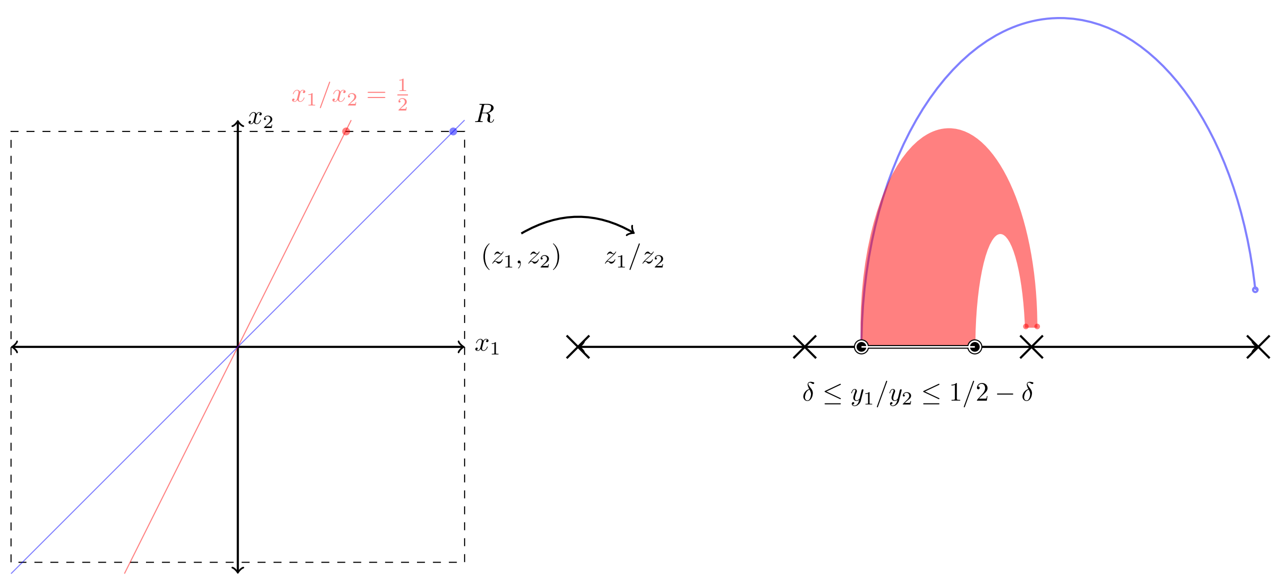

Figure 5.3: Proof that $I_n(z_1, z_2)$ is slowly increasing. The left side of the figure demonstrates the (x_1, x_2) plane; the right hand side is demonstrating the image of the map $(z_1, z_2) \mapsto z_1/z_2$ when we fix various values of (y_1, y_2). The red filled region is showing the image of the line $x_1 = x_2/2$ as (y_1, y_2) varies over K, with $\max\{|x_1|, |x_2|\} \leq R$. The blue curve is showing the image of the line $x_1 = x_2$ (fixing (y_1, y_2) such that $y_1/y_2 = \delta$). The poles of $\csc(2\pi z)$ are demonstrated with $\times$.

First, we notice that if we fix y_1/y_2 as above, then for every R sufficiently large we have:

$$\max\{|x_1|, |x_2|\} = R \Rightarrow |\csc(2\pi z_1/z_2)| \leq \left| \csc\left(2\pi \frac{R^2/2 + y_1 y_2}{R^2 + y_2^2} + i\frac{Ry_1 - Ry_2/2}{R^2 + y_2^2}\right) \right|$$

This estimate follows from the observation that the maximum of $\csc(2\pi z_1/z_2)$ on the box occurs at the point (x_1, x_2) such that the distance from z_1/z_2 to a pole of $\csc(2\pi z)$ is minimized; this condition is satisfied on the line $x_1 = x_2/2$. A uniform bound over K can be found because of our previous estimate on y_1/y_2. Figure 5.3 demonstrates these estimates. The proof is completed by noticing that $\csc(2\pi z_1/z_2)$ has linear growth (which is dominated by any exponential) as $x_2 \to \infty$ because all of its poles are simple. $\square$

By Proposition 5.8.3, $L_{\Omega SU(2)}(x_1, x_2)$ is a slowly increasing hyperfunction, so we may take its Fourier transform. Let S_n be a contour in $\mathrm{Lie}(T \times S^1)_{\mathbb{C}}$ chosen so that $(y_1, y_2) \in \gamma_{p_n}$. After choosing a holomorphic partition of unity $\chi_\sigma(z)$, we may write the following expression for the Duistermaat-Heckman hyperfunction:

$$DH(\xi_1, \xi_2) = \frac{1}{(2\pi i)^2} \sum_{\sigma \in \Sigma} \sum_{n \in \mathbb{Z}} b_{-\sigma^\circ} \left(\int_{S_n} e^{-i(\zeta_1 - n)z_1 - i(\zeta_2 - n^2/2)z_2} \frac{2\pi(n + z_1/z_2)}{\sin(2\pi z_1/z_2)} \chi_\sigma(z_1, z_2)\, dz_1\, dz_2 \right)$$

One might try and proceed with the computation of this integral, as in the example of section 5.3; however, if one uses the standard holomorphic partition of unity then the computation of the contour integrals by a method of iterated residues becomes very complicated. The difficulty essentially arises from the fact that the integrand of the resulting multivariable contour integral has a polar locus consisting of triples of lines that intersect. If one uses the following partition of unity:

$$1 = \frac{1}{1+e^{z_1}} \frac{1}{1+e^{\pi z_2}} + \frac{1}{1+e^{-z_1}} \frac{1}{1+e^{\pi z_2}} + \frac{1}{1+e^{z_1}} \frac{1}{1+e^{-\pi z_2}} + \frac{1}{1+e^{-z_1}} \frac{1}{1+e^{-\pi z_2}}$$

then polar locus of the integrand defining the Fourier transform consists of isolated singularities which are locally cut out by a pair of equations. The residues near such singularities are readily computed, but do not appear to re-sum in any obvious way. We leave a further examination of the form of the Duistermaat-Heckman hyperfunction of $\Omega SU(2)$ as an open problem.

Bibliography

[ABV12] J. Adams, D. Barbasch, and D. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, vol. 104, Springer Science & Business Media, 2012.

[AP83] M.F. Atiyah and A.N. Pressley, *Convexity and loop groups*, Arithmetic and geometry, Springer, 1983, pp. 33–63.

[Art89] J. Arthur, *Unipotent automorphic representations: conjectures*, Astérisque (1989), no. 171-172, 13–71, Orbites unipotentes et représentations, II. MR 1021499

[Art06] ———, *A note on L-packets*, Pure and Applied Mathematics Quarterly **2** (2006), no. 1.

[Art13] ———, *The endoscopic classification of representations orthogonal and symplectic groups*, vol. 61, American Mathematical Soc., 2013.

[Ati85] M.F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque **131** (1985), 43–59.

[BD91] A. Beilinson and V. Drinfeld, *Quantization of Hitchins integrable system and Hecke eigensheaves*, 1991.

[BL06] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Springer, 2006.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, vol. 2, 1979, pp. 27–61.

[Cas80] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Mathematica **40** (1980), no. 3, 387–406.

[CFM+17a] C. Cunningham, A. Fiori, J. Mracek, A. Moussaoui, and B. Xu, *Arthur packets and Adams-Barbasch-Vogan packets for p-adic groups, 1: Background and conjectures*, arXiv:1705.01885, 2017.

[CFM+17b] ———, *Arthur packets and Adams-Barbasch-Vogan packets for p-adic groups, 2: Examples*, In Prep., 2017.

[CM93] D. Collingwood and W. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, CRC Press, 1993.

[Del74] P. Deligne, *Théorie de Hodge: III*, Publications mathématiques de l'IHÉS **44** (1974), 5–77.

[DH82] J.J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Inventiones mathematicae **69** (1982), no. 2, 259–268.

[GH14] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 2014.

[Gin86] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Inventiones mathematicae **84** (1986), no. 2, 327–402.

[GLS88] V. Guillemin, E. Lerman, and S. Sternberg, *On the Kostant multiplicity formula*, Journal of Geometry and Physics **5** (1988), no. 4, 721–750.

[GR14] D. Gaitsgory and N. Rozenblyum, *Crystals and d-modules*, arXiv preprint arXiv:1111.2087 (2014).

[Har98] M. Harris, *The local Langlands conjecture for $GL(n)$ over a p-adic field, $n < p$*, Inventiones mathematicae **134** (1998), no. 1, 177–210.

[Hen86] G. Henniart, *On the local Langlands conjecture for $GL(n)$: the cyclic case*, Annals of Mathematics **123** (1986), no. 1, 145–203.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, vol. 151, Princeton university press, 2001.

[HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361

[JK95a] L. Jeffrey and F. Kirwan, *Intersection pairings in moduli spaces of holomorphic bundles on a Riemann surface*, Electronic Research Announcements of the American Mathematical Society **1** (1995), no. 2, 57–71.

[JK95b] ———, *Localization for nonabelian group actions*, Topology **34** (1995), no. 2, 291–327.

[Kan89] A. Kaneko, *Introduction to the theory of hyperfunctions*, vol. 3, Springer Science & Business Media, 1989.

[Kas83] M. Kashiwara, *Systems of microdifferential equations*, Birkhäuser, 1983.

[Kas95] ———, *Algebraic study of systems of partial differential equations*, Mémoires de la Société Mathématique de France **63** (1995), 1–14.

[Kaw70] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients.*

[KKS] M. Kashiwara, T. Kawai, and M. Sato, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations: proceedings of a conference at Katata (H. Komatsu, ed.), Springer.

[KL87] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Inventiones mathematicae **87** (1987), no. 1, 153–215.

[Kot84] R. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J **51** (1984), no. 3, 611–650.

[KS99a] G. Kato and D.C. Struppa, *Fundamentals of algebraic microlocal analysis*, vol. 217, CRC Press, 1999.

[KS99b] R.E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999).

[KS13] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, vol. 292, Springer Science & Business Media, 2013.

[KW08] B. Khesin and R. Wendt, *The geometry of infinite-dimensional groups*, vol. 51, Springer Science & Business Media, 2008.

[Lan79] R. Langlands, *Stable conjugacy: definitions and lemmas*, Canad. J. Math **31** (1979), no. 4, 700–725.

[LL79] J.P. Labesse and R. Langlands, *L-indistinguishability for* $SL(2)$, Canad. J. Math **31** (1979), no. 4, 726–785.

[LS85] G. Lusztig and N. Spaltenstein, *On the generalized Springer correspondence for classical groups*, Advanced Studies in Pure Math **6** (1985), 289–316.

[LS87] R. Langlands and D. Shelstad, *On the definition of transfer factors*, Mathematische Annalen **278** (1987), no. 1-4, 219–271.

[Lus84] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Inventiones mathematicae **75** (1984), no. 2, 205–272.

[Lus85] ———, *Character sheaves I*, Advances in Mathematics **56** (1985), no. 3, 193–237.

[Lus88] ———, *Cuspidal local systems and graded Hecke algebras, I*, Publications Mathématiques de l'IHÉS **67** (1988), 145–202.

[Lus89] ———, *Affine Hecke algebras and their graded version*, Journal of the American Mathematical Society **2** (1989), no. 3, 599–635.

[Lus95a] ———, *Cuspidal local systems and graded Hecke algebras, II*, Representations of groups, 1995, pp. 217–275.

[Lus95b] ———, *Study of perverse sheaves arising from graded lie algebras*, Advances in Mathematics **112** (1995), no. 2, 147–217.

[MV88] I. Mirković and K. Vilonen, *Characteristic varieties of character sheaves*, Inventiones mathematicae **93** (1988), no. 2, 405–418.

[Ngô10] B.C. Ngô, *Le lemme fondamental pour les algebres de Lie*, Publications mathématiques de l'IHÉS **111** (2010), no. 1, 1–169.

[NZ09] D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, Journal of the American Mathematical Society **22** (2009), no. 1, 233–286.

[Pic89] R.F. Picken, *The propagator for quantum mechanics on a group manifold from an infinite-dimensional analogue of the Duistermaat-Heckman integration formula*, Journal of Physics A: Mathematical and General **22** (1989), no. 13, 2285.

[PS86] A.N. Pressley and G.B. Segal, *Loop groups*, Clarendon Press, 1986.

[Rai16] C. Raicu, *Characters of equivariant D-modules on spaces of matrices*, Compositio Mathematica **152** (2016), no. 9, 1935–1965.

[Sai17] T. Saito, *Characteristic cycle of the external product of constructible sheaves*, Manuscripta mathematica (2017), 1–12.

[Sat59] M. Sato, *Theory of hyperfunctions, I.*, Journal of the Faculty of Science **8** (1959).

[Sch13] P. Scholze, *The local Langlands correspondence for $GL(n)$ over p-adic fields*, Inventiones mathematicae **192** (2013), no. 3, 663–715.

[Ser13] J.P. Serre, *Galois cohomology*, Springer Science & Business Media, 2013.

[She79] D. Shelstad, *Notes on L-indistinguishability*, Automorphic Forms, Representations and L-functions (1979), 193–204.

[Sho88] T. Shoji, *Geometry of orbits and Springer correspondence*, Astérisque (1988), no. 168, 61–140.

[Spa85] N. Spaltenstein, *On the generalized Springer correspondence for exceptional groups*, Advanced studies in pure math **6** (1985), 317–338.

[STZ16] V. Shende, D. Treumann, and E. Zaslow, *Legendrian knots and constructible sheaves*, Inventiones mathematicae (2016), 1–103.

[SZ14] A.J. Silberger and E.W. Zink, *Langlands classification for L-parameters*, arXiv preprint arXiv:1407.6494 (2014).

[Tam08] D. Tamarkin, *Microlocal condition for non-displaceablility*, arXiv preprint arXiv:0809.1584 (2008).

[Vog93] D. Vogan, *The local Langlands conjecture*, Contemporary Mathematics **145** (1993), 305–305.

[Wal97] J.L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Mathematica **105** (1997), no. 2, 153–236.

[Wit92] E. Witten, *Two dimensional gauge theories revisited*, Journal of Geometry and Physics **9** (1992), no. 4, 303–368.