ARTHUR PACKETS AND ADAMS-BARBASCH-VOGAN PACKETS FOR p-ADIC GROUPS, 1: BACKGROUND AND CONJECTURES

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Abstract. This paper begins the project of adapting the 1992 book by Adams, Barbasch and Vogan on the Langlands classification of admissible representations of real groups, to p-adic groups, continuing in the direction charted by Vogan in his 1993 paper on the Langlands correspondence. This paper presents three theorems in that direction. The first theorem shows how Lusztig's work on perverse sheaves arising from graded Lie algebras may be brought to bear on the problem; the second theorem proves that Arthur parameters determine strongly regular conormal vectors to a parameter space of certain Langlands parameters; the third theorem shows how to replace the microlocalisation functor as it appears in the work of Adams, Barbasch and Vogan with a functor built from Deligne's vanishing cycles functor. The paper concludes with three conjectures, the first of which is the prediction that Arthur packets are Adams-Barbasch-Vogan packets for p-adic groups. This paper is the first in a series.

Contents

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Introduction

Let F be a local field of characteristic zero and G be a connected reductive linear algebraic group over F. According to the local Langlands conjecture, the set $\Pi(G(F))$ of isomorphism classes of irreducible admissible representations of $G(F)$ can be naturally partitioned into finite subsets, called L-packets. Moreover, the local Langlands conjecture predicts that if an L-packet contains one tempered representation, then all the representations in that L-packet are tempered, so tempered L-packets provide a partition of tempered irreducible admissible representations. Tempered L-packets enjoy some other very nice properties. For instance, every tempered L-packet determines a stable distribution on $G(F)$ by a non-trivial linear combination of the distribution characters of the representations in the packet. Tempered L-packets also have an endoscopy theory, which leads to a parametrisation of the distribution characters of the representations in the packet. These properties fail for non-tempered L-packets. To remedy this, in 1989 Arthur introduced what are now know as Arthur packets, which enlarge the non-tempered L-packets in such a way that these last two properties do extend to the non-tempered case. Arthur's motivation was global, arising from the classification of automorphic representations, so the local meaning of Arthur packets was unclear when they first appeared.

Shortly after Arthur packets were introduced, Adams, Barbasch and Vogan suggested a purely local description of Arthur packets for real groups, in 1992, using microlocal analysis of certain stratified complex varieties built from Langlands parameters. In 1993, Vogan used similar tools to make a prediction for a local description of Arthur packets for p-adic groups. The packets of admissible representations they described may be referred to as ABV packets, in both the real and p-adic cases. Since these constructions are purely local, and since the initial description of Arthur packets was global in nature, it was not easy to compare ABV packets with Arthur packets. The conjecture that Arthur packets are ABV packets has remained open since the latter were introduced.

When Arthur finished his monumental work on the classification of automorphic representations of symplectic and special orthogonal groups in 2013, the situation changed dramatically. Not only did he prove his own conjectures on Arthur packets given in [5], but he also gave a local characterization of them, using twisted endoscopy. This opened the door to comparing Arthur packets with ABV packets and motivated us to compare Arthur's work with Vogan's constructions in the p-adic case. This paper is the first in a series making that comparison.

We now describe the main results in this paper. From now on, we assume F is p-adic.

To begin, let us review Arthur's main local result in the endoscopic classification of representations. Suppose now that the connected reductive algebraic group G over F is quasi-split. An Arthur parameter for G is a homomorphism, $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$, where L_F is the local Langlands group, to the Langlands group ${}^LG = \widehat{G} \rtimes W_F$, satisfying a number of conditions. One important condition is that the image of $\psi(W_F)$ under the projection onto $\widehat{G}$ must have compact closure. When G is symplectic or special orthogonal, Arthur [2, Theorem 1.5.1] assigns to any ψ a multiset $\Pi_\psi(G(F))$ over $\Pi(G(F))$, known as the Arthur packet of G associated with ψ. It is a deep result of Moeglin [28] that $\Pi_\psi(G(F))$ is actually a subset of $\Pi(G(F))$. Endoscopy theory [2, Theorem 2.2.1] in this case gives rise to a canonical map

$$
\begin{aligned}
\Pi_\psi(G(F)) &\to \widehat{\mathcal{S}_\psi} \\
\pi &\mapsto \langle\,\cdot\,,\pi\rangle_\psi
\end{aligned}
\tag{1}
$$

to $\widehat{\mathcal{S}_\psi}$, the set of irreducible characters of $\mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0 Z(\widehat{G})^{\Gamma_F}$. If the Arthur parameter $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ is trivial on $\mathrm{SL}(2,\mathbb{C})$ then $\Pi_\psi(G(F))$ is a tempered L-packet and the map (1) is a bijection. In general, $\Pi_\psi(G(F))$ contains the L-packet $\Pi_{\phi_\psi}(G(F))$, where ϕ_ψ is the Langlands parameter given by $\phi_\psi(u) := \psi(u, d_u)$, where for $u \in L_F$ we set $d_u = \mathrm{diag}(|u|^{1/2}, |u|^{-1/2})$ with $|\ |$ the pullback of the norm map on W_F.

The map (1) determines a stable distribution on $G(F)$ by

$$\Theta^G_\psi = \sum_{\pi\in\Pi_\psi(G(F))} \langle z_\psi, \pi\rangle_\psi \; \Theta_\pi. \tag{2}$$

where z_ψ is the image of $\psi(1,-1)$ in $\mathcal{S}_\psi$ with $(1,-1) \in L_F \times \mathrm{SL}(2,\mathbb{C})$ where -1 is the non-trivial central element of $\mathrm{SL}(2,\mathbb{C})$.

In this paper we express Arthur's conjectural generalisation of (1) for inner twists of G using pure rational forms of G as articulated by Vogan. A pure rational form (also known as a *pure inner form*) of G is a cocycle $\delta \in Z^1(F, G)$. An *inner rational form* is a cocycle $\sigma \in Z^1(F, \mathrm{Inn}(G))$. Using the maps

$$Z^1(F, G) \to Z^1(F, G_{\mathrm{ad}}) = Z^1(F, \mathrm{Inn}(G)) \to Z^1(F, \mathrm{Aut}(G)),$$

every pure rational form of G determines an inner rational form of G and every inner rational form of G determines a rational form of G. Following [32], a representation of a pure rational form of G is defined to be a pair (π, δ), where δ is a pure rational form of G and π is an equivalence class of admissible representations of $G_\delta(F)$. Then $G(\bar{F})$-conjugation defines an equivalence relation on such pairs, which is compatible with the equivalence relation on pure rational forms $Z^1(F, G)$ producing $H^1(F, G)$. Again following [32], we write $\Pi_{\mathrm{pure}}(G/F)$ for the equivalence classes of such pairs. Then, after choosing a representative for each class in $H^1(F, G)$, we may write

$$\Pi_{\mathrm{pure}}(G/F) = \bigsqcup_{[\delta]\in H^1(F,G)} \Pi(G_\delta(F), \delta),$$

where $\Pi(G_\delta(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi(G_\delta(F))\}$.

An *inner twist* of G is a pair (G, φ), where G is a rational form of G and φ is an isomorphism between G and G such that $\gamma \mapsto \varphi \circ \gamma(\varphi)^{-1}$ is a 1-cocycle in $Z^1(\Gamma_F, \mathrm{Inn}(G))$. Every inner rational form σ of G determines an *inner twist* $(G_\sigma, \varphi_\sigma)$ such that the action of $\gamma \in \Gamma_F$ on $G_\sigma(\bar{F})$ is given through the σ-twisted action on $G(\bar{F})$. We use the notation $(G_\delta, \varphi_\delta)$ for the inner twist of G determined by the pure rational form δ. An Arthur parameter ψ for G is *relevant* to G_δ if any Levi subgroup of ${}^LG$ that ψ factors through is the dual group of a Levi subgroup of G_δ. In [2, Conjecture 9.4.2], Arthur assigns to any relevant ψ a multiset $\Pi_\psi(G_\delta(F))$ over $\Pi(G_\delta(F))$, which is called the Arthur packet for G_δ associated to ψ. Moeglin's work shows that, since G_δ comes from a pure rational form, $\Pi_\psi(G_\delta(F))$ is again a subset of $\Pi(G_\delta(F))$. To extend (1) to this case, Arthur replaces the group $\mathcal{S}_\psi$ with a generally non-abelian group $\mathcal{S}_{\psi,\mathrm{sc}}$ [2, Section 9.2], which is a central extension of $\mathcal{S}_\psi$ by $\widehat{Z}_{\psi,\mathrm{sc}}$; compare with (25). Let $\widetilde{\zeta}_{G_\delta}$ be a character of $\widehat{Z}_{\psi,\mathrm{sc}}$ and let $\mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \widetilde{\zeta}_{G_\delta})$ be the set of isomorphism classes of $\widetilde{\zeta}_{G_\delta}$-equivariant representations of $\mathcal{S}_{\psi,\mathrm{sc}}$ and $\langle\,\cdot\,,\pi\rangle_{\psi,\mathrm{sc}}$ is the character of the associated representation of $\mathcal{S}_{\psi,\mathrm{sc}}$.

Endoscopy theory [2, Conjecture 9.4.2] gives a map

$$\Pi_\psi(G_\delta(F)) \to \mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \widetilde{\zeta}_{G_\delta}); \tag{3}$$

the character of the representation attached to an irreducible representation π of the inner twist $(G_\delta, \varphi_\delta)$ is denoted by $\langle\,\cdot\,,\pi\rangle_{\psi,\mathrm{sc}}$. The map (3) depends only on (1) and the pure rational form δ. For any Arthur parameter ψ for G and any pure rational form δ of G we define

$$\Pi_\psi(G_\delta(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi_\psi(G_\delta(F))\}$$

where, if ψ is not relevant to G_δ then $\Pi_\psi(G^*_\delta(F))$ and thus $\Pi_\psi(G_\delta(F),\delta)$ is empty. Now we introduce

$$\Pi_\psi(G/F) := \{[\pi,\delta] \in \Pi_{\mathrm{pure}}(G/F) \mid (\pi,\delta) \in \Pi_\psi(G_\delta(F),\delta)\}. \tag{4}$$

After choosing a representative pure rational form δ for every class in $H^1(F,G)$, we have

$$\Pi_{\mathrm{pure},\psi}(G/F) = \bigsqcup_{[\delta]\in H^1(F,G)} \Pi_\psi(G_\delta(F),\delta).$$

Now, set

$$A_\psi := \pi_0(Z_{\widehat{G}}(\psi)) = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0$$

and let $\chi_\delta : \pi_0(Z(\widehat{G})^{\Gamma_F}) \to \mathbb{C}^\times$ be the character matching $[\delta] \in H^1(F,G)$ under the Kottwitz isomorphism $H^1(F,G) \cong \mathrm{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1)$. Let $\mathrm{Rep}(A_\psi,\chi_\delta)$ denote the set of equivalence classes of representations of A_ψ such that the pullback of the representations along

$$\pi_0(Z(\widehat{G})^{\Gamma_F}) \to \pi_0(Z_{\widehat{G}}(\psi))$$

is χ_δ. In Proposition 1.10.3 we show that (3) defines a canonical map

$$\Pi_{\mathrm{pure},\psi}(G/F) \to \mathrm{Rep}(A_\psi) \tag{5}$$

and we write $\langle\,\cdot\,,[\pi,\delta]\rangle_\psi$ for the representation attached to $[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)$. built from canonical maps

$$\Pi_\psi(G_\delta(F),\delta) \to \mathrm{Rep}(A_\psi,\chi_\delta). \tag{6}$$

The maps (6) depend only on δ and (1), as discussed in Section 1.10. When $\delta = 1$, (6) recovers (1) and if $\psi = \phi$ is tempered (6) gives a canonical bijection

$$\Pi_\phi(G_\delta(F),\delta) \to \Pi(A_\phi,\chi_\delta), \tag{7}$$

where $\Pi(A_\phi,\chi_\delta)$ denotes the set of χ_δ-equivariant characters of A_ψ.

In this paper we give a geometric and categorical approach to calculating a generalisation of (5), and therefore of (6) also, which applies to all quasi-split connected reductive algebraic groups G over p-adic fields, by assuming the local Langlands correspondence for its pure rational forms, as articulated by Vogan in [32]. Our approach is based on ideas developed for real groups in [1] and on results from [32] for p-adic groups. After specializing to the case of quasi-split symplectic and special orthogonal p-adic groups, we conjecture that this geometric approach produces a map that coincides with (6) from Arthur. The generalisation of (6) that we propose leads quickly to what should be a generalisation of Arthur packets. To acknowledge the debt we owe to [1] and [32], we refer to the packets appearing in this paper as ABV packets for p-adic groups. Much of this paper is concerned with assembling the tools needed to give a precise and workable definition of ABV packets for p-adic groups and a precise and testable conjecture that they generalise Arthur packets.

We now sketch our generalisation of (6). Let F be a p-adic field and let G be any quasi-split connected reductive algebraic group over F. Every Langlands parameter ϕ for G determines an "infinitesimal parameter" $\lambda_\phi : W_F \to {}^LG$ by $\lambda_\phi(w) := \phi(w,d_w)$ where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$. The map $\phi \mapsto \lambda_\phi$ is not injective, but the preimage of any infinitesimal parameter falls into finitely many equivalence classes of Langlands parameters under $\widehat{G}$-conjugation. Set $\lambda_\psi := \lambda_{\phi_\psi}$. Let $\Pi_{\mathrm{pure},\lambda_\psi}(G/F)$ be the set of $[\pi,\delta] \in \Pi_{\mathrm{pure}}(G/F)$ such that the Langlands parameter ϕ, whose associated L-packet contains π, satisfies $\lambda_\phi = \lambda_\psi$. The generalisation of (6) that we define, following [1] and [32], takes the form of a map

$$\Pi_{\mathrm{pure},\lambda_\psi}(G/F) \to \mathrm{Rep}(A_\psi). \tag{8}$$

The genesis of the map (8) is the interesting part, as it represents a sort of geometrisation and categorification of (6).

To order to define (8), in Section 2 we review the definition of a variety V_λ, following [32], that parametrises the set $P_\lambda({}^LG)$ of Langlands parameters ϕ for G such that $\lambda_\phi = \lambda$. The variety V_λ is equipped with an action of $Z_{\widehat{G}}(\lambda)$. Then, again following [32], we consider the category $\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)$ of equivariant perverse sheaves on V_λ. Together with (7), the version of the Langlands correspondence that applies to G and its pure rational forms determines a *bijection* between $\Pi_{\mathrm{pure},\lambda}(G/F)$ and isomorphism classes of simple objects in $\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)$:

$$\begin{aligned}\Pi_{\mathrm{pure},\lambda}(G/F) &\to \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}},\\ [\pi,\delta] &\mapsto \mathcal{P}(\pi,\delta).\end{aligned} \tag{9}$$

Inspired by an analogous result in [1] for real groups, in Proposition 4.6.1 we show that every Arthur parameter ψ determines a particular element in the conormal bundle to V_λ

$$(x_\psi,\xi_\psi) \in T^*_{C_\psi}(V_{\lambda_\psi}),$$

where $C_\psi \subseteq V_{\lambda_\psi}$ is the $Z_{\widehat{G}}(\lambda_\psi)$-orbit of $x_\psi \in V_\lambda$, such that the $Z_{\widehat{G}}(\lambda_\psi)$-orbit of (x_ψ,ξ_ψ) is the unique open orbit $T^*_{C_\psi}(V_{\lambda_\psi})_{\mathrm{sreg}}$ in $T^*_{C_\psi}(V_{\lambda_\psi})$. Then we use (x_ψ,ξ_ψ) to show that A_ψ is the equivariant fundamental group of $T^*_{C_\psi}(V_\lambda)_{\mathrm{reg}}$. Thus, (x_ψ,ξ_ψ) determines an equivalence of categories

$$\mathsf{Loc}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi,\bar\eta}(V_\lambda)_{\mathrm{sreg}}) \to \mathsf{Rep}(A_\psi),$$

where $\mathsf{Rep}(A_\psi)$ denotes the category of representations of A_ψ. This means that the spectral transfer factors $\langle\,\cdot\,,\pi\rangle_{\psi,\mathrm{sc}}$ for ψ appearing in (3) can be interpreted as equivariant local systems on $T^*_{C_\psi}(V_{\lambda_\psi})_{\mathrm{sreg}}$

In Section 5.1 we use the vanishing cycles functor to define an exact functor

$$\mathsf{Ev}_{C_\psi,\bar\eta} : \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda) \to \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi}(V_\lambda)_{\mathrm{reg}}) \tag{10}$$

which plays the role of the microlocalisation functor as it appears in [1] for real groups. Vanishing cycles of perverse sheaves on V_λ are fundamental tools for understanding the singularities on the boundaries of strata in V_λ and their appearance here is quite natural. The restriction of $\mathcal{H}^{-\dim V_\lambda}\,\mathsf{Ev}_{C_\psi,\bar\eta}\,\mathcal{P}$ to $T^*_{C_\psi}(V_{\lambda_\psi})_{\mathrm{sreg}}$ is an equivariant local system on $T^*_{C_\psi}(V_{\lambda_\psi})_{\mathrm{sreg}}$ and thus a representation of A_ψ. Using deep facts about vanishing cycles, we show that if $\mathcal{P}$ is an equivariant perverse sheaf on V_λ, then $\mathsf{Ev}_{C_\psi,\bar\eta}\,\mathcal{P}$ is cohomologically concentrated in degree V_λ, allowing us to introduce the exact functor

$$\mathsf{Ev}^0_{C_\psi,\bar\eta} := \mathsf{Ev}_{C_\psi,\bar\eta}[-\dim V_\lambda] : \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda) \to \mathsf{Loc}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi,\bar\eta}(V_\lambda)_{\mathrm{reg}}). \tag{11}$$

When combined with restriction

$$\mathsf{Loc}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi,\bar\eta}(V_\lambda)_{\mathrm{reg}}) \to \mathsf{Loc}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi,\bar\eta}(V_\lambda)_{\mathrm{sreg}})$$

and the equivalence

$$\mathsf{Loc}_{Z_{\widehat{G}}(\lambda)}(T^*_{C_\psi,\bar\eta}(V_\lambda)_{\mathrm{sreg}}) \to \mathsf{Rep}(A_\psi)$$

from above, this defines an exact functor

$$\mathsf{Ev}_\psi : \mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda) \to \mathsf{Rep}(A_\psi). \tag{12}$$

Passing to isomorphism classes of objects, this functor defines a map

$$\mathsf{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)^{\text{simple}}_{/\text{iso}} \to \mathsf{Rep}(A_\psi)_{/\text{iso}}.$$

When composed with (9), this defines (8).

We now explain the conjectured relation between (5) and (8). With reference to (12), consider the support of (8), called the *ABV packet* for ψ:

$$\Pi^{\text{ABV}}_{\text{pure},\psi}(G/F) := \{[\pi,\delta] \in \Pi_{\text{pure},\lambda}(G/F) \mid \mathsf{Ev}_\psi\, \mathcal{P}(\pi,\delta) \neq 0\}. \tag{13}$$

We can break the ABV packet $\Pi^{\text{ABV}}_{\text{pure},\psi}(G/F)$ apart according to pure rational forms of G:

$$\Pi^{\text{ABV}}_{\text{pure},\psi}(G/F) = \bigsqcup_{[\delta]\in H^1(F,G)} \Pi^{\text{ABV}}_{\psi}(G_\delta(F),\delta),$$

where

$$\Pi^{\text{ABV}}_{\psi}(G_\delta(F),\delta) := \{(\pi,\delta) \in \Pi(G_\delta(F),\delta) \mid [\pi,\delta] \in \Pi^{\text{ABV}}_{\psi}(G/F)\},$$

so

$$\Pi^{\text{ABV}}_{\psi}(G_\delta(F),\delta) = \{(\pi,\delta) \in \Pi(G_\delta(F),\delta) \mid \mathsf{Ev}_\psi\, \mathcal{P}(\pi,\delta) \neq 0\}.$$

We may now state the main conjecture of this paper, given in a slightly stronger form as Conjectures 1 and 2 in Section 6.

Conjecture. Let ψ be an Arthur parameter for a quasi-split symplectic or special orthogonal p-adic group G. Then

$$\Pi_{\text{pure},\psi}(G/F) = \Pi^{\text{ABV}}_{\text{pure},\psi}(G/F).$$

Moreover, for all pure rational forms δ of G and for all $[\pi,\delta] \in \Pi_{\text{pure},\lambda_\psi}(G/F)$,

$$\langle\,\cdot\,,[\pi,\delta]\rangle_\psi = \operatorname{trace} \mathsf{Ev}_\psi\, \mathcal{P}(\pi,\delta).$$

The pithy version of this conjecture is *Arthur packets are ABV packets for p-adic groups*, but that statement obscures the fact that Arthur packets are defined separately for each inner rational form (more precisely the corresponding inner twist), while ABV packets treat all pure rational forms in one go. More seriously, this pithy version of the conjecture obscures the fact that the conjecture proposes a completely geometric approach to calculating the characters $\langle\,\cdot\,,\pi\rangle_{\psi,\text{sc}}$ appearing in Arthur's endoscopic classification of representations.

To simplify the discussion, in this introduction we have only described ABV packets for Arthur parameters; however, as we see in this paper, it is possible to attach an ABV packet to each Langlands parameter. Consequently, there are more ABV packets than Arthur packets. So, while the conjecture above asserts that every Arthur packet in an ABV packet, it is certainly not true that every ABV packet is an Arthur packet. If the conjecture is true, it gives credence to the idea that ABV packets may be thought of as generalised Arthur packets.

The main features of this paper are:

(1) in Section 1, a quick review of the main local result from [2] as it specialises to pure rational forms of quasi-split connected reductive groups over p-adic fields;

(2) in Section 2, a brief description of Vogan's parameter variety for p-adic groups and a review of Vogan's perspective on the local Langlands conjecture for pure rational forms of quasi-split connected reductive groups over p-adic fields, using based on [32];
(3) Theorem 3.1.1, showing that the Vogan variety for an arbitrary infinitesimal parameter coincides with the Vogan variety for an unramified infinitesimal parameter and also showing that the category of equivariant perverse sheaves is related to the category of equivariant perverse sheaves on a graded Lie algebra, thereby putting tools from [25] at our disposal;
(4) Theorem 4.1.1, showing that Arthur parameters determine conormal vectors to Vogan's parameter space and further that representations of the component group attached to the Arthur parameter correspond exactly to equivariant local systems on the orbit of that conormal vector, as in the case of real groups [1];
(5) Theorem 5.3.1 on a functor of vanishing cycles, replacing microlocalisation;
(6) Vogan's conjectures from [32] expressed in terms of vanishing cycles as Conjectures 1, 2 and 3, in Section 6.

Although we do not prove the conjecture above in this paper, we do have in mind a strategy for a proof using twisted spectral endoscopic transfer and its geometric counterpart for perverse sheaves on Vogan varieties; we will do this for unipotent representations of odd orthogonal groups in a subsequent paper. In this paper we have more modest goals: following [32], adapting conjectures from [1] to p-adic groups and casting them in a form amenable to calculations. In [10] we provide additional evidence for these conjectures by verifying them in examples chosen to illustrate features of the three theorems above.

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1. Arthur packets and pure rational forms

The goal of this section is primarily to set some notation, recall some definitions, and set the stage for the geometric description of the characters of A_ψ appearing in the introduction.

1.1. **Local Langlands group.**

Let F be a p-adic field; let $q = q_F$ be the cardinality of the residue field for F. Let $\bar{F}$ be an algebraic closure of F and set $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$. There is an exact sequence

$$1 \longrightarrow I_F \longrightarrow \Gamma_F \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1,$$

where I_F is the inertia subgroup of Γ_F and $\bar{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. Since $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$, it contains a dense subgroup $W_{k_F} \cong \mathbb{Z}$, in which 1 corresponds to the automorphism $x \mapsto x^{q_F}$ in $\bar{\mathbb{F}}_q$. We fix a lift Fr in Γ_F of $x \mapsto x^{q_F}$ in W_{k_F}. The Weil group W_F of F is the preimage of W_{k_F} in Γ_F,

$$\begin{tikzcd} 1 \arrow[r] & I_F \arrow[r] & W_F \arrow[r] & W_{k_F} \arrow[l, bend right] \arrow[r] & 1, \end{tikzcd}$$

topologised so that the compact subgroup I_F is open in W_F. Let

$$|\ |_F : W_F \longrightarrow \mathbb{R}^{\times}$$

be the norm homomorphism, trivial on I_F and sending Fr to q_F. Then $|\ |_F$ is continuous with respect to this topology for W_F.

The *local Langlands group* of F is the trivial extension of W_F by $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SL}(2,\mathbb{C}) \arrow[r] & L_F \arrow[r] & W_F \arrow[l, bend right] \arrow[r] & 1. \end{tikzcd}$$

1.2. **L-groups.** Let G be a connected reductive linear algebraic group over F. Let

$$\Psi_0(G) = (X^*, \Delta, X_*, \Delta^\vee)$$

be the based root datum of G. The dual based root datum is

$$\Psi_0^\vee(G) := (X_*, \Delta^\vee, X^*, \Delta).$$

A *dual group* of G is a complex connected reductive algebraic group $\widehat{G}$ together with a bijection

$$\eta_{\widehat{G}} : \Psi_0^\vee(G) \cong \Psi_0(\widehat{G}).$$

The Galois group Γ_F acts on $\Psi_0(G)$ and $\Psi_0^\vee(G)$; see [7, §1.3]. This action induces a homomorphism

$$\mu : \Gamma_F \longrightarrow \mathrm{Aut}(\Psi_0(G)) \cong \mathrm{Aut}(\Psi_0^\vee(G)).$$

Let $\widehat{G}$ be a dual group of G. Then we can compose $\eta_{\widehat{G}}$ with μ and get a homomorphism

$$\mu_{\widehat{G}} : \Gamma_F \longrightarrow \mathrm{Aut}(\Psi_0(\widehat{G})).$$

An *L-group data* for G is a triple $(\widehat{G}, \rho, \mathrm{Spl}_{\widehat{G}})$, where $\widehat{G}$ is a dual group of G, $\rho : \Gamma_F \longrightarrow \mathrm{Aut}(\widehat{G})$ is a continuous homomorphism and $\mathrm{Spl}_{\widehat{G}} := (B, T, \{X_\alpha\})$ is a splitting of $\widehat{G}$ such that ρ preserves $\mathrm{Spl}_{\widehat{G}}$ and induces $\mu_{\widehat{G}}$ on $\Psi_0(\widehat{G})$ (see [7, Sections 1, 2] for details.)

The *L-group* of G determined by the L-group data $(\widehat{G}, \rho, \mathrm{Spl}_{\widehat{G}})$ is

$${}^LG := \widehat{G} \rtimes W_F,$$

where the action of W_F on $\widehat{G}$ factors through ρ. Since ρ induces $\mu_{\widehat{G}}$ on $\Psi_0(\widehat{G})$ and since $\mathrm{Aut}(\Psi_0(\widehat{G}))$ is finite, the action of W_F on $\widehat{G}$ factors through a finite quotient of W_F. We remark that the L-group, ${}^LG$, only depends on $\widehat{G}$ and ρ and is unique up to conjugation by elements in $\widehat{G}$ fixed by Γ_F.

Henceforth we fix an L-group, ${}^LG$, of G and make ${}^LG$ a topological group by giving $\widehat{G}$ the discrete topology and W_F the profinite topology.

1.3. **Langlands parameters.** If $\phi : L_F \to {}^LG$ is a group homomorphism that commutes with the projections $L_F \to W_F$ and ${}^LG \to W_F$, then we may define $\phi^\circ : L_F \to \widehat{G}$ by $\phi(w,x) = \phi^\circ(w,x) \rtimes w$. Then we have the following map of split short exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SL}(2,\mathbb{C}) \arrow[r] \arrow[d] & L_F \arrow[r] \arrow[d, "\phi"] \arrow[dl, "\phi^\circ"'] & W_F \arrow[r] \arrow[l, bend right] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \widehat{G} \arrow[r] & {}^LG \arrow[r] & W_F \arrow[r] \arrow[l, bend right] & 1. \end{tikzcd}$$

A *Langlands parameter* for G is a homomorphism $\phi : L_F \to {}^LG$ such that

(P.i) ϕ is continuous;
(P.ii) ϕ commutes with the projections $L_F \to W_F$ and ${}^LG \to W_F$;
(P.iii) $\phi^\circ|_{\mathrm{SL}(2)} : \mathrm{SL}(2) \to \widehat{G}$ is a morphism of algebraic groups;
(P.iv) the image of $\phi|_{W_F}$ consists of semisimple elements in ${}^LG$.

Let $P({}^LG)$ be the set of Langlands parameters for G. For $\phi \in P({}^LG)$, we refer to

$$A_\phi := \pi_0(Z_{\widehat{G}}(\phi)) = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^0$$

as the *component group for ϕ*.

Langlands parameters are *equivalent* if they are conjugate under $\widehat{G}$. The set of equivalence classes of Langlands parameters of G is denoted by $\Phi(G/F)$; it is independent of the choice of L-group ${}^LG$ made above.

1.4. **Arthur parameters.** If $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^LG$ is a group homomorphism that commutes with the projections $L_F \times \mathrm{SL}(2,\mathbb{C}) \to L_F \to W_F$ and ${}^LG \to W_F$, then we define $\psi^\circ : L_F \times \mathrm{SL}(2) \to \widehat{G}$ by $\psi(w,x,y) = \phi^\circ(w,x,y) \rtimes w$, where $(w,x) \in L_F$ and $y \in \mathrm{SL}(2)$.

An *Arthur parameter* for G is a homomorphism $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^LG$ such that

(Q.i) $\psi|_{L_F}$ is a Langlands parameter for G;
(Q.ii) $\psi^\circ|_{\mathrm{SL}(2)} : \mathrm{SL}(2) \to \widehat{G}$ is a morphism of algebraic groups;
(Q.iii) the image $\psi^\circ|_{W_F} : W_F \to \widehat{G}$ is bounded (its closure is compact) in the complex topology for $\widehat{G}$.

The set of Arthur parameters for G will be denoted by $Q({}^LG)$. The set of $\widehat{G}$-conjugacy classes of Arthur parameters will be denoted by $\Psi(G/F)$.

For $\psi \in Q({}^LG)$, we refer to

$$A_\psi := \pi_0(Z_{\widehat{G}}(\psi)) = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0$$

as the *component group for ψ*.

1.5. **Langlands parameters of Arthur type.** Define $d : W_F \to \mathrm{SL}(2,\mathbb{C})$ by

$$d_w := \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}. \tag{14}$$

Note that $w \mapsto (w, d_w)$ is a section of $L_F \to W_F$. For $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$, define $\phi_\psi : L_F \to {}^LG$ by

$$\phi_\psi(w,x) = \psi(w,x,d_w).$$

This defines a map

$$Q({}^LG) \rightarrow P({}^LG) \qquad (15)$$
$$\psi \mapsto \phi_\psi.$$

We will refer to ϕ_ψ as the *Langlands parameter for* ψ. The function $\psi \mapsto \phi_\psi$ is neither injective nor surjective. Langlands parameters in the image of the map $Q({}^LG) \rightarrow P({}^LG)$ are called *Langlands parameters of Arthur type*. The function

$$\Psi(G/F) \rightarrow \Phi(G/F),$$

induced from $Q({}^LG) \rightarrow P({}^LG)$, is injective.

1.6. **Pure rational forms.** We suppose now that the connected reductive algebraic group over F is quasi-split.

An *inner rational form* σ of G is a 1-cocycle of Γ_F in G_{ad}, where G_{ad} is the adjoint group of G. It determines an *inner twist* $(G_\sigma, \varphi_\sigma^*)$ of G as follows. Let $G_\sigma(\bar{F}) := G^*(\bar{F})$ and φ_σ be the identity map. The action of $\gamma \in \Gamma_F$ on $G_\sigma(\bar{F})$ is given through the twisted Galois action on $G(\bar{F})$, *i.e.*, $\gamma : g \mapsto \mathrm{Ad}(\sigma(\gamma))(\gamma \cdot g)$ for $g \in G(\bar{F})$, where $\gamma \cdot g$ refers to the action of Γ_F on $G(\bar{F})$ defining G over F. We will represent the inner twist by G_σ, and identify $G_\sigma(F)$ as a subgroup of $G(\bar{F})$ through φ_σ. Two inner rational forms σ_1, σ_2 of G are equivalent if they give the same cohomology class in $H^1(F, G_{\mathrm{ad}})$, or equivalently $G_{\sigma_1}(F)$ and $G_{\sigma_2}(F)$ are conjugate under $G(\bar{F})$. There is a canonical isomorphism

$$H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}, \mathbb{C}^1)$$

where $\widehat{G}_{\mathrm{sc}}$ is the simply connected cover of the derived group of $\widehat{G}$. The character of $Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}$ determined by $[\sigma] \in H^1(F, G_{\mathrm{ad}})$ will be denoted ζ_σ.

A *pure rational form* δ of G is a 1-cocycle of Γ_F in G. It determines an inner rational form $\sigma := \delta(\sigma)$ by the canonical map

$$Z^1(F, G) \rightarrow Z^1(F, G_{\mathrm{ad}}). \qquad (16)$$

We will denote the inner twist G_σ by G_δ. Two pure rational forms of G are equivalent if they give the same cohomology class in $H^1(F, G_{\mathrm{ad}})$. There is also a canonical isomorphism

$$H^1(F, G) \cong \mathrm{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1).$$

The character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ corresponding to the equivalence class of δ will be denoted by χ_δ. By [21, Proposition 6.4], the homomorphism $G \rightarrow G_{\mathrm{ad}}$ induces a commuting diagram:

$$\begin{tikzcd} H^1(F, G) \arrow[r] \arrow[d, "\cong"] & H^1(F, G_{\mathrm{ad}}) \arrow[d, "\cong"] \\ \mathrm{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1) \arrow[r] & \mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}, \mathbb{C}^1). \end{tikzcd}$$

So ζ_σ is the image of χ_δ and we will also denote it by ζ_δ.

1.7. **Langlands packets for pure rational forms.** An isomorphism class of representations of a pure rational form of G is a pair (π, δ), where π is an isomorphism class of admissible representations of $G_\delta(F)$. Then $G(\bar{F})$-conjugation defines an equivalence relation on such pairs, which is compatible with the equivalence relation on pure rational forms $Z^1(F, G)$. We denote the equivalence class of (π, δ) by $[\pi, \delta]$, and following [32], write $\Pi_{\mathrm{pure}}(G/F)$ for the set of these equivalence classes. The local Langlands correspondence for pure rational forms of G can be stated as in the following conjecture. *There is a natural bijection between* $\Pi_{pure}(G/F)$ *and* $\widehat{G}$*-conjugacy classes of pairs* (ϕ, ρ) *with* $\phi \in P({}^LG)$ *and* $\rho \in \mathrm{Irrep}(A_\phi)$. We will call the pair (ϕ, ρ) in this conjecture a complete Langlands parameter. For $\phi \in P({}^LG)$, we define the corresponding *pure Langlands packet*

$$\Pi_{\mathrm{pure},\phi}(G/F)$$

to be consisting of $[\pi, \delta]$ in $\Pi_{\mathrm{pure}}(G/F)$, such that they are associated with $\widehat{G}$-conjugacy classes of (ϕ, ρ) for any $\rho \in \mathrm{Irrep}(A_\phi)$ under the local Langlands correspondence for pure rational forms. This is also known as the Langlands-Vogan packet.

1.8. **Arthur packets for quasi-split symplectic or special orthogonal groups.** From now on until the end of Section 1, we will assume G is a quasi-split symplectic or special orthogonal group over F. In [2, Theorem 1.5.1], Arthur assigns to $\psi \in Q({}^LG)$ a multiset $\Pi_\psi(G(F))$ over $\Pi(G(F))$, which is usually referred to as the Arthur packet of G associated with ψ. It is a deep result of Moeglin [28] that $\Pi_\psi(G(F))$ is actually a subset of $\Pi(G(F))$. Arthur [2, Theorem 2.2.1] also associates $\Pi_\psi(G(F))$ with a canonical map

$$\begin{aligned} \Pi_\psi(G(F)) &\to \widehat{\mathcal{S}_\psi} \\ \pi &\mapsto \langle\,\cdot\,, \pi\rangle_\psi \end{aligned} \tag{17}$$

where

$$\mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0 Z(\widehat{G})^{\Gamma_F}, \tag{18}$$

and $\widehat{\mathcal{S}_\psi}$ denotes the set of irreducible characters of $\mathcal{S}_\psi$. We use (17) to define a stable virtual representation of $G(F)$ by

$$\eta^G_\psi := \sum_{\pi\in\Pi_\psi(G(F))} \langle z_\psi, \pi\rangle_\psi\ \pi, \tag{19}$$

where $z_\psi \in \mathcal{S}_\psi$ is the image of $\psi(1,-1)$ under the mapping $Z_{\widehat{G}}(\psi) \to \mathcal{S}_\psi$, here $(1,-1) \in L_F$ with -1 is the non-trivial central element in $\mathrm{SL}(2,\mathbb{C})$. Every semisimple $s \in Z_{\widehat{G}}(\psi)$ determines an element x of $\mathcal{S}_\psi$ and thus a new virtual representation

$$\eta^G_{\psi,s} := \sum_{\pi\in\Pi_\psi(G(F))} \langle z_\psi x, \pi\rangle_\psi\ \pi. \tag{20}$$

Turning to the stable distributions on $G(F)$, we set

$$\Theta^G_\psi := \sum_{\pi\in\Pi_\psi(G(F))} \langle z_\psi, \pi\rangle_\psi\ \Theta_\pi, \tag{21}$$

and

$$\Theta^G_{\psi,s} := \sum_{\pi\in\Pi_\psi(G(F))} \langle z_\psi x, \pi\rangle_\psi\ \Theta_\pi. \tag{22}$$

The pair (ψ, s) also determines an endoscopic datum $(G', {}^LG', s, \xi)$ for G and an Arthur parameter ψ' for G' so that $\psi = \xi \circ \psi'$. In fact, G' is a product group, whose factors consist of symplectic, special orthogonal and general linear groups. So one can extend the above discussions about G to G' without difficulty as done in [2].

Arthur's main local result shows that, for locally constant compactly supported function f on $G(F)$, we have

$$\Theta^{G}_{\psi,s}(f) = \Theta^{G'}_{\psi'}(f'), \tag{23}$$

where f' is the Langlands-Shelstad transfer of f from $G(F)$ to $G'(F)$. It is in this sense that the maps (17) are compatible with spectral endoscopic transfer to $G(F)$.

On the other hand, there is an involution θ of $G := \mathrm{GL}(N)$ over F such that $(G, {}^{L}G, s, \xi_N)$ is a twisted endoscopic datum for $G^{+}(F) := \mathrm{GL}(N, F) \rtimes \langle\theta\rangle$ in the sense of [23, Section 2.1], for suitable semisimple $s \in \widehat{G}^{\theta}$, the component of $\widehat{\theta}$ in $\widehat{G}^{+} := \widehat{G} \rtimes \langle\widehat{\theta}\rangle$, where $\widehat{\theta}$ is the dual involution. Arthur's main local result also shows that, for locally constant compactly supported function f^{θ} on $G^{\theta}(F) := G(F) \rtimes \theta$,

$$\Theta^{G}_{\psi}(f) = \Theta^{G^{+}}_{\psi_N,s}(f^{\theta}), \tag{24}$$

where f is the Langlands-Kottwitz-Shelstad transfer of f^{θ} from $G^{\theta}(F)$ to $G^{*}(F)$ and $\Theta^{G^{+}}_{\psi_N,s}$ is the twisted character of a particular extension of the Speh representation of $\mathrm{GL}(N, F)$ associated with Arthur parameter $\psi_N := \xi_N \circ \psi$ to the disconnected group $G^{+}(F)$. It is in this sense that the maps (17) are compatible with twisted spectral endoscopic transfer from $G(F)$.

Arthur shows that the map (17) is uniquely determined by: the stability of Θ^{G}_{ψ}; property (23) for all endoscopic data G'; and property (24) for twisted endoscopy of $\mathrm{GL}(N)$. In particular, the endoscopic character identities that are used to pin down $\langle \,\cdot\, , \pi\rangle_{\psi}$ involve values at *all* elements of $\mathcal{S}_{\psi}$.

When ψ is trivial on the second $\mathrm{SL}(2, \mathbb{C})$, it becomes a tempered Langlands parameter. In this case, Arthur shows (17) is a bijection. By the Langlands classification of $\Pi(G(F))$, which is in terms of tempered representations, this bijection extends to all Langlands parameters of G. Moreover, it follows from Arthur's results that there is a bijection between $\Pi(G(F))$ and $\widehat{G}$-conjugacy classes of pairs (ϕ, ϵ) for $\phi \in P({}^{L}G)$ and $\epsilon \in \widehat{\mathcal{S}_{\phi}}$.

1.9. **Arthur packets for inner rational forms.** A conjectural description of Arthur packets for *inner twists* of G is presented in [2, Chapter 9], though the story is far from complete. Let σ be an inner rational forms of G. An Arthur parameter ψ of G_{σ} is said to be *relevant* if any Levi subgroup of ${}^{L}G_{\sigma}$ that ψ factors through is the L-group of a Levi subgroup of G_{σ}. We denote the subset of relevant Arthur parameter by $Q_{\mathrm{rel}}(G_{\sigma})$. In [2, Conjecture 9.4.2], Arthur assigns to $\psi \in Q_{\mathrm{rel}}(G_{\sigma})$ a multiset $\Pi_{\psi}(G_{\sigma}(F))$ over $\Pi(G_{\sigma}(F))$, which is called the Arthur packet of G_{σ} associated with ψ. This time Moeglin's results [28] only show $\Pi_{\psi}(G_{\sigma}(F))$ is a subset of $\Pi(G_{\sigma}(F))$ in case when σ comes from a pure rational form; see also [2, Conjecture 9.4.2, Remark 2]. For the purpose of comparison with the geometric construction of Arthur packets, in this paper we define $\Pi_{\psi}(G_{\sigma}(F))$ simply as the image of this multiset in $\Pi(G_{\sigma}(F))$.

To extend (17) to this case, one must replace the group $\mathcal{S}_{\psi}$ with a larger, finite, generally non-abelian group $\mathcal{S}_{\psi,\mathrm{sc}}$, which is a central extension

$$1 \longrightarrow \widehat{Z}_{\psi,\mathrm{sc}} \longrightarrow \mathcal{S}_{\psi,\mathrm{sc}} \longrightarrow \mathcal{S}_{\psi} \longrightarrow 1 \tag{25}$$

of $\mathcal{S}_{\psi}$ by the finite abelian group

$$\widehat{Z}_{\psi,\mathrm{sc}} := Z(\widehat{G}^{*}_{\mathrm{sc}})/Z(\widehat{G}^{*}_{\mathrm{sc}}) \cap S^{0}_{\psi,\mathrm{sc}}.$$

To explain the group in this exact sequence, we introduce the following notations. Set

$$S_\psi := Z_{\widehat{G}^*}(\psi) \quad \text{and} \quad \bar{S}_\psi := Z_{\widehat{G}^*}(\psi)/Z(\widehat{G})^{\Gamma_F}.$$

So $\bar{S}_\psi$ is the image of S_ψ in $\widehat{G}^*_{ad}$, whose preimage in $\widehat{G}^*$ is $S_\psi Z(\widehat{G})$. Let $S_{\psi,\mathrm{sc}}$ be the preimage of $\bar{S}_\psi$ under the projection $\widehat{G}^*_{\mathrm{sc}} \to \widehat{G}^*_{ad}$, which is the same as the preimage of $S_\psi Z(\widehat{G})$ in $\widehat{G}^*_{\mathrm{sc}}$. Let $S^\sharp_{\psi,\mathrm{sc}}$ be the preimage of S_ψ in $\widehat{G}^*_{\mathrm{sc}}$ and $\widehat{Z}^\sharp_{\mathrm{sc}}$ be the preimage of $Z(\widehat{G})^{\Gamma_F}$ in $\widehat{G}^*_{\mathrm{sc}}$. Let us write $Z(\widehat{G}^*)$ (resp. $Z(\widehat{G}^*_{\mathrm{sc}})$) for $\widehat{Z}$ (resp. $\widehat{Z}_{\mathrm{sc}}$). It is clear that $\widehat{Z}^{\Gamma_F}_{\mathrm{sc}} \hookrightarrow \widehat{Z}^\sharp_{\mathrm{sc}}$. Then we have the following commutative diagram, which is exact on each row:

$$\begin{tikzcd}
1 \arrow[r] & \widehat{Z}^{\Gamma_F} \arrow[r] & S_\psi \arrow[r] & \bar{S}_\psi \arrow[r] & 1 \\
1 \arrow[r] & \widehat{Z}^\sharp_{\mathrm{sc}} \arrow[r] \arrow[u, two heads] \arrow[d, hook] & S^\sharp_{\psi,\mathrm{sc}} \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \bar{S}_\psi \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & \widehat{Z}_{\mathrm{sc}} \arrow[r] & S_{\psi,\mathrm{sc}} \arrow[r] & \bar{S}_\psi \arrow[r] & 1.
\end{tikzcd}$$

Note $S_{\psi,\mathrm{sc}} = S^\sharp_{\psi,\mathrm{sc}}\widehat{Z}_{\mathrm{sc}}$, and hence $S^0_{\psi,\mathrm{sc}} = (S^\sharp_{\psi,\mathrm{sc}})^0$. After passing to the component groups, we have the following commutative diagram, which is again exact on each row:

$$\begin{tikzcd}
1 \arrow[r] & \widehat{Z}^{\Gamma_F}_\psi \arrow[r] & A_\psi \arrow[r] & \mathcal{S}_\psi \arrow[r] & 1 \\
1 \arrow[r] & \widehat{Z}^\sharp_{\psi,\mathrm{sc}} \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \mathcal{S}^\sharp_{\psi,\mathrm{sc}} \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \mathcal{S}_\psi \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & \widehat{Z}_{\psi,\mathrm{sc}} \arrow[r] & \mathcal{S}_{\psi,\mathrm{sc}} \arrow[r] & \mathcal{S}_\psi \arrow[r] & 1.
\end{tikzcd}$$

Here $A_\psi, \mathcal{S}_\psi, \mathcal{S}^\sharp_{\psi,\mathrm{sc}}, \mathcal{S}_{\psi,\mathrm{sc}}$ are the corresponding component groups and

$$\widehat{Z}^{\Gamma_F}_\psi := \widehat{Z}^{\Gamma_F}/\widehat{Z}^{\Gamma_F} \cap S^0_\psi$$
$$\widehat{Z}^\sharp_{\psi,\mathrm{sc}} := \widehat{Z}^\sharp_{\mathrm{sc}}/\widehat{Z}^\sharp_{\mathrm{sc}} \cap S^0_{\psi,\mathrm{sc}}$$
$$\widehat{Z}_{\psi,\mathrm{sc}} := \widehat{Z}_{\mathrm{sc}}/\widehat{Z}_{\mathrm{sc}} \cap S^0_{\psi,\mathrm{sc}}$$

Let ζ_σ be the character of $\widehat{Z}^{\Gamma_F}_{\mathrm{sc}}$ corresponding to the equivalence class of σ. We will also fix an extension of ζ_σ to $\widehat{Z}_{\mathrm{sc}}$ and denote that by $\tilde{\zeta}_\sigma$. By [3, Lemma 2.1], an Arthur parameter ψ of G_σ is relevant if and only if the restriction of ζ_σ to $\widehat{Z}^{\Gamma_F}_{\mathrm{sc}} \cap S^0_{\psi,\mathrm{sc}}$ is trivial.

Lemma 1.9.1. *$\widehat{Z}^{\Gamma_F}_{sc} \cap S^0_{\psi,sc} = \widehat{Z}_{sc} \cap S^0_{\psi,sc}$.*

Proof. It suffices to show $\widehat{Z}_{\mathrm{sc}} \cap S^0_{\psi,\mathrm{sc}} \subseteq \widehat{Z}^{\Gamma_F}_{\mathrm{sc}}$. Let $L_F \times \mathrm{SL}(2,\mathbb{C})$ act on $\widehat{G}^*_{\mathrm{sc}}$ by conjugation of the preimage of $\psi(L_F \times \mathrm{SL}(2,\mathbb{C}))$ in ${}^LG_{\mathrm{sc}}$. Then we can define the group cohomology $H^0_\psi(L_F \times \mathrm{SL}(2,\mathbb{C}), \widehat{G}^*_{\mathrm{sc}})$, which is the group of fixed points in $\widehat{G}_{\mathrm{sc}}$ under the action of $L_F \times \mathrm{SL}(2,\mathbb{C})$. It is clear that $H^0_\psi(L_F \times \mathrm{SL}(2,\mathbb{C}), \widehat{G}^*_{\mathrm{sc}}) \subseteq S^\sharp_{\psi,\mathrm{sc}}$. In fact, it is also not hard to show that

$$(H^0_\psi(L_F \times \mathrm{SL}(2,\mathbb{C}), \widehat{G}^*_{\mathrm{sc}}))^0 = (S^\sharp_{\psi,\mathrm{sc}})^0.$$

As a result, we have

$$\widehat{Z}_{\mathrm{sc}} \cap S^0_{\psi,\mathrm{sc}} \subseteq \widehat{Z}_{\mathrm{sc}} \cap (H^0_{\psi}(L_F \times \mathrm{SL}(2,\mathbb{C}), \widehat{G}^*_{\mathrm{sc}}))^0 \subseteq \widehat{Z}_{\mathrm{sc}} \cap H^0_{\psi}(L_F \times \mathrm{SL}(2,\mathbb{C}), \widehat{G}^*_{\mathrm{sc}}) = \widehat{Z}^{\Gamma_F}_{\mathrm{sc}}$$

This finishes the proof. □

So, if ψ is relevant, it follows from Lemma 1.9.1 that $\tilde{\zeta}_\sigma$ descends to a character of $\widehat{Z}_{\psi,\mathrm{sc}}$. Let $\mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}_\sigma)$ be the set of isomorphism classes of $\tilde{\zeta}_\sigma$-equivariant representations of $\mathcal{S}_{\psi,\mathrm{sc}}$. In [2, Conjecture 9.4.2], Arthur conjectures a map

$$\Pi_\psi(G_\sigma(F)) \to \mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}_\sigma) \tag{26}$$

and writes $\langle \,\cdot\, , \pi\rangle_{\psi,\mathrm{sc}}$ for the character of the associated representation of $\mathcal{S}_{\psi,\mathrm{sc}}$. Because of our definition of $\Pi_\psi(G_\sigma(F))$ here, one can not replace $\mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}_\sigma)$ by the subset $\Pi(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}_\sigma)$ of $\tilde{\zeta}_\sigma$-equivariant irreducible characters of $\mathcal{S}_{\psi,\mathrm{sc}}$ as in Arthur's original formulation. The map (26) is far from being canonical for it depends on (17) and various other choices implicitly.

When $\psi = \phi$ is a tempered Langlands parameter, Arthur states all these results as a theorem [2, Theorem 9.4.1]. In particular, he claims (26) gives a bijection

$$\Pi_\phi(G_\sigma(F)) \to \Pi(\mathcal{S}_{\phi,\mathrm{sc}}, \tilde{\zeta}_\sigma). \tag{27}$$

By the Langlands classification of $\Pi(G_\sigma(F))$, which is in terms of tempered representations, this bijection extends to all relevant Langlands parameters of G_σ. Moreover, it follows from [2, Theorem 9.4.1] that there is a bijection between $\Pi(G_\sigma(F))$ and $\widehat{G}$-conjugacy classes of pairs (ϕ, ϵ) for $\phi \in P_{\mathrm{rel}}({}^L G^*_\sigma)$ and $\epsilon \in \Pi(\mathcal{S}_{\phi,\mathrm{sc}}, \tilde{\zeta}_\sigma)$.

1.10. **Pure Arthur packets.** Let δ be a pure rational form of G and ψ be an Arthur parameter of G_δ. Let χ_δ be the character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ corresponding to the equivalence class of δ. We will also denote its pull-back to $Z(\widehat{G})^{\Gamma_F}$ by χ_δ. Let $\zeta_\delta := \zeta_{\sigma(\delta)}$ be the character of $Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}$, which is also the pull-back of χ_δ along

$$Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F} \to \pi_0(Z(\widehat{G})^{\Gamma_F}).$$

Lemma 1.10.1. *χ_δ is trivial on $\widehat{Z}^{\Gamma_F} \cap S^0_\psi$ if and only if ζ_δ is trivial on $\widehat{Z}^{\Gamma_F}_{sc} \cap S^0_{\psi,sc}$.*

Proof. One just needs to notice that S^0_ψ is the product of $(\widehat{Z}^{\Gamma_F})^0$ with the image of $S^0_{\psi,\mathrm{sc}}$ in S_ψ. □

As a direct consequence, we have the following corollary.

Corollary 1.10.2. *An Arthur parameter ψ of G_δ is relevant if and only if χ_δ is trivial on $\widehat{Z}^{\Gamma_F} \cap S^0_\psi$.*

Let us assume ψ is relevant. Then χ_δ descends to a character of $\widehat{Z}^{\Gamma_F}_\psi$. Let $\mathrm{Rep}(A_\psi, \chi_\delta)$ be the set equivalence classes of χ_δ-equivariant representations of A_ψ. Let $\tilde{\zeta}_\delta$ be a character of $\widehat{Z}_{\mathrm{sc}}$ extending ζ_δ, so that its restriction to $\widehat{Z}^\sharp_{\mathrm{sc}}$ is the pull-back of χ_δ. Since ψ is relevant, $\tilde{\zeta}_\delta$ descends to a character of $\widehat{Z}_{\psi,\mathrm{sc}}$. Let $\mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}_\delta)$ be the set of equivalence classes of $\tilde{\zeta}_\delta$-equivariant representations of $\mathcal{S}_{\psi,\mathrm{sc}}$.

Proposition 1.10.3. *Let χ a character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$. Let $\tilde{\zeta}$ be a character of $Z(\widehat{G}_{sc})$ Suppose the pull-back of χ along $\widehat{Z}^\sharp_{sc} \to Z(\widehat{G})^{\Gamma_F} \to \pi_0(Z(\widehat{G})^{\Gamma_F})$ coincides with the restriction of $\tilde{\zeta}$ to $\widehat{Z}^\sharp_{sc} \hookrightarrow Z(\widehat{G}_{sc})$. Then there is a canonical bijection*

$$\mathrm{Rep}(A_\psi, \chi) \to \mathrm{Rep}(\mathcal{S}_{\psi,sc}, \tilde{\zeta}). \tag{28}$$

Proof. Since

$$\mathrm{Ker}(\mathcal{S}^\sharp_{\psi,\mathrm{sc}} \to A_\psi) = \mathrm{Ker}(\widehat{Z}^\sharp_{\psi,\mathrm{sc}} \to \widehat{Z}^{\Gamma_F}_\psi),$$

there is a canonical bijection

$$\mathrm{Rep}(A_\psi, \chi) \to \mathrm{Rep}(\mathcal{S}^\sharp_{\psi,\mathrm{sc}}, \zeta^\sharp),$$

where $\zeta^\sharp$ is the pull-back of χ_δ to $\widehat{Z}^\sharp_{\psi,\mathrm{sc}}$. Since

$$\mathcal{S}_{\psi,\mathrm{sc}} = \widehat{Z}_{\psi,\mathrm{sc}}\ \mathcal{S}^\sharp_{\psi,\mathrm{sc}} \qquad \text{and} \qquad \widehat{Z}_{\psi,\mathrm{sc}} \cap \mathcal{S}^\sharp_{\psi,\mathrm{sc}} = \widehat{Z}^\sharp_{\psi,\mathrm{sc}},$$

there is also a canonical bijection

$$\mathrm{Rep}(\mathcal{S}_{\psi,\mathrm{sc}}, \tilde{\zeta}) \to \mathrm{Rep}(\mathcal{S}^\sharp_{\psi,\mathrm{sc}}, \zeta^\sharp).$$

Combining the two isomorphisms above, we obtain the canonical bijection promised above. □

Let us take δ among various other choices to be made in defining (26). To emphasize this choice, we will define

$$\Pi_\psi(G_\delta(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi_\psi(G_\delta(F))\}.$$

Then by composing (26) with (28) modulo isomorphisms, we can have a canonical map

$$\begin{aligned} \Pi_\psi(G_\delta(F), \delta) &\to \mathrm{Rep}(A_\psi, \chi_\delta) \\ (\pi, \delta) &\mapsto \langle\ \cdot\ , (\pi, \delta)\rangle_\psi \end{aligned} \tag{29}$$

which only depends on δ and (17). In particular, it becomes (17) when $\delta = 1$. For equivalent pure rational forms δ_1 and δ_2 of G, it follows from the construction of (26) that the following diagram commutes.

$$\begin{tikzcd} \Pi_\psi(G_{\delta_1}(F), \delta_1) \arrow[r, "\cong"] \arrow[d] & \Pi_\psi(G_{\delta_2}(F), \delta_2) \arrow[d] \\ \mathrm{Rep}(A_\psi, \chi_{\delta_1})_{/\mathrm{iso}} \arrow[r, equal] & \mathrm{Rep}(A_\psi, \chi_{\delta_2})_{/\mathrm{iso}} \end{tikzcd}$$

As a result, (29) is also well-defined for the equivalence class $[\pi, \delta]$.

Let ψ be an Arthur parameter of G. For pure rational form δ such that ψ is not relevant, we will define $\Pi_\psi(G_\delta(F), \delta)$ to be empty. Then we can define the *pure Arthur packet* associated with ψ to be

$$\Pi_{\mathrm{pure},\psi}(G/F) = \bigsqcup_{[\delta] \in H^1(F,G)} \Pi_\psi(G_\delta(F), \delta) \tag{30}$$

as a subset of $\Pi_{\mathrm{pure}}(G/F)$. It is equipped with a canonical map

$$\begin{aligned} \Pi_{\mathrm{pure},\psi}(G/F) &\to \mathrm{Rep}(A_\psi) \\ [\pi, \delta] &\mapsto \langle \cdot, [\pi, \delta]\rangle_\psi \end{aligned} \tag{31}$$

When $\psi = \phi$ is a tempered Langlands parameter, this induces a bijection

$$\begin{aligned} \Pi_{\mathrm{pure},\phi}(G/F) &\to \Pi(A_\phi) \\ [\pi,\delta] &\mapsto \langle \cdot, [\pi,\delta]\rangle_\phi. \end{aligned}$$

This bijection also extends to all Langlands parameters ϕ of G, according to the discussion in the end of Section 1.9. Combined with the local Langlands correspondence for each pure rational form of G, we can conclude the local Langlands correspondence for pure rational forms of G appearing in Section 1.7.

1.11. **Virtual representations of pure rational forms.** Let $\mathsf{K}\Pi_{\mathrm{pure}}(G/F)$ be the free abelian group generated by the set $\Pi_{\mathrm{pure}}(G/F)$. Define $\eta_\psi \in \mathsf{K}\Pi_{\mathrm{pure}}(G/F)$ by

$$\eta_\psi := \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\psi}(G/F)} e(\delta)\ \langle a_\psi, [\pi,\delta]\rangle_\psi\ [\pi,\delta], \tag{32}$$

where $e(\delta) = e(G_\delta)$ is the Kottwitz sign [22] of the group G_δ, and a_ψ is the image of $\psi(1,-1)$ in A_ψ. Using (30) we have

$$\eta_\psi = \sum_{[\delta]\in H^1(F,G)} e(\delta)\ \eta_\psi^\delta$$

where, for each pure rational form δ of G,

$$\eta_\psi^\delta := \sum_{(\pi,\delta)\in\Pi_\psi(G_\delta(F),\delta)} \langle a_\psi, (\pi,\delta)\rangle_\psi\ [\pi,\delta].$$

For semisimple $s \in Z_{\widehat{G}}(\psi)$, we define $\eta_{\psi,s} \in \mathsf{K}\Pi_{\mathrm{pure}}(G/F)$ by

$$\eta_{\psi,s} = \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\psi}(G/F)} e(\delta)\ \langle a_\psi a_s, (\pi,\delta)\rangle_\psi\ [\pi,\delta],$$

where a_s is the image of s in A_ψ. As above, we can break this into summands indexed by pure rational form by writing

$$\eta_{\psi,s} = \sum_{[\delta]\in H^1(F,G)} e(\delta)\ \eta_{\psi,s}^\delta$$

where, for each pure rational form δ of G,

$$\eta_{\psi,s}^\delta := \sum_{(\pi,\delta)\in\Pi_\psi(G_\delta(F),\delta)} \langle a_\psi a_s, (\pi,\delta)\rangle_\psi\ [\pi,\delta].$$

Then $\eta_{\psi,1}^\delta = \eta_\psi^\delta$ and $\eta_{\psi,1} = \eta_\psi$. We note that, with reference to (19) and (20),

$$\eta_\psi^1 = \eta_\psi^G \qquad \text{and} \qquad \eta_{\psi,s}^1 = \eta_{\psi,s}^G.$$

Turning from virtual representations to distributions, we see that each η_ψ^δ and $\eta_{\psi,s}^\delta$ determines a distribution on $G_\delta(F)$ by

$$\Theta_{\psi,s}^\delta := \sum_{(\pi,\delta)\in\Pi_\psi(G_\delta(F),\delta)} \langle a_\psi a_s, (\pi,\delta)\rangle_\psi\ \Theta_\pi.$$

This extends (21) and (22) from $G(F)$ to $G_\delta(F)$ arising from pure rational forms δ of G:

$$\Theta_\psi^1 = \Theta_\psi^G \qquad \text{and} \qquad \Theta_{\psi,s}^1 = \Theta_{\psi,s}^G.$$

1.12. **A quick preview of the rest of the paper.** Inspired by ideas developed for real groups in [1] and without assuming G is symplectic or special orthogonal, the remainder of this paper is devoted to offering a geometric, categorical and calculable description of a map

$$\begin{aligned} \Pi_{\mathrm{pure},\lambda_\psi}(G/F) &\to \mathsf{Rep}(A_\psi)_{/\mathrm{iso}}, \\ [\pi,\delta] &\mapsto \mathsf{Ev}_\psi\, \mathcal{P}(\pi,\delta), \end{aligned} \tag{33}$$

for any quasi-split G and for any Arthur parameter $\psi : L_F \times \mathrm{SL}(2) \to {}^LG$, and also to explaining the conjecture that the map provides a generalisation of (31). Here $\mathsf{Rep}(A_\psi)$ denotes the category of representations of A_ψ so $\mathsf{Rep}(A_\psi)_{/\mathrm{iso}}$ includes the representation of A_ψ on the vector space 0, in particular. In fact, more generally, we will define a map

$$\begin{aligned} \Pi_{\mathrm{pure},\lambda_\phi}(G/F) &\to \mathsf{Rep}(\pi_1(A^{\mathrm{mic}}_{C_\phi}))_{/\mathrm{iso}}, \\ [\pi,\delta] &\mapsto \mathsf{Ev}_\phi\, \mathcal{P}(\pi,\delta), \end{aligned} \tag{34}$$

for any Langlands parameter ϕ for G, such that when $\phi = \phi_\psi$, it coincides with (33). Then

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi}(G/F) := \{[\pi,\delta] \in \Pi_{\mathrm{pure},\lambda}(G/F) \mid \mathsf{Ev}_\phi\, \mathcal{P}(\pi,\delta) \neq 0\}. \tag{35}$$

Note that this extends the definition given in (13) to all Langlands parameters. After defining (34), we build virtual representations

$$\eta^{\mathrm{ABV}}_\phi := \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\lambda_\phi}(G/F)} (-1)^{\dim(C_\phi)-d(\pi,\delta)} e(\delta)\ \operatorname{rank} \mathsf{Ev}_\phi\, \mathcal{P}(\pi,\delta)\ [\pi,\delta], \tag{36}$$

where $d(\pi,\delta) := \dim \operatorname{supp}(\mathcal{P}(\pi,\delta))$, and more generally,

$$\eta^{\mathrm{ABV}}_{\phi,s} = \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\lambda_\phi}(G/F)} (-1)^{\dim(C_\phi)-d(\pi,\delta)} e(\delta)\ \operatorname{trace} \mathsf{Ev}_\phi\, \mathcal{P}(\pi,\delta)(a_s)\ [\pi,\delta],$$

for $s \in Z_{\widehat{G}}(\psi)$ and a_s is the image of s in A_ψ.

The conjectures in Section 6 can all be phrased in terms of these virtual representations.

(1) Conjecture 1: if ψ is an Arthur parameter for symplectic or special orthogonal G, then $\eta^{\mathrm{ABV}}_\psi = \eta_\psi$.
(2) Conjecture 2: if ψ is an Arthur parameter for symplectic or special orthogonal G then $\eta^{\mathrm{ABV}}_{\psi,s} = \eta_{\psi,s}$, for all $s \in Z_{\widehat{G}}(\psi)$. This implies the statement above, taking the case $s = 1$.
(3) Without assuming G is symplectic or special orthogonal, though still connected and quasi-split, Conjecture 3 asserts that the virtual representations η^{ABV}_ϕ, as ϕ ranges over $\Phi(G/F)$, forms a basis for the space of strongly stable virtual representations as defined in [32, 1.6].

In [10] we provide evidence for all three conjectures by providing examples. We prove Conjecture 2 (and therefore Conjecture 1 also) for Arthur parameters for unipotent representations of $G = \mathrm{SO}(2n+1)$ in [11].

2. Equivariant perverse sheaves on parameter varieties

In this section we drop the quasi-split hypothesis and let G be an arbitrary connected reductive algebraic group over a p-adic field F.

2.1. **Infinitesimal parameters.** An *infinitesimal parameter* for G is a homomorphism $\lambda : W_F \to {}^LG$ such that

(R.i) λ is continuous;
(R.ii) λ is a section of ${}^LG \to W_F$;
(R.iii) the image of λ consists of semisimple elements in ${}^LG$.

Let $R({}^LG)$ be the set of infinitesimal parameters for G. We will use the notation $\lambda^\circ : W_F \to \widehat{G}$ for the function defined by $\lambda(w) = \lambda^\circ(w) \rtimes w$. The *component group for* λ is

$$A_\lambda := \pi_0(Z_{\widehat{G}}(\lambda)) = Z_{\widehat{G}}(\lambda)/Z_{\widehat{G}}(\lambda)^0. \tag{37}$$

The set of $\widehat{G}$-conjugacy classes of infinitesimal parameters is denoted by $\Lambda(G/F)$.

For any Langlands parameter $\phi : L_F \to {}^LG$, define the *infinitesimal parameter of* ϕ by

$$\begin{array}{rcl} \lambda_\phi : W_F & \to & {}^LG \\ w & \mapsto & (w, d_w), \end{array}$$

where $d : W_F \to \mathrm{SL}(2,\mathbb{C})$ was defined in Section 1.1. This defines

$$\begin{array}{rcl} P({}^LG) & \to & R({}^LG) \\ \phi & \mapsto & \lambda_\phi. \end{array} \tag{38}$$

The function $\phi \mapsto \lambda_\phi$ is surjective but not, in general, injective. For any fixed $\lambda \in R({}^LG)$, set

$$P_\lambda({}^LG) := \{\phi \in P({}^LG) \mid \lambda_\phi = \lambda\}.$$

We write $\Phi_\lambda(G/F)$ for the set of $Z_{\widehat{G}}(\lambda)$-conjugacy classes of Langlands parameters with infinitesimal parameter λ.

With reference to Section 1.7, for any quasi-split G over F, we set

$$\Pi_{\mathrm{pure},\lambda}(G/F) := \bigcup_{\phi \in P_\lambda({}^LG)} \Pi_{\mathrm{pure},\phi}(G/F),$$

with the union taken in $\Pi_{\mathrm{pure}}(G/F)$. Then, after choosing a representative for each class in $\Phi_\lambda({}^LG)$, we have

$$\Pi_{\mathrm{pure},\lambda}(G/F) = \bigsqcup_{[\phi] \in \Phi_\lambda({}^LG)} \Pi_{\mathrm{pure},\phi}(G/F).$$

Now the local Langlands correspondence for pure rational forms of G (cf. Section 1.7) provides a bijection

$$\Pi_{\mathrm{pure},\lambda}(G/F) \leftrightarrow \{(\phi,\rho) \mid \phi \in P_\lambda({}^LG), \rho \in \mathrm{Irrep}(A_\phi)\}_{/\sim}, \tag{39}$$

where the equivalence on pairs (ϕ,ρ) is defined by $Z_{\widehat{G}}(\lambda)$-conjugation.

2.2. **Vogan varieties.** Fix $\lambda \in R({}^LG)$. Define

$$H_\lambda := Z_{\widehat{G}}(\lambda) := \{g \in \widehat{G} \mid (g \rtimes 1)\lambda(w)(g \rtimes 1)^{-1} = \lambda(w),\ \forall w \in W_F\} \tag{40}$$

and

$$K_\lambda := Z_{\widehat{G}}(\lambda(I_F)) := \{g \in \widehat{G} \mid (g \rtimes 1)\lambda(w)(g \rtimes 1)^{-1} = \lambda(w),\ \forall w \in I_F\}. \tag{41}$$

The centraliser K_λ of $\lambda(I_F)$ in $\widehat{G}$ consists of fixed points in $\widehat{G}$ under a finite group of semisimple automorphisms of $\widehat{G}$, so K_λ is a reductive algebraic group. Since H_λ can be viewed as the group of fixed points in K_λ under the semisimple automorphism $\mathrm{Ad}(\lambda(\mathrm{Fr}))$, then K_λ is also a reductive algebraic group. Neither H_λ nor K_λ is connected, in general.

Following [32, (4.4)(e)], define

$$V_\lambda := V_\lambda({}^LG) := \{x \in \operatorname{Lie} K_\lambda \mid \operatorname{Ad}(\lambda(\mathrm{Fr}))x = q_F x\}, \tag{42}$$

called the *Vogan variety* for λ. Then H_λ acts on V_λ by conjugation.

Lemma 2.2.1. *V_λ is a conical subvariety in the nilpotent cone of* $\operatorname{Lie} K_\lambda$.

Proof. Set $\mathfrak{k}_\lambda = \operatorname{Lie} K_\lambda$. Decompose $\mathfrak{k}_\lambda$ according to the eigenvalues of $\operatorname{Ad}(\lambda(\mathrm{Fr}))$:

$$\mathfrak{k}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{k}_\lambda(\nu). \tag{43}$$

Then, using the Lie bracket in $\mathfrak{k}_\lambda$, we have

$$[\ ,\ ] : \mathfrak{k}_\lambda(\nu_1) \times \mathfrak{k}_\lambda(\nu_2) \to \mathfrak{k}_\lambda(\nu_1\nu_2). \tag{44}$$

It follows that all elements in V_λ are ad-nilpotent in $\hat{\mathfrak{g}}$. So it is enough to show that V_λ does not intersect the centre $\hat{\mathfrak{z}}$ of $\hat{\mathfrak{g}}$. Since the adjoint action of $\lambda(W_F)$ on $\hat{\mathfrak{z}}$ factors through a finite quotient of Γ_F, the $\operatorname{Ad}(\lambda(\mathrm{Fr}))$-eigenvalues on $\hat{\mathfrak{z}}$ are all roots of unity. In particular, they can not be q_F, so V_λ does not intersect $\hat{\mathfrak{z}}$. This shows that all elements in V_λ are nilpotent in $\hat{\mathfrak{g}}$. It is clear from (42) that $V_\lambda({}^LG)$ is closed under scalar multiplication by $\mathbb{C}^\times$ in $\mathfrak{k}_\lambda^{\mathrm{nilp}}$. □

With reference to decomposition of $\mathfrak{k}_\lambda = \operatorname{Lie} K_\lambda$ in the proof of Lemma 2.2.1, observe that

$$\mathfrak{k}_\lambda(q_F) = V_\lambda \qquad \text{and} \qquad \mathfrak{k}_\lambda(1) = \operatorname{Lie} H_\lambda.$$

Proposition 2.2.2. *For each infinitesimal parameter $\lambda \in R({}^LG)$, the H_λ-equivariant function*

$$\begin{aligned} P_\lambda({}^LG) &\longrightarrow V_\lambda({}^LG),\\ \phi &\mapsto x_\phi := d\varphi \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \end{aligned}$$

where $\varphi := \phi^\circ|_{\mathrm{SL}(2,\mathbb{C})} : \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$, is surjective. The fibre of $P_\lambda({}^LG) \to V_\lambda({}^LG)$ over any $x \in V_\lambda({}^LG)$ is a principal homogeneous space for the unipotent radical of $Z_{H_\lambda}(x)$. The induced map between the sets of H_λ-orbits

$$\begin{aligned} \Phi_\lambda({}^LG) &\longrightarrow V_\lambda({}^LG)/H_\lambda,\\ [\phi] &\mapsto C_\phi \end{aligned}$$

is a bijection.

Proof. Fix $x \in V_\lambda = \mathfrak{k}_\lambda(q_F)$. By Lemma 2.2.1, x is nilpotent. There exists an $\mathfrak{sl}_2$-triple (x, y, h) in $\mathfrak{k}_\lambda$ such that

$$x \in V_\lambda = \mathfrak{k}_\lambda(q_F) \qquad \text{and} \qquad z \in \mathfrak{h}_\lambda = \mathfrak{k}_\lambda(1) \qquad \text{and} \qquad y \in \mathfrak{k}_\lambda(q_F^{-1}); \tag{45}$$

see, for example, [17, Lemma 2.1]. Let $\varphi : \mathrm{SL}(2,\mathbb{C}) \to K_\lambda$ be the homomorphism defined by

$$\mathrm{d}\varphi \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} = x, \qquad \mathrm{d}\varphi \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} = h, \qquad \mathrm{d}\varphi \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} = y,$$

and define $\phi : W_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ by

$$\phi(w, g) = \varphi(g)\varphi(d_w^{-1})\lambda(w).$$

Then $\phi \in P_\lambda({}^LG)$ and

$$\mathrm{d}(\phi^\circ|_{SL(2,\mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{d}\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x.$$

This shows the map $P_\lambda({}^LG) \to V_\lambda({}^LG)$ is surjective.

Now, suppose that ϕ_1 is also mapped to x under the map $P_\lambda({}^LG) \to V_\lambda({}^LG)$ and set $\varphi_1 := \phi_1^\circ|_{\mathrm{SL}(2,\mathbb{C})}$. Then φ_1 determines an $\mathfrak{sl}_2$-triple (x, y_1, z_1) in $\mathfrak{k}_\lambda$ such that

$$z_1 \in \mathfrak{h}_\lambda = \mathfrak{k}_\lambda(1) \qquad \text{and} \qquad y_1 \in \mathfrak{k}_\lambda(q_F^{-1}).$$

The two $\mathfrak{sl}_2$-triples (x, y, z) and (x, y_1, z_1) are conjugate by an element of $Z_{H_\lambda}(x)$; see, for example, the second part of [17, Lemma 2.1]. Thus, φ and φ_1 are conjugate under $Z_{H_\lambda}(x)$. We can also write ϕ_1 as

$$\phi_1(w, g) = \varphi_1(g)\varphi_1(d_w^{-1})\lambda(w).$$

It is then clear that ϕ and ϕ_1 are also conjugate under $Z_{H_\lambda}(x)$. This shows that the map $P_\lambda({}^LG) \to V_\lambda({}^LG)$ induces a bijection between H_λ-orbits and also that the fibre above any $x \in V_\lambda$ is in bijection with $Z_{H_\lambda}(x)/Z_{H_\lambda}(\phi)$ for $\phi \mapsto x$ and that $Z_{H_\lambda}(x) = Z_{H_\lambda}(\phi)U$ where U is the unipotent radical of $Z_{H_\lambda}(x)$. □

We remark that Proposition 2.2.2 is analogous to [1, Proposition 6.17] for real groups. However, Proposition 2.2.2 might appear to contradict with [32, Corollary 4.6]. The apparent discrepancy is explained by the two different incarnations of the Weil-Deligne group: we use $L_F = W_F \times \mathrm{SL}(2,\mathbb{C})$ while [32] uses $W'_F = W_F \rtimes \mathbb{G}_{\mathrm{add}}(\mathbb{C})$ and we use pullback along $W_F \to L_F$ given by $w \mapsto (w, d_w)$ to define the infinitesimal parameter of a Langlands parameter while [32] uses restriction of a parameter $W'_F \to {}^LG$ to W_F to define its infinitesimal parameter. We find L_F preferable to W'_F here because it stresses the analogy to the real groups case. However, there is a cost. In the optic of [32], V_λ is exactly a moduli space for Langlands parameters $\phi : W'_F \to {}^LG$ with $\phi|_{W_F} = \lambda$, while in this paper the map $P_\lambda({}^LG) \to V_\lambda({}^LG)$ from Langlands parameters $\phi : L_F \to {}^LG$ with $\lambda_\phi = \lambda$ to V_λ is not a bijection, as we saw in Proposition 2.2.2.

2.3. **Parameter varieties.** Recall from Section 2.1 that elements of $\Lambda(G/F)$ are $\widehat{G}$-conjugacy class of elements of $R({}^LG)$. We will use the notation $[\lambda] \in \Lambda(G/F)$ for the class of $\lambda \in R({}^LG)$; then $[\lambda]$ is an *infinitesimal character* in the language of [32]. Consider the variety

$$X_\lambda := X_\lambda({}^LG) := \widehat{G} \times_{H_\lambda} V_\lambda({}^LG). \tag{46}$$

Then $[\lambda] = [\lambda']$ implies $X_\lambda({}^LG) \cong X_{\lambda'}({}^LG)$. Set

$$P_{[\lambda]}({}^LG) := \{\phi \in P({}^LG) \mid \lambda_\phi = \mathrm{Ad}(g)\lambda, \exists g \in \widehat{G}\}.$$

It follows immediately from Proposition 2.2.2 that the function

$$P_{[\lambda]}({}^LG) \to X_\lambda({}^LG), \tag{47}$$

induced from $P_\lambda({}^LG) \to V_\lambda({}^LG)$ is $\widehat{G}$-equivariant, surjective, and the fibre over any $x \in X_\lambda({}^LG)$ is a principal homogeneous space for the unipotent radical of $Z_{\widehat{G}}(x)$.

Let $\mathrm{Hom}_{W_F}(W_F, {}^LG)$ be the set of homomorphisms that satisfy conditions (R.i) and (R.ii). Observe that

$$R({}^LG) = \{\lambda \in \mathrm{Hom}_{W_F}(W_F, {}^LG) \mid \lambda(\mathrm{Fr}) \in {}^LG_{\mathrm{ss}}\}$$

where ${}^LG_{\mathrm{ss}} \subseteq {}^LG$ denotes the set of semisimple elements in ${}^LG$. Now let $\mathrm{Hom}_{W_F}(I_F, {}^LG)$ be the set of continuous homomorphisms that commute with the natural maps $I_F \to W_F$ and ${}^LG \to W_F$. As explained in [29, Section 10], the set $\mathrm{Hom}_{W_F}(W_F, {}^LG)$ naturally carries the structure of (locally finite-type) variety over $\mathbb{C}$ and its components are indexed by $\widehat{G}$-conjugacy classes of those $\phi_0 \in \mathrm{Hom}_{W_F}(I_F, {}^LG)$ that lie in the image of $\mathrm{Hom}_{W_F}(W_F, {}^LG) \to \mathrm{Hom}_{W_F}(I_F, {}^LG)$ given by restriction. We remark that $\widehat{G}$-orbits in $\mathrm{Hom}_{W_F}(W_F, {}^LG)$ are closed subvarieties.

Now consider the (locally finite-type) variety

$$X({}^LG) \quad := \quad \{(\lambda, x) \in \mathrm{Hom}_{W_F}(W_F, {}^LG) \times \mathrm{Lie}\,\widehat{G} \mid x \in V_\lambda({}^LG)\}$$

This (locally finite-type) variety comes equipped with morphisms

$$\begin{array}{ccccc} X({}^LG) & \to & \mathrm{Hom}_{W_F}(W_F, {}^LG) & \to & \mathrm{Hom}_{W_F}(I_F, {}^LG) \\ (\lambda, x) & \mapsto & \lambda & \mapsto & \lambda|_{I_F}. \end{array}$$

The components of $X({}^LG)$ are again indexed by $\widehat{G}$-conjugacy classes of those $\phi_0 \in \mathrm{Hom}_{W_F}(I_F, {}^LG)$ that lie in the image of $\mathrm{Hom}_{W_F}(W_F, {}^LG) \to \mathrm{Hom}_{W_F}(I_F, {}^LG)$. The fibre of $X({}^LG) \to \mathrm{Hom}_{W_F}(W_F, {}^LG)$ above $\lambda \in R({}^LG) \subseteq X({}^LG)$ is precisely the affine variety $X_\lambda({}^LG)$ defined in (46).

Now, with reference to the definition of λ_ϕ from (38) and the definition of x_ϕ in Proposition 2.2.2, consider the map

$$\begin{array}{ccc} P({}^LG) & \to & X({}^LG) \\ \phi & \mapsto & (\lambda_\phi, x_\phi). \end{array} \tag{48}$$

It follows from Proposition 2.2.2 that the image of this map is $\{(\lambda, x) \in X({}^LG) \mid \lambda \in R({}^LG)\}$ and the fibre of $P({}^LG) \to X({}^LG)$ above any (λ, x) in its image is a principal homogeneous space for the unipotent radical of $Z_{\widehat{G}}(x)$, and moreover that $P({}^LG) \to X({}^LG)$ induces a bijection

$$\begin{gathered} \Phi({}^LG) \longrightarrow X({}^LG)/\widehat{G}, \\ [\phi] \mapsto S_\phi. \end{gathered}$$

Though the map (48) is neither injective nor surjective, in general, and though $X({}^LG)$ is not of finite type over $\mathbb{C}$, in general, we refer to $X({}^LG)$ as the *parameter variety* for G.

We note that $X({}^LG)$ is stratified into $\widehat{G}$-orbit varieties, locally closed in $X({}^LG)$; this stratification is not finite, in general, but it is closure-finite. For each $\widehat{G}$-orbit $S \subseteq X({}^LG)$, there is some $\lambda \in \mathrm{Hom}_{W_F}(W_F, {}^LG)$ such that $S \subseteq X_\lambda({}^LG)$. Then $\bar{S}$, the closure of S in $X({}^LG)$, is also contained in $X_\lambda({}^LG)$. It is essentially for this reason that this paper is concerned with the affine varieties $X_\lambda({}^LG)$, for $[\lambda] \in \Lambda(G/F)$, rather than the full parameter variety $X({}^LG)$.

2.4. **Equivariant perverse sheaves.** The definitive reference for perverse sheaves is [6], and we will use notation from that paper here, but equivariant perverse sheaves do not appear in [6], so we now briefly describe that category and some properties that will be important to us. Our treatment is consistent with [9, Section 5].

Let $m : H \times V \to V$ be a group action in the category of algebraic varieties. So, in particular, H is an algebraic group, but need not be connected. Consider the morphisms

$$\begin{tikzcd} H \times H \times V \arrow[r, shift left=2, "m_1{,}m_2{,}m_3"] \arrow[r] \arrow[r, shift right=2] & H \times V \arrow[r, shift left, "m"] \arrow[r, shift right, "m_0"'] & V \arrow[l, bend right, "s"'] \end{tikzcd}$$

where $m_0 : H \times V \to V$ is projection, $s : V \to H \times V$ is defined by $s(x) = (1, x)$ and $m_1, m_2, m_3 : H \times H \times V \to H \times V$ are defined by

$$\begin{aligned} m_1(h_1, h_2, x) &= (h_1 h_2, x) \\ m_2(h_1, h_2, x) &= (h_1, m(h_2, x)) \\ m_3(h_1, h_2, x) &= (h_2, x). \end{aligned}$$

These are all smooth morphisms. An object in $\mathsf{Per}_H(V)$ is a pair $(\mathcal{A}, \alpha)$ where $\mathcal{A} \in \mathsf{Per}(V)$ and

$$\alpha : m^*[\dim H]\mathcal{A} \to m_0^*[\dim H]\mathcal{A} \tag{49}$$

is an isomorphism in $\mathsf{Per}(H \times V)$ such that

$$s^*(\alpha) = \mathrm{id}_{\mathcal{A}}[\dim H] \tag{50}$$

and such that the following diagram in $\mathsf{Per}(H \times H \times V)$, which makes implicit use of [6, 1.3.17] commutes:

$$\begin{tikzcd} m_2^*[\dim H]m^*[\dim H]\mathcal{A} \arrow[r, "m_2^*[\dim H](\alpha)"] \arrow[d, "m\circ m_1 = m\circ m_2"] & m_2^*[\dim H]m_0^*[\dim H]\mathcal{A} \arrow[d, "m_0\circ m_2 = m\circ m_3"'] \\ m_1^*[\dim H]m^*[\dim H]\mathcal{A} \arrow[d, "m_1^*[\dim H](\alpha)"] & m_3^*[\dim H]m^*[\dim H]\mathcal{A} \arrow[d, "m_3^*[\dim H](\alpha)"'] \\ m_1^*[\dim H]m_0^*[\dim H]\mathcal{A} & m_3^*[\dim H]m_0^*[\dim H]\mathcal{A}. \arrow[l, "m_0\circ m_3 = m_0\circ m_1"'] \end{tikzcd} \tag{51}$$

We remark that ${}^p\mathrm{H}^{\dim H}\, m^* = m^*[\dim H]$ on $\mathsf{Per}(V)$ and ${}^p\mathrm{H}^{\dim H}\, m_i^* = m_i^*[\dim H]$ on $\mathsf{Per}(H \times V)$ for $i = 1, 2, 3$; see [6, 4.2.4]. This does *not* require connected H.

Morphisms of H-equivariant perverse sheaves $(\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$ are morphisms of perverse sheaves $\phi : A \to B$ for which the diagram

$$\begin{tikzcd} m^*[\dim H]A \arrow[r, "m^*[\dim H](\phi)"] \arrow[d, "\alpha"'] & m^*[\dim H]B \arrow[d, "\beta"] \\ m_0^*[\dim H]A \arrow[r, "m_0^*[\dim H](\phi)"] & m_0^*[\dim H]B \end{tikzcd} \tag{52}$$

commutes. This defines $\mathsf{Per}_H(V)$, the category of H-equivariant perverse sheaves on V.

The category $\mathsf{Per}_H(V)$ comes equipped with the forgetful functor

$$\mathsf{Per}_H(V) \to \mathsf{Per}(V)$$

trivial on morphisms and given on objects by $(\mathcal{A}, \alpha) \to \mathcal{A}$. This is a special case of a more general construction called equivariant pullback. Let $m : H \times V \to V$ and $m' : H' \times V' \to V'$ be actions. Let $u : H' \to H$ be a morphism in the category of algebraic groups and suppose H' acts on V and H acts on V. A morphism $f : V' \to V$ is *equivariant* (with respect to u) if

$$\begin{tikzcd} H' \times V' \arrow[r, "m'"] \arrow[d, "u\times f"'] & V' \arrow[d, "f"] \\ H \times V \arrow[r, "m"] & V \end{tikzcd}$$

commutes. Then for every $i \in \mathbb{Z}$ there is a functor ${}^p\mathrm{H}^i_u\, f^* : \mathsf{Per}_H(V) \to \mathsf{Per}_{H'}(V')$ making

$$\begin{tikzcd} \mathsf{Per}_{H'}(V') \arrow[d, "\text{forget}"'] & \mathsf{Per}_H(V) \arrow[l, "{}^p\mathrm{H}^i_u\, f^*"'] \arrow[d, "\text{forget}"] \\ \mathsf{Per}(V') & \mathsf{Per}(V) \arrow[l, "{}^p\mathrm{H}^i\, f^*"'] \end{tikzcd}$$

commute; we call this equivariant pullback. The forgetful functor above is just ${}^p\mathrm{H}^0_1\, \mathrm{id}_V^*$, where $u : 1 \to H$.

The category $\mathsf{Per}_H(V)$ also comes equipped with the forgetful functor

$$\mathsf{Per}_H(V) \to \mathsf{Per}_{H^0}(V)$$

where H^0 is the identity component of H. The category $\mathsf{Per}_{H^0}(V)$ is easier to study than $\mathsf{Per}_H(V)$, since the functor $\mathsf{Per}_{H^0}(V) \to \mathsf{Per}(V)$ is faithful, which is generally not the case for $\mathsf{Per}_H(V) \to \mathsf{Per}(V)$ when H is not connected. The following lemma shows how $\mathsf{Per}_H(V)$ is related to $\mathsf{Per}_{H^0}(V)$.

Lemma 2.4.1. *Let $m : H \times V \to V$ be a group action in the category of algebraic varieties. Suppose V is smooth and connected. We have a sequence of functors*

$$\begin{tikzcd} \mathsf{Rep}(\pi_0(H)) \arrow[r, "E\mapsto E_V{[\dim V]}"] & \mathsf{Per}_H(V) \arrow[r, shift left, "\textit{forget}:\ \mathcal{P}\mapsto\mathcal{P}_0"] & \mathsf{Per}_{H^0}(V) \arrow[l, shift left, "\pi_*"] \end{tikzcd}$$

such that:

- ***(a)*** *for every $E \in \mathsf{Rep}(\pi_0(H))$, $(E_V[\dim V])_0 \cong \mathbb{1}_V^{\dim E}[\dim V]$;*
- ***(b)*** *the functor $\mathsf{Rep}(\pi_0(H)) \to \mathsf{Per}_H(V)$ is fully faithful and its essential image is the category of perverse local systems $\mathcal{L}[\dim V] \in \mathsf{Per}_H(V)$ such that $(\mathcal{L}[\dim V])_0 \cong \mathbb{1}_V^{\dim \mathcal{L}}[\dim V]$;*
- ***(c)*** *the forgetful functor $\mathsf{Per}_H(V) \to \mathsf{Per}_{H^0}(V)$ is exact and admits isomorphic left and right adjoints $\pi_* : \mathsf{Per}_{H^0}(V) \to \mathsf{Per}_H(V)$;*
- ***(d)*** *every $\mathcal{P} \in \mathsf{Per}_H(V)$ is a summand of $\pi_*\mathcal{P}_0$.*

Proof. The identity $\mathrm{id}_V : V \to V$ is equivariant with respect to the inclusion $u : H^0 \to H$ of the identity component of H. Consider the functor

$${}^p\mathrm{H}^0_u\, \mathrm{id}_V^* : \mathsf{Per}_H(V) \to \mathsf{Per}_{H^0}(V).$$

The trivial map $0 : V \to 0$ is equivariant with respect to the quotient $\pi_0 : H \to \pi_0(H)$.

$$\begin{tikzcd} H \times V \arrow[r] \arrow[d] & V \arrow[d] \\ \pi_0(H) \times 0 \arrow[r] & 0 \end{tikzcd}$$

Consider the functor

$${}^p\mathrm{H}^{\dim H}_{\pi_0}\, 0^* : \mathsf{Per}_{\pi_0(H)}(0) \to \mathsf{Per}_H(V).$$

Then

$$({}^p\mathrm{H}^{\dim H}_{\pi_0}\, 0^*)\ ({}^p\mathrm{H}^0_u\, \mathrm{id}_V^*) \cong {}^p\mathrm{H}^{\dim H}_0\, 0^*$$

and we have a sequence of functors

$$\begin{tikzcd} \mathsf{Per}_{\pi_0(H)}(0) \arrow[r] & \mathsf{Per}_H(V) \arrow[r, "\text{forget}"] & \mathsf{Per}_{H^0}(V) \end{tikzcd}$$

The tensor category $\mathsf{Per}_{\pi_0(H)}(0)$ is equivalent to $\mathsf{Rep}(\pi_0(H))$, the category of representations of the finite group $\pi_0(H)$. Property (a) now follows from the canonical isomorphism of functors above.

Since V is smooth, the functor $\mathsf{Rep}(\pi_0(H)) \to \mathsf{Per}_H(V)$ is given explicitly by $E \mapsto E_V[\dim V]$; this functor is full and faithful by, for example, [6, Corollaire 4.2.6.2], from which we also find the adjoint functors $\mathsf{Per}_H(V) \to \mathsf{Rep}(\pi_0(H))$ and Property (b). Connectedness of V plays a role here.

To see Property (c), set $\tilde{V} = H \times_{H^0} V$ and consider the closed embedding $i : V \to \tilde{V}$ given by $i(x) = [1, x]_{H^0}$. By descent, equivariant pullback

$$ {}^p\mathrm{H}^0_u\, i^* : \mathsf{Per}_H(\tilde{V}) \to \mathsf{Per}_{H^0}(V) $$

is an equivalence. Now consider the morphism

$$ \begin{aligned} c : \tilde{V} &\to V \\ [h, x]_{H^0} &\mapsto h \cdot x. \end{aligned} $$

Then $c : \tilde{V} \to V$ is an H-equivariant finite etale cover with group $\pi_0(H) = H/H^0$. In fact, $\tilde{V} \cong V \times H/H_0$ and c is simply the composition of this isomorphism with projection $V \times H/H_0 \to V$. Since c is proper and semismall, the adjoint to pullback

$$ {}^p\mathrm{H}^0\, c^* : \mathsf{Per}(V) \to \mathsf{Per}(\tilde{V}) $$

takes perverse sheaves to perverse sheaves,

$$ {}^p\mathrm{H}^0\, c_* : \mathsf{Per}(\tilde{V}) \to \mathsf{Per}(V) $$

and coincides with ${}^p\mathrm{H}^0\, c_!$; see also [6, Corollaire 2.2.6]. To see that the adjoint extends to a functor of equivariant perverse sheaves, define

$$ {}^p\mathrm{H}^0_H\, c_* : \mathsf{Per}_H(\tilde{V}) \to \mathsf{Per}_H(V) $$

as follows. On objects, ${}^p\mathrm{H}^0_H\, c_*(\tilde{\mathcal{A}}, \tilde{\alpha}) = (\mathcal{A}, \alpha)$ with $\mathcal{A} = {}^p\mathrm{H}^0\, c_*\tilde{\mathcal{A}}$ while the isomorphism $\alpha : {}^p\mathrm{H}^{\dim H}\, m^*\mathcal{A} \to {}^p\mathrm{H}^{\dim H}\, m_0^*\mathcal{A}$ in $\mathsf{Per}(H \times V)$ is defined by the following diagram of isomorphisms.

$$ \begin{tikzcd} {}^p\mathrm{H}^{\dim H}\, m^*\mathcal{A} \arrow[r, "\alpha"] \arrow[d, equal] & {}^p\mathrm{H}^{\dim H}\, m_0^* A \arrow[d, equal] \\ {}^p\mathrm{H}^{\dim H}\, m^*({}^p\mathrm{H}^0\, c_*\tilde{\mathcal{A}}) \arrow[d, "\text{smooth base change}"] & {}^p\mathrm{H}^{\dim H}\, m_0^*({}^p\mathrm{H}^0\, c_*\tilde{\mathcal{A}}) \arrow[d, "\text{smooth base change}"'] \\ {}^p\mathrm{H}^0(\mathrm{id}_H \times c)_*\ {}^p\mathrm{H}^{\dim H}(\tilde{m})^* A \arrow[r, "{}^p\mathrm{H}^0(\mathrm{id}_H \times c)_*(\tilde{\alpha})"] & {}^p\mathrm{H}^0(\mathrm{id}_H \times c)_*\ {}^p\mathrm{H}^{\dim H}(\tilde{m}_0)^* A \end{tikzcd} $$

It is straightforward to verify that α satisfies (50) and (51) as they apply here and also that if $\tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ is a map in $\mathsf{Per}_H(\tilde{V})$ then ${}^p\mathrm{H}^i\, c_!(\tilde{\mathcal{A}} \to \tilde{\mathcal{B}})$ satisfies condition (52), so is a map in $\mathsf{Per}_H(V)$. By this definition of ${}^p\mathrm{H}^i_H\, c_* : \mathsf{Per}_H(\tilde{V}) \to \mathsf{Per}_H(V)$, it follows immediately that the diagram

$$ \begin{tikzcd} \mathsf{Per}_H(\tilde{V}) \arrow[r, "{}^p\mathrm{H}^0_H\, c_*"] \arrow[d, "\text{forget}"'] & \mathsf{Per}_H(V) \arrow[d, "\text{forget}"] \\ \mathsf{Per}(\tilde{V}) \arrow[r, "c_* = {}^p\mathrm{H}^0\, c_*"] & \mathsf{Per}(V) \end{tikzcd} $$

commutes. Now, we define the adjoint $\pi_* : \mathsf{Per}_{H^0}(V) \to \mathsf{Per}_H(V)$ by the following diagram

$$\begin{tikzcd}
\mathsf{Per}_{H^0}(V) \arrow[rr, "\pi_*"] & & \mathsf{Per}_H(V) \\
& \mathsf{Per}_H(\tilde{V}). \arrow[ul, "\text{equiv.}", "{}^p\mathrm{H}^0_u\, i^*"'] \arrow[ur, "{}^p\mathrm{H}^0_H\, c_*"'] &
\end{tikzcd}$$

This shows Property (c).

Property (d) follows from the Decomposition Theorem applied to $c : \tilde{V} \to V$. ☐

2.5. Equivariant perverse sheaves on parameter varieties.

Our fundamental object of study is the category $\mathsf{Per}_{\widehat{G}}(X_\lambda)$ of $\widehat{G}$-equivariant perverse sheaves on $X({}^LG)$, for fixed $[\lambda] \in \Lambda(G/F)$. Consider the closed embedding

$$\begin{array}{ccc} V_\lambda & \to & X_\lambda \\ x & \mapsto & [1,x]_{H_\lambda}. \end{array}$$

By a simple application of equivariant descent, the functor obtained by equivariant pullback along $V_\lambda \to X_\lambda$,

$$\mathsf{Per}_{H_\lambda}(V_\lambda) \leftarrow \mathsf{Per}_{\widehat{G}}(X_\lambda),$$

is an equivalence. Consequently, it may equally be said that our fundamental object of study is the category $\mathsf{Per}_{H_\lambda}(V_\lambda)$ of H_λ-equivariant perverse sheaves on V_λ.

Now define

$$\tilde{X}_\lambda := \widehat{G} \times_{H_\lambda^0} V_\lambda. \tag{53}$$

Then

$$\begin{array}{ccc} V_\lambda & \to & \tilde{X}_\lambda \\ x & \mapsto & [1,x]_{H_\lambda^0} \end{array}$$

induces an equivalence

$$\mathsf{Per}_{\widehat{G}}(\tilde{X}_\lambda) \to \mathsf{Per}_{H_\lambda^0}(V_\lambda).$$

Define

$$\begin{aligned} c_\lambda : \tilde{X}_\lambda &\to X_\lambda \\ [h,x]_{H_\lambda^0} &\mapsto [h,x]_{H_\lambda}. \end{aligned}$$

Arguing as in Section 2.4, it follows that there is a sequence of exact functors

$$\begin{tikzcd}
\mathsf{Rep}(A_\lambda) \arrow[r, "E \mapsto E_{X_\lambda}[\dim X_\lambda]"] & \mathsf{Per}_{\widehat{G}}(X_\lambda) \arrow[r, shift left, "(c_\lambda)^*"] \arrow[d, "\text{equiv}"] & \mathsf{Per}_{\widehat{G}}(\tilde{X}_\lambda) \arrow[l, shift left, "(c_\lambda)_*"] \arrow[d, "\text{equiv}"] \\
& \mathsf{Per}_{H_\lambda}(V_\lambda) & \mathsf{Per}_{H_\lambda^0}(V_\lambda)
\end{tikzcd}$$

enjoying the properties of Lemma 2.4.1.

2.6. **Langlands component groups are equivariant fundamental groups.** Now that we have a precise definition of $\mathsf{Per}_{H_\lambda}(V_\lambda)$, we consider its simple objects.

Every simple object in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ takes the form $\mathcal{IC}(C,\mathcal{L})$, where C is an H_λ-orbit in V_λ and $\mathcal{L}$ is a simple equivariant local system on C. Thus, simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ are parametrised by pairs (C,ρ) where C is an H_λ-orbit in V_λ and ρ is an isomorphism class of irreducible representations of the equivariant fundamental group A_C of C. To calculate that group, we may pick a base point $x \in C$ so

$$A_C \cong \pi_1(C,x)_{H_\lambda^0}. \tag{54}$$

We are left with a canonical bijection:

$$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} \leftrightarrow \{(C,\gamma) \mid H_\lambda\text{-orbit } C \subseteq V_\lambda,\ \rho \in \mathrm{Irrep}(A_C)\}.$$

Lemma 2.6.1. *For any Langands parameter $\phi : L_F \to {}^LG$,*

$$A_{C_\phi} = A_\phi,$$

where $C_\phi \subseteq V_{\lambda_\phi}$ is the H_{λ_ϕ}-orbit of x_ϕ; see Proposition 2.2.2.

Proof. Recall from Section 1.3 that the component group for a Langlands parameter ϕ is given by$A_\phi = \pi_0(Z_{\widehat{G}}(\phi)) = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^0$. Since $\lambda_\phi(W_F) \subseteq \phi(L_F)$, $A_\phi = \pi_0(Z_{H_{\lambda_\phi}}(\phi))$. On the other hand, the equivariant fundamental group of C_ϕ is $\pi_1(C_\phi,x_\phi)_{H_{\lambda_\phi}} = \pi_0(Z_{H_{\lambda_\phi}}(x_\phi))$. From the proof of Proposition 2.2.2 we see that $Z_{H_{\lambda_\phi}}(x_\phi) = Z_{H_{\lambda_\phi}}(\phi)U$, where U is a connected unipotent group. It follows that

$$\pi_0(Z_{H_{\lambda_\phi}}(x_\phi)) = \pi_0(Z_{H_{\lambda_\phi}}(\phi)U) = \pi_0(Z_{H_{\lambda_\phi}}(\phi)),$$

which concludes the proof. □

The following proposition is one of the fundamental ideas in [32]. Because our set up is slightly different, however, we include a proof here.

Proposition 2.6.2. *Suppose G is quasi-split. The local Langlands correspondence for pure rational forms determines a bijection between the set of isomorphism classes of simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ and those of $\Pi_{pure,\lambda}(G/F)$ as defined in Section 2.1:*

$$\mathsf{Per}_{H_\lambda}(V_\lambda)^{simple}_{/iso} \leftrightarrow \Pi_{pure,\lambda}(G/F).$$

Proof. We have already seen (39) that the local Langlands correspondence for pure rational forms gives a bijection between $\Pi_{\text{pure},\lambda}(G/F)$ and

$$\{([\phi],\epsilon) \mid [\phi] \in \Phi_\lambda({}^LG), \epsilon \in \mathrm{Irrep}(A_\phi)\}$$

Proposition 2.2.2 gives a canonical bijection between $\Phi_\lambda({}^LG)$ and the set of H_λ-orbits in V_λ. When $C \leftrightarrow [\phi]$ under this bijection, Lemma 2.6.1, gives a bijection between $\mathrm{Irrep}(A_C)$ and $\mathrm{Irrep}(A_\phi)$. □

We introduce some convenient notation for use below. For $[\pi,\delta] \in \Pi_\lambda(G/F)$, let $\mathcal{P}(\pi,\delta)$ be a simple perverse sheaf in the isomorphism class determined by $[\pi,\delta]$ using Proposition 2.6.2:

$$\begin{array}{rcl} \Pi_{\text{pure},\lambda}(G/F) & \to & \mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} \\ [\pi,\delta] & \mapsto & \mathcal{P}(\pi,\delta). \end{array}$$

Conversely, for a simple perverse sheaf $\mathcal{P} = \mathcal{IC}(C, \mathcal{L})$ in $\mathsf{Per}_{H_\lambda}(V_\lambda)$, let $\chi_{\mathcal{P}}$ be the character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ obtained by pullback along

$$\pi_0(Z(\widehat{G})^{\Gamma_F}) \to \pi_0(Z_{\widehat{G}}(x)) \tag{55}$$

from the representation of $\pi_0(Z_{\widehat{G}}(x))$ determined by the choice of a base point $x \in C$ and the equiviariant local system $\mathcal{L}$ on C. Let $\delta_{\mathcal{P}} \in Z^1(F, G)$ be a pure rational form of G representing the class determined by $\chi_{\mathcal{P}}$ under the Kottwitz isomorphism. Let $\pi_{\mathcal{P}}$ be an admissible representation of $G_{\delta_{\mathcal{P}}}(F)$ such that $[\pi_{\mathcal{P}}, \delta_{\mathcal{P}}]$ matches $\mathcal{P}$ under Proposition 2.6.2:

$$\begin{array}{rcl} \mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} & \to & \Pi_{\text{pure},\lambda}(G/F) \\ \mathcal{P} & \mapsto & [\pi_{\mathcal{P}}, \delta_{\mathcal{P}}]. \end{array}$$

3. Reduction to unramified parameters

Let G be an arbitrary connected reductive algebraic group over a p-adic field F.

3.1. **Unramification.** In this section we show that the study of $\mathsf{Per}_{\widehat{G}}(X_\lambda)$ may be reduced to the study of $\mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{\text{nr}}})$ for a split connected reductive group G_λ and an unramified infinitesimal parameter $\lambda_{\text{nr}} : W_F \to {}^LG_\lambda$. Moreover, we show how the tools developed in [25] may be brought to bear on $\mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{\text{nr}}})$. The group G_λ that appears in Theorem 3.1.1 is sometimes an endoscopic group for G , but not in general; nonetheless, the principle of functoriality applies here through the inclusion of L-groups $r_\lambda : {}^LG_\lambda \to {}^LG$.

Theorem 3.1.1. *Let $\lambda : W_F \to {}^LG$ be an infinitesimal parameter.*

(a) *There is a connected reductive group G_λ, split over F, and an infinitesimal parameter $\lambda_{nr} : W_F \to {}^LG_\lambda$ for G_λ, trivial on I_F, and an inclusion of L-groups $r_\lambda : {}^LG_\lambda \to {}^LG$ such that the following diagram commutes*

$$\begin{tikzcd} W_F \arrow[r, "\lambda"] & {}^LG \\ W_F \arrow[u] \arrow[r, "\lambda_{nr}"] & {}^LG_\lambda, \arrow[u, "r_\lambda"'] \end{tikzcd}$$

where $W_F \to W_F$ is trivial on I_F and $\mathrm{Fr} \mapsto \mathrm{Fr}$ (chosen in Section 1.1).

(b) *By equivariant pullback, the inclusion of L-groups $r_\lambda : {}^LG_\lambda \to {}^LG$ defines an equivalence*

$$\mathsf{Per}_{\widehat{G}}(\tilde{X}_\lambda) \to \mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}})$$

where $\tilde{X}_\lambda$ is defined in Section 2.5, (53).

(c) *There is a sequence of exact functors*

$$\mathsf{Rep}(A_\lambda) \xrightarrow{E \mapsto E_{X_\lambda}[\dim X_\lambda]} \mathsf{Per}_{\widehat{G}}(X_\lambda) \underset{(c_\lambda)_*}{\overset{(c_\lambda)^*}{\rightleftarrows}} \mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}})$$

enjoying the properties of Lemma 2.4.1, where A_λ is defined by (37).

(d) *There is a connected complex reductive algebraic group M_λ, a co-character $\iota : \mathbb{G}_m \to M_\lambda$ and an integer n such that*

$$\mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}}) \equiv \mathsf{Per}_{M_\lambda^\iota}(\mathfrak{m}_{\lambda,n}),$$

where $\mathfrak{m}_{\lambda,n}$ is the weight-n space of $\mathrm{Ad}(\iota)$ acting on $\mathfrak{m}_\lambda = \mathrm{Lie}\, M_\lambda$.

The proof of Theorem 3.1.1 will be given in Section 3.5.

3.2. **Elliptic and hyperbolic semisimple elements in L-groups.** Recall that a semisimple element x of a complex reductive group is H is called *hyperbolic* (resp. *elliptic*) if for every torus D containing x and every rational character $\chi : D \to \mathbb{G}_m(\mathbb{C})$ of D, $\chi(x)$ is a positive real number (resp. $\chi(x)$ has complex norm 1). An arbitrary semisimple element can be uniquely decomposed as a commuting product of hyperbolic and elliptic semisimple elements. An element commutes with x if and only if it commutes with its hyperbolic and elliptic parts separately.

Recall that an element $g \in {}^L G$ is semisimple if $\mathrm{Int}(g)$ is a semisimple automorphism of $\widehat{G}$. Then $g = f \rtimes w \in {}^L G$ is semisimple if and only if $f' \in \widehat{G}$ is semisimple where $(f \rtimes w)^N = f' \rtimes w^N$ and w^N acts trivially on $\widehat{G}$.

The hyperbolic and elliptic parts of a semisimple $g = f \rtimes \mathrm{Fr} \in {}^L G$ are defined as follows. Let N be as above, so $(f \rtimes \mathrm{Fr})^N = f' \rtimes \mathrm{Fr}^N$ and Fr^N acts trivially on $\widehat{G}$. Then $f' \in \widehat{G}$ is semisimple. Let $s' \in \widehat{G}$ be the hyperbolic part of f' and let $t' \in \widehat{G}$ be the elliptic part of f'. Let s be the unique hyperbolic element of $\widehat{G}$ such that $s^N = s'$. It is clear that s is independent of N. Set $t = s^{-1} f$. We call $s \rtimes 1$ the *hyperbolic part of* $f \rtimes \mathrm{Fr}$ and $t \rtimes \mathrm{Fr}$ the *elliptic part of* $f \rtimes w$. Then $\mathrm{Ad}(s) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$ is the hyperbolic part of the semisimple automorphism $\mathrm{Ad}(f \rtimes \mathrm{Fr}) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$ and $\mathrm{Ad}(t \rtimes \mathrm{Fr}) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$ is the elliptic part of the semisimple automorphism $\mathrm{Ad}(f \rtimes \mathrm{Fr}) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$. Moreover, ${}^{\mathrm{Fr}}s = t^{-1} s t$, so

$$(s \rtimes 1)(t \rtimes \mathrm{Fr}) = (t \rtimes \mathrm{Fr})(s \rtimes 1).$$

Lemma 3.2.1. *Write* $\lambda(\mathrm{Fr}) = f_\lambda \rtimes \mathrm{Fr}$*; let* $s_\lambda \rtimes 1$ *be the hyperbolic part of* $\lambda(\mathrm{Fr})$ *and let* $t_\lambda \rtimes \mathrm{Fr}$ *be the elliptic part of* $\lambda(\mathrm{Fr})$*. Then* $s_\lambda \in H_\lambda^0$ *and* K_λ *is normalised by* $f_\lambda \rtimes \mathrm{Fr}$ *and by* $t_\lambda \rtimes \mathrm{Fr}$.

Proof. let I_F' be the kernel of $\rho : \Gamma_F \to \mathrm{Aut}(\widehat{G})$ restricted to I_F. Then I_F' is an open subgroup of I_F and I_F' is normalised by Fr^N in W_F, with N as above. Set $I_F^0 = \lambda^{-1}(1 \rtimes I_F') \subseteq I_F'$. By continuity of λ, I_F^0 is an open subgroup of I_F. Then $\lambda(\mathrm{Fr}^N)$ normalises $\lambda(I_F^0)$. Since $\lambda(\mathrm{Fr}^N)$ also normalises $\lambda(I_F)$, we see $\lambda(\mathrm{Fr}^N)$ acts on the finite group $\lambda(I_F)/\lambda(I_F^0)$. In particular, replacing N by a larger integer if necessary, it follows that $\lambda(\mathrm{Fr}^N)$ acts on $\lambda(I_F)/\lambda(I_F^0)$ trivially.

Recall the notation $\lambda(\mathrm{Fr}) = f_\lambda \rtimes \mathrm{Fr}$ and $\lambda(\mathrm{Fr}^N) = f' \rtimes \mathrm{Fr}^N$. We now show $f' \in Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda$. For any $h \rtimes w \in \lambda(I_F)$,

$$\lambda(\mathrm{Fr}^N)(h \rtimes w)(\lambda(\mathrm{Fr}^N))^{-1} = h \rtimes ww'$$

for some $w' \in I_F^0$. Since $\lambda(\mathrm{Fr}^N) = f' \rtimes \mathrm{Fr}^N = (1 \rtimes \mathrm{Fr}^N)(f' \times 1)$, we get

$$\mathrm{Fr}^N f'(h \rtimes w) f'^{-1} \mathrm{Fr}^{-N} = h \rtimes ww'.$$

This implies

$$f' h w (f'^{-1}) \rtimes w = \mathrm{Fr}^{-N}(h \rtimes ww')\mathrm{Fr}^N = h \rtimes \mathrm{Fr}^{-N} ww' \mathrm{Fr}^N.$$

Therefore, $f' h w (f'^{-1}) = h$ and $w = \mathrm{Fr}^{-N} ww' \mathrm{Fr}^N$. From the first equality, we can conclude $f'(h \rtimes w) f'^{-1} = h \rtimes w$. Hence $f' \in Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda$.

Since some power of f' will lie in $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$, replacing N by a larger integer if necessary, we may conclude that f' actually belongs to $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$. In particular, we can take both s' and t' in K_λ^0.

Since $\lambda(\mathrm{Fr}^N) = \lambda(\mathrm{Fr})^{-1} \lambda(\mathrm{Fr}^N) \lambda(\mathrm{Fr})$, we have

$$f' \rtimes \mathrm{Fr}^N = (f_\lambda \rtimes \mathrm{Fr})^{-1}(f' \rtimes \mathrm{Fr}^N)(f_\lambda \rtimes \mathrm{Fr}) = \big((f_\lambda \rtimes \mathrm{Fr})^{-1} f' (f_\lambda \rtimes \mathrm{Fr})\big) \rtimes \mathrm{Fr}^N.$$

Thus, $f' = \lambda(\mathrm{Fr})^{-1} f' \lambda(\mathrm{Fr})$. Since $\lambda(\mathrm{Fr})$ normalises $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$, we have

$$f' = \lambda(\mathrm{Fr})^{-1} f' \lambda(\mathrm{Fr}) = (\lambda(\mathrm{Fr})^{-1} s' \lambda(\mathrm{Fr}))(\lambda(\mathrm{Fr})^{-1} t' \lambda(\mathrm{Fr})),$$

where, as above, s' is the hyperbolic part of f' and t' is the elliptic part of f'. Since the decomposition of a semisimple element of $\widehat{G}$ into hyperbolic and elliptic parts is unique, we have

$$s' = \lambda(\mathrm{Fr})^{-1} s' \lambda(\mathrm{Fr}) \qquad \text{and} \qquad t' = \lambda(\mathrm{Fr})^{-1} t' \lambda(\mathrm{Fr}).$$

In particular, it now follows that $s' \in Z_{\widehat{G}}(\lambda)^0 = H_\lambda^0$. Since $s_\lambda^N = s'$, it follows that $s_\lambda \in Z_{\widehat{G}}(\lambda)^0 = H_\lambda^0$, also.

The Frobenius element Fr normalises I_F, so $\lambda(\mathrm{Fr}) = f_\lambda \rtimes \mathrm{Fr}$ normalises $\lambda(I_F)$ and hence normalises K_λ as well. Since $s_\lambda \in H_\lambda^0 = Z_{\widehat{G}}(\lambda)^0 \subseteq Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda^0$, it follows now that s_λ normalises K_λ; likewise, $t_\lambda \times \mathrm{Fr}$ normalises K_λ. □

3.3. Construction of the unramified parameter.

Define

$$J_\lambda := Z_{\widehat{G}}(\lambda(I_F)) \cap Z_{\widehat{G}}(t_\lambda \rtimes \mathrm{Fr}) = Z_{K_\lambda}(t_\lambda \rtimes \mathrm{Fr}). \tag{56}$$

Lemma 3.2.1 shows that J_λ is a complex reductive algebraic group. It follows from Section 3.2 that $s_\lambda \in J_\lambda^0$ and t normalises J_λ^0.

We now have the following complex reductive groups attached to $\lambda \in R({}^LG)$:

$$H_\lambda \subseteq J_\lambda \subseteq K_\lambda \subseteq \widehat{G}.$$

Let G_λ be the split connected reductive algebraic group over F so that

$${}^LG_\lambda = J_\lambda^0 \times W_F. \tag{57}$$

Define

$$r_\lambda : {}^LG_\lambda \to {}^LG \qquad \text{by} \qquad h \times 1 \mapsto h \rtimes 1 \quad \text{and} \quad 1 \times \mathrm{Fr} \mapsto t_\lambda \rtimes \mathrm{Fr}. \tag{58}$$

Then $r_\lambda : {}^LG_\lambda \to {}^LG$ is a homomorphism of L-groups. Using Lemma 3.2.1, we define an unramified (*i.e.*, trivial on I_F) homomorphism

$$\begin{aligned} \lambda_{\mathrm{nr}} : W_F &\longrightarrow {}^LG_\lambda \\ \mathrm{Fr} &\mapsto s_\lambda \times \mathrm{Fr}. \end{aligned} \tag{59}$$

Lemma 3.3.1. *Let $\lambda : W_F \to {}^LG$ be an infinitesimal parameter. Define $\lambda_{nr} : W_F \to {}^LG_\lambda$ as above. Then*

$$V_{\lambda_{nr}} = V_\lambda \qquad \text{and} \qquad H_{\lambda_{nr}} = H_\lambda^0.$$

Consequently,

$$\mathsf{Per}_{H_{\lambda_{nr}}}(V_{\lambda_{nr}}) = \mathsf{Per}_{H_\lambda^0}(V_\lambda).$$

Proof. Applying (40) to $\lambda_{\mathrm{nr}} : W_F \to {}^LG_\lambda$ gives

$$H_{\lambda_{\mathrm{nr}}} = Z_{J_\lambda^0}(\lambda_{\mathrm{nr}}) = Z_{J_\lambda^0}(s_\lambda) = H_\lambda^0.$$

Applying (41) to $\lambda_{\mathrm{nr}} : W_F \to {}^LG_\lambda$ gives

$$K_{\lambda_{\mathrm{nr}}} = Z_{J_\lambda^0}(\lambda_{\mathrm{nr}}|_{I_F}) = J_\lambda^0.$$

Applying (42) to $\lambda_{\mathrm{nr}} : W_F \to {}^LG_\lambda$ gives

$$V_{\lambda_{\mathrm{nr}}} = V_{\lambda_{\mathrm{nr}}}({}^LG_\lambda) = \{x \in \mathrm{Lie}\, Z_{\widehat{G}_\lambda}(\lambda_{\mathrm{nr}}|_{I_F}) \mid \mathrm{Ad}(\lambda_{\mathrm{nr}}(\mathrm{Fr}))x = q_F\, x\}.$$

Since $\widehat{G}_\lambda = J^0_\lambda$ and $\lambda_{\mathrm{nr}}|_{I_F} = 1$, and since Fr acts trivially on J^0_λ in ${}^LG_\lambda$, we have

$$V_{\lambda_{\mathrm{nr}}} = \{x \in \mathfrak{j}_\lambda \mid \operatorname{Ad}(s_\lambda)x = q_F\, x\}. \tag{60}$$

Then $V_\lambda = V_{\lambda_{\mathrm{nr}}}$ because $\operatorname{Ad}(f_\lambda \rtimes \mathrm{Fr})x = qx$ if and only if $\operatorname{Ad}(t_\lambda \rtimes \mathrm{Fr})x = x$ and $\operatorname{Ad}(s_\lambda)x = qx$. □

Lemma 3.3.1 tells us that the category $\mathsf{Per}_{H^0_\lambda}(V_\lambda)$ determined by $\lambda : W_F \to {}^LG$ can always be apprehended as the category for an *unramified* infinitesimal parameter $\lambda_{\mathrm{nr}} : W_F \to {}^LG_\lambda$. Note, however, that it is $\mathsf{Per}_{H_\lambda}(V_\lambda)$, not $\mathsf{Per}_{H^0_\lambda}(V_\lambda)$ which is needed to study Arthur packets of admissible representations of pure rational forms of $G(F)$; fortunately, Lemma 2.4.1 describes the relation between these two categories.

Remark 3.3.2. Without defining G^+_λ itself, let us set ${}^LG^+_\lambda := J_\lambda \times W_F$ and define $\lambda^+_{\mathrm{nr}} : W_F \to {}^LG^+_\lambda$ by the composition of λ_{nr} and ${}^LG_\lambda \hookrightarrow {}^LG^+_\lambda$. Then (57) may also be used to define $r^+_\lambda : {}^LG^+_\lambda \hookrightarrow {}^LG$ and extends r_λ. Arguing as in the proof of Lemma 3.3.1, it follows that

$$V_{\lambda^+_{\mathrm{nr}}} = V_\lambda \qquad \text{and} \qquad H_{\lambda^+_{\mathrm{nr}}} = H_\lambda,$$

so

$$\mathsf{Per}_{H_{\lambda^+_{\mathrm{nr}}}}(V_{\lambda^+_{\mathrm{nr}}}) = \mathsf{Per}_{H_\lambda}(V_\lambda).$$

We pursue this perspective elsewhere.

3.4. Construction of the cocharacter.

From Section 3.3, recall the definition of $s_\lambda \in \widehat{G}$ and the fact that s_λ lies in the identity component of the subgroup $J_\lambda \subseteq \widehat{G}$. Decompose the Lie algebra $\mathfrak{j}_\lambda$ of J_λ according to $\operatorname{Ad}(s_\lambda)$-eigenvalues:

$$\mathfrak{j}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{j}_\lambda(\nu), \qquad \mathfrak{j}_\lambda(\nu) := \{x \in \mathfrak{j}_\lambda \mid \operatorname{Ad}(s_\lambda)(x) = \nu x\}.$$

Following [25], define

$$\mathfrak{j}^\dagger_\lambda := \bigoplus_{r \in \mathbb{Z}} \mathfrak{j}_\lambda(q_F^r).$$

Lemma 3.4.1. *There is a connected reductive algebraic subgroup M_λ of J^0_λ and a cocharacter $\iota : \mathbb{G}_m \longrightarrow M_\lambda$ such that*

$$M^\iota_\lambda = H_{\lambda_{nr}} \qquad \textit{and} \qquad \mathfrak{m}_\lambda = \mathfrak{j}^\dagger_\lambda,$$

where $\mathfrak{m}_\lambda := \operatorname{Lie} M_\lambda$ and an integer n so that, for every $r \in \mathbb{Z}$,

$$\mathfrak{m}_{\lambda, rn} = \mathfrak{j}_\lambda(q_F^r),$$

where $\mathfrak{m}_{\lambda,rn} := \{x \in \mathfrak{m} \mid \operatorname{Ad}(\iota(t))x = t^{rn}x,\ \forall t \in \mathbb{G}_m\}$. In particular,

$$V_\lambda = \mathfrak{j}_\lambda(q_F) = \mathfrak{m}_{\lambda,n}.$$

Proof. Decompose the Lie algebra $\mathfrak{j}_\lambda$ of J_λ according to $\operatorname{Ad}(s_\lambda)$-eigenvalues:

$$\mathfrak{j}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{j}_\lambda(\nu).$$

Fix a maximal torus S of J^0_λ such that $s_\lambda \in S$ and denote the set of roots determined by this choice by $R(S, J^0_\lambda)$. For $\alpha \in R(S, J^0_\lambda)$, denote the root space in $\mathfrak{j}_\lambda$ by $\mathfrak{u}_\alpha$. Then

$$\mathfrak{j}_\lambda(\nu) = \bigoplus_{\substack{\alpha \in R(S, J^0_\lambda) \\ \alpha(s_\lambda) = \nu}} \mathfrak{u}_\alpha. \tag{61}$$

Let $\langle\cdot,\cdot\rangle$ be the natural pairing between $X^*(S)$ and $X_*(S)$. First, let us consider all $\alpha \in R(S, J^0_\lambda)$ such that $\alpha(s_\lambda)$ are integral powers of q. For these roots we can choose $\chi \in X_*(S)\otimes_{\mathbb{Z}}\mathbb{Q}$ so that $\langle\alpha,\chi\rangle = r$ if $\alpha(s_\lambda) = q^r$ for some integer r. Let n be an integer such that $n\chi \in X_*(S)$, and we set $t = (n\chi)(\zeta q^{1/n}) \in S$, where ζ is a primitive n-th root of unity. Now for $\alpha \in R(S, J^0_\lambda)$ such that $\alpha(s_\lambda) = q^r$, we have

$$\alpha(t) = \alpha((n\chi)(\zeta q^{1/n})) = \alpha(\chi(\zeta q^{1/n}))^n = (\zeta q^{1/n})^{rn} = q^r = \alpha(s_\lambda).$$

Next, consider those $\alpha \in R(S, J^0_\lambda)$ such that $\alpha(s_\lambda)$ are not integral powers of q. We have two cases: if $\langle\alpha,\chi\rangle \in \mathbb{Z}$, then $\alpha(t)$ is an integral power of q; if $\langle\alpha,\chi\rangle \notin \mathbb{Z}$, then $\alpha(t) \in \zeta^l\mathbb{R}_{>0}$ for some $0 < l < n$. Since s_λ is hyperbolic, $\alpha(s_\lambda) \in \mathbb{R}_{>0}$ for all $\alpha \in R(S, J^0_\lambda)$, so $\alpha(s_\lambda) \neq \alpha(t)$ in either case. Therefore, we can define $M_\lambda = Z_{J_\lambda}(s_\lambda t^{-1})^0$and take $\iota = n\chi$. □

3.5. **Proof of Theorem 3.1.1.** The essential facts about the groups K_λ, H_λ, J_λ and M_λ are summarised in the following diagram.

$$\begin{tikzcd}
 & & \widehat{G} & \\
 & & K_\lambda := Z_{\widehat{G}}(\lambda(I_F)) \arrow[u, hook] & \\
M^0_\lambda = M_\lambda \arrow[r, tail] & J^0_\lambda \arrow[r, tail] & J_\lambda := Z_{K_\lambda}(t_\lambda \rtimes \mathrm{Fr}) \arrow[r, two heads] \arrow[u, hook] & \pi_0(J_\lambda) \\
M^\iota_\lambda \arrow[r, equal] \arrow[u, hook] & H^0_\lambda \arrow[r, tail] \arrow[u, hook] & H_\lambda = Z_{J_\lambda}(s_\lambda) \arrow[r, two heads] \arrow[u, hook] & \pi_0(H_\lambda) \arrow[u, hook]
\end{tikzcd}$$

From the definitions of G_λ (57), λ_{nr} (59) and $r_\lambda : {}^LG_\lambda \to {}^LG$ (58), we have

$$r_\lambda(\lambda_{\mathrm{nr}}(\mathrm{Fr})) = r_\lambda(s_\lambda \times \mathrm{Fr}) = (s_\lambda \rtimes 1)(t_\lambda \rtimes \mathrm{Fr}) = f_\lambda \rtimes \mathrm{Fr} = \lambda(\mathrm{Fr}). \tag{62}$$

Now, Theorem 3.1.1 follows from a direct application of Lemmas 2.4.1 and 3.4.1, as in the diagram below.

$$\begin{tikzcd}
\mathsf{Rep}(A_\lambda) \arrow[d, equal] & \mathsf{Per}_{\widehat{G}}(X_\lambda) \arrow[r, shift left, "(c_\lambda)^*"] \arrow[d, "\text{equiv}"'] & \mathsf{Per}_{\widehat{G}_\lambda}(X_{\lambda_{\mathrm{nr}}}) \arrow[l, shift left, "(c_\lambda)_*"] \arrow[d, "\text{equiv}"] \\
\mathsf{Rep}(\pi_0(H_\lambda)) \arrow[r] & \mathsf{Per}_{H_\lambda}(V_\lambda) \arrow[r, "\text{forget}"] & \mathsf{Per}_{H^0_\lambda}(V_\lambda) \arrow[d, equal] \\
 & & \mathsf{Per}_{M^\iota_\lambda}(\mathfrak{m}_{\lambda,n})
\end{tikzcd}$$

3.6. **Further properties of Vogan varieties.** From the proof of Theorem 3.1.1 we get a very concrete description of V_λ as a variety, for any $\lambda \in R({}^LG)$, following (61):

$$V_\lambda \cong \mathbb{A}^d, \qquad \text{for} \quad d = |\{\alpha \in R(S, J^0_\lambda) \mid \alpha(s_\lambda) = q_F\}|.$$

Proposition 3.6.1. *The space V_λ is stratified into H_λ-orbits, of which there are finitely many, with a unique open orbit.*

Proof. With Proposition 3.4.1 in hand, this follows immediately from [25, Proposition 3.5] and [25, Section 3.6]. □

A different proof is given in [32, Proposition 4.5].

Proposition 3.6.2. *Every H_λ-orbit in V_λ is a conical variety.*

Proof. By Proposition 3.4.1, it suffices to prove that every M_λ^ι-orbit C in $\mathfrak{m}_{\lambda,n}$ is a conical variety. Arguing as in the proof of [17, Lemma 2.1], for $x \in C$, we can find a homomorphism $\varphi : \mathrm{SL}(2,\mathbb{C}) \to M_\lambda$ such that for $t \in \mathbb{C}^*$

$$\varphi\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \in M_\lambda^\iota \qquad \text{and} \qquad \mathrm{d}\varphi\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} = x.$$

Then

$$\mathrm{Ad}\left(\varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\right)(x) = d\varphi\begin{pmatrix} 0 & t^2 \\ 0 & 0\end{pmatrix} = t^2 x,$$

so $t^2 x \in C$. □

4. Arthur parameters and the conormal bundle

In this section we see how Arthur parameters may be apprehended as certain regular conormal vectors $\xi \in T^*_{C,x}(V_\lambda)_{\mathrm{reg}}$.

Let G is an arbitrary connected reductive linear algebraic group over the p-adic field F.

4.1. **Regular conormal vectors.** For $\lambda \in R({}^L G)$ and every H_λ-orbit $C \subseteq V_\lambda$, let $T^*_C(V_\lambda)_{\mathrm{reg}} \subset T^*_C(V_\lambda)$ be the subvariety defined by

$$T^*_C(V_\lambda)_{\mathrm{reg}} := T^*_C(V_\lambda) \setminus \bigcup_{C \subsetneq \overline{C}_1} \overline{T^*_{C_1}(V_\lambda)}. \tag{63}$$

Also define

$$T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}} := \bigcup_C T^*_C(V_\lambda)_{\mathrm{reg}},$$

the union taken over all H_λ-orbits C in V_λ. Then $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$ is open subvariety of $T^*_{H_\lambda}(V_\lambda)$ and each $T^*_C(V_\lambda)_{\mathrm{reg}}$ is a component in $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$.

We may compose (15) and (38):

$$\begin{array}{ccccc} Q({}^L G) & \to & P({}^L G) & \to & R({}^L G) \\ \psi & \mapsto & \phi_\psi & \mapsto & \lambda_{\phi_\psi}. \end{array} \tag{64}$$

To simplify notation, we set $\lambda_\psi := \lambda_{\phi_\psi}$. We will refer to λ_ψ as the *infinitesimal parameter of ψ*. Using Proposition 2.2.2, define

$$x_\psi := x_{\phi_\psi} \in V_{\lambda_\psi}$$

and let $C_\psi \subseteq V_{\lambda_\psi}$ be the H_λ-orbit of $x_\psi \in V_{\lambda_\psi}$.

Theorem 4.1.1. *Let $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^L G$ be an Arthur parameter. Let $\lambda_\psi : W_F \to {}^L G$ be its infinitesimal parameter. Then ψ determines a regular conormal vector*

$$\xi_\psi \in T^*_{C_\psi, x_\psi}(V_\lambda)_{reg},$$

with the property that the H_λ-orbit of (x_ψ, ξ_ψ) in $T^*_{C_\psi}(V_\lambda)$ is open and dense in $T^*_{C_\psi}(V_\lambda)_{reg}$. The equivariant fundamental group of this orbit is A_ψ.

The proof of Theorem 4.1.1 will be given in Section 4.8.

4.2. **Cotangent space to the Vogan variety.** Consider

$$ {}^tV_\lambda := \{x \in \mathfrak{k}_\lambda \mid \operatorname{Ad}(\lambda(\mathrm{Fr}))(x) = q_F^{-1}x\}, \tag{65} $$

which clearly comes equipped with an action of H_λ just as V_λ comes equipped with an action of H_λ. Compare ${}^tV_\lambda$ with V_λ defined in (42). In fact, the variety ${}^tV_\lambda$ has already appeared: see the proof of Proposition 2.2.2. We note

$$ {}^tV_\lambda = \mathfrak{k}_\lambda(q_F^{-1}) = \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda,-n}, $$

where $\mathfrak{k}$ and $\mathfrak{m}_n$ are defined in Sections 2.2 and 3.4, respectively.

For $\phi : L_F \to {}^LG$, we can define

$$ \begin{aligned} P_\lambda({}^LG) &\longrightarrow {}^tV_\lambda, \\ \phi &\mapsto x_\phi := \mathrm{d}\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{66} $$

where $\varphi := \phi^\circ|_{\mathrm{SL}(2,\mathbb{C})} : \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$. This map satisfies all the properties of the map $P_\lambda({}^LG) \to V_\lambda({}^LG)$ in Proposition 2.2.2, from which it follows that there is a canonical bijection between H_λ-orbits in V_λ and H_λ-orbits in ${}^tV_\lambda$, so that the following diagram commutes.

$$ \begin{tikzcd} P_\lambda({}^LG)/H_\lambda \arrow[r, equal] \arrow[d] & P_\lambda({}^LG)/H_\lambda \arrow[d] \\ V_\lambda/H_\lambda \arrow[r, "\cong"] & {}^tV_\lambda/H_\lambda \end{tikzcd} $$

Proposition 4.2.1. *There is an H_λ-equivariant isomorphism*

$$ T^*(V_\lambda) \simeq V_\lambda \times {}^tV_\lambda, $$

and consequently,

$$ T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda,n} \oplus \mathfrak{m}_{\lambda,-n}. $$

Proof. As V_λ is an affine H_λ-space there is a standard H_λ-equivariant isomorphism $T^*(V_\lambda) \simeq V_\lambda \times V_\lambda^*$, so it suffices to exhibit an H_λ-equivariant isomorphism

$$ V_\lambda^* \cong {}^tV_\lambda. $$

To do this, let J_λ be the reductive group defined in (56) and write $\mathfrak{j}_\lambda$ for $\operatorname{Lie} J_\lambda$, as in Section 3.3. From Proposition 3.4.1, we have

$$ V_\lambda = \mathfrak{j}_\lambda(q_F) \qquad \text{and} \qquad \mathfrak{h}_\lambda = \mathfrak{j}_\lambda(1) \qquad \text{and} \qquad {}^tV_\lambda = \mathfrak{j}_\lambda(q_F^{-1}). $$

As J_λ is reductive, its Lie algebra decomposes into a direct sum of its centre and a semisimple Lie algebra, $\mathfrak{j}_\lambda \simeq Z(\mathfrak{j}_\lambda) \oplus [\mathfrak{j}_\lambda, \mathfrak{j}_\lambda]$. We choose any non-degenerate symmetric bilinear form on $Z(\mathfrak{j}_\lambda)$ and extend to a bilinear form on $\mathfrak{j}_\lambda$ using the Cartan-Killing form, while insisting that the direct sum decomposition above is orthogonal, that is, the components in the direct sum are pairwise perpendicular. The result is a non-degenerate, symmetric, J_λ-invariant bilinear pairing

$$ (\ |\ ) : \mathfrak{j}_\lambda \times \mathfrak{j}_\lambda \to \mathbb{A}^1. $$

Now, if $\mathfrak{j}_\lambda(\nu)$ and $\mathfrak{j}_\lambda(\nu')$ are two $\operatorname{Ad}(s_\lambda)$-weight spaces, then the invariance of the pairing implies that $(\,\mathfrak{j}_\lambda(\nu) \,|\, \mathfrak{j}_\lambda(\nu')\,) \neq 0$ if and only if $\nu' = \nu^{-1}$. Since the pairing is non-degenerate this gives an $Z_{J_\lambda}(s_\lambda) = H_\lambda$-equivariant isomorphism

$$ V_\lambda^* = \mathfrak{j}_\lambda(q_F)^* \cong \mathfrak{j}_\lambda(q_F^{-1}) = {}^tV_\lambda. $$

A similar argument using the cocharacter $\iota : \mathbb{G}_{\mathrm{m}} \to M_\lambda$ and the graded Lie algebra

$$\begin{aligned}\mathfrak{m}_\lambda &= \cdots \oplus \mathfrak{m}_{\lambda,2n} \oplus (\mathfrak{m}_{\lambda,n} \oplus \mathfrak{m}_{\lambda,0} \oplus \mathfrak{m}_{\lambda,-n}) \oplus \mathfrak{m}_{\lambda,-2n} \oplus \cdots \\ &= \cdots \oplus \mathfrak{m}_{\lambda,2n} \oplus (V_\lambda \oplus \mathfrak{h}_\lambda \oplus {}^tV_\lambda) \oplus \mathfrak{m}_{\lambda,-2n} \oplus \cdots\end{aligned}$$

produces an $M_\lambda^\iota = H_\lambda^0$-equivariant isomorphism

$$V_\lambda^* = \mathfrak{m}_{\lambda,n}^* \cong \mathfrak{m}_{\lambda,-n} = {}^tV_\lambda. \tag{67}$$

This allows us to view $T^*(V_\lambda)$ as a subspace of $\mathfrak{m}_\lambda$, even with H_λ-action, and gives H_λ-equivariant isomorphisms

$$T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda,n} \oplus \mathfrak{m}_{\lambda,-n},$$

as desired. □

4.3. Conormal bundle to the Vogan variety.

Proposition 4.3.1. *Let $C \subseteq V_\lambda$ be an H_λ-orbit in V_λ; then*

$$T_C^*(V_\lambda) = \{(x,\xi) \in T^*(V_\lambda) \mid x \in C, [x,\xi] = 0\},$$

where $[\ ,\ ]$ denotes the Lie bracket on $\mathfrak{j}_\lambda$ and where we use Proposition 4.2.1 to identify $T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1})$. Consequently,

$$T_{H_\lambda}^*(V_\lambda) = \{(x,\xi) \in T^*(V_\lambda) \mid [x,\xi] = 0\}.$$

Proof. The map $\mathfrak{h}_\lambda \to T_x(C)$ given by $X \mapsto [x,X]$ is a surjection. So for any $\xi \in \mathfrak{j}_\lambda(q_F^{-1})$, we have $\xi \in T_{C,x}^*(V_\lambda)$ if and only if $0 = (\,\xi \,|\, [x,X]\,) = (\,[\xi,x] \,|\, X\,)$ for all $X \in \mathfrak{h}_\lambda$. As we saw in the proof of Proposition 4.2.1, the pairing restricts non-degenerately to $\mathfrak{h}_\lambda$, so this is also equivalent to require $[x,\xi] = 0$. □

Corollary 4.3.2. $T_{H_\lambda}^*(V_\lambda) \hookrightarrow (\,\cdot\,|\,\cdot\,)^{-1}(0)$.

Proof. If $(x,\xi) \in V_\lambda \times V_\lambda^*$ lies in $T_{H_\lambda}^*(V_\lambda)$ then $[x,\xi] = 0$. Choose an $\mathfrak{sl}_2$-triple (x,y,z) such that $y \in {}^tV_\lambda$, and $z \in \mathfrak{h}_\lambda$. Then,

$$(\,x \,|\, \xi\,) = \frac{1}{2}(\,[z,x] \,|\, \xi\,) = \frac{1}{2}(\,z \,|\, [x,\xi]\,) = 0.$$ □

4.4. Orbit duality.

Using the H_λ-equivariant isomorphism $V_\lambda^* \to {}^tV_\lambda$ of Proposition 4.2.1, we define an H_λ-equivariant isomorphism

$$\begin{aligned} T^*(V_\lambda) &\to T^*({}^tV_\lambda) \\ (x,\xi) &\mapsto (\xi,x), \end{aligned} \tag{68}$$

which we refer to as *transposition*. Just as every H_λ-orbit $C \subset V_\lambda$ determines the conormal bundle

$$T_C^*(V_\lambda) = \left\{(x,\xi) \in V_\lambda \times {}^tV_\lambda \mid x \in C,\ [x,\xi] = 0\right\},$$

every H_λ-orbit $B \subset {}^tV_\lambda$ determines a conormal bundle in $T^*({}^tV_\lambda)$:

$$T_B^*({}^tV_\lambda) = \left\{(\xi,x) \in {}^tV_\lambda \times V_\lambda \mid \xi \in B,\ [\xi,x] = 0\right\}.$$

Lemma 4.4.1. *For every H_λ-orbit C in V_λ there is a unique H_λ-orbit C^* in ${}^tV_\lambda$ so that transposition (68) restricts to an isomorphism*

$$\overline{T_C^*(V_\lambda)} \cong \overline{T_{C^*}^*({}^tV_\lambda)}.$$

The rule $C \mapsto C^*$ is a bijection from H_λ-orbits in V_λ to H_λ-orbits in ${}^tV_\lambda$.

Proof. This is a well-known result. See [30, Corollary 2] for the case when H_λ is connected. The result extends easily to the case when H_λ is not connected. ☐

The orbit C^* is called the *dual orbit* of $C \subseteq V_\lambda$; likewise, the dual orbit of $B \subseteq {}^tV_\lambda$ is denoted by B^*.

Lemma 4.4.2. *If $(x,\xi) \in T^*_C(V_\lambda)_{reg}$ then $\xi \in C^*$, so*

$$T^*_C(V_\lambda)_{reg} \subseteq \{(x,\xi) \in C \times C^* \mid [x,\xi] = 0\}.$$

Proof. Since $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{reg}}$, then (x,ξ) is not contained in any other closures of conormal bundles except for that of C. On the other hand, $(\xi, x) \in T^*_{B_\xi}(V^t_\lambda)$ where B_ξ is the H_λ -orbit of ξ in ${}^tV_\lambda$,, so $\overline{T^*_C(V_\lambda)} \cong \overline{T^*_{B_\xi}(V^t_\lambda)}$. Hence $B_\xi = C^*$, *i.e.*, $\xi \in C^*$. ☐

Proposition 4.4.3. *If $(x,\xi) \in T^*_C(V_\lambda)$ then $(x,\xi) \in C \times C^*$ and $[x,\xi] = 0$ and $(\, x \,|\, \xi \,) = 0$.*

Proof. Combine Lemma 4.3.2 with 4.4.2. ☐

We remark that $(x,\xi) \in C \times C^*$ implies neither $[x,\xi] = 0$ nor $(\, x \,|\, \xi \,) = 0$ in general; several examples to illustrate this fact appear in [10].

We denote the canonical bijection between H_λ-orbits in V_λ and H_λ-orbits in ${}^tV_\lambda$, and vice versa, by

$$C \quad\mapsto\quad {}^tC \qquad\text{and}\qquad B \quad\mapsto\quad B^t.$$

Note the equivariant fundamental groups (54) are preserved:

$$A_C \quad\cong\quad A_{{}^tC} \qquad\text{and}\qquad A_B \quad\cong\quad A_{B^t}.$$

For $C \subseteq V_\lambda$ (resp, $B \subseteq {}^tV_\lambda$) we refer to ${}^tC$ (resp. B^t) as the *transposed orbit* of C (resp. B). Composing orbit transposition with orbit duality defines an involution

$$C \mapsto \widehat{C} := {}^tC^* \tag{69}$$

on the set of H_λ-orbits in V_λ.

4.5. **Strongly regular conormal vectors.** We say that $(x,\xi) \in T^*_C(V_\lambda)$ is *strongly regular* if its H_λ-orbit is open and dense in $T^*_C(V_\lambda)$. We write $T^*_C(V_\lambda)_{\mathrm{sreg}}$ for the strongly regular part of $T^*_C(V_\lambda)_{\mathrm{reg}}$. We set

$$T^*_{H_\lambda}(V_\lambda)_{\mathrm{sreg}} := \bigcup_C T^*_C(V_\lambda)_{\mathrm{sreg}}.$$

Proposition 4.5.1.

$$T^*_{H_\lambda}(V_\lambda)_{sreg} \subseteq T^*_{H_\lambda}(V_\lambda)_{reg}$$

and if $(x,\xi) \in T^*_C(V_\lambda)$ is strongly regular then its H_λ-orbit is $T^*_C(V_\lambda)_{sreg}$.

Proof. First we show $T^*_C(V_\lambda)_{\mathrm{sreg}} \subseteq T^*_C(V_\lambda)_{\mathrm{reg}}$. From the definition of $T^*_C(V_\lambda)_{\mathrm{reg}}$ (63) it is clear that it is open and dense in $T^*_C(V_\lambda)$. Fix $(x,\xi) \in T^*_C(V_\lambda)$ and let $\mathcal{O}_{H_\lambda}(x,\xi)$ denote the H_λ-orbit of (x,ξ). If (x,ξ) is not regular, then $(x,\xi) \in \overline{T^*_{C_1}(V_\lambda)}$ for some $C_1 \neq C$ with $C \subset \bar{C}_1$, so all of $\mathcal{O}_{H_\lambda}(x,\xi)$ and its closure also does not intersect $T^*_C(V_\lambda)_{\mathrm{reg}}$. Suppose, for a contradiction, that (x,ξ) is strongly regular also. Then the closure of $\mathcal{O}_{H_\lambda}(x,\xi)$ is $T^*_C(V_\lambda)$, which certainly does intersect $T^*_C(V_\lambda)_{\mathrm{reg}}$. So, if (x,ξ) is not regular, then it is not strongly regular.

Now suppose $T^*_{C,x}(V_\lambda)_{\mathrm{sreg}}$ is not empty, then it is enough to show $T^*_{C,x}(V_\lambda)_{\mathrm{sreg}}$ forms a single $Z_{H_\lambda}(x)$-orbit. Note

$$T^*_{C,x}(V_\lambda)_{\mathrm{sreg}} = \{\xi \in T^*_{C,x}(V_\lambda) \mid [\mathrm{Lie}(Z_{H_\lambda}(x)), \xi] = T^*_{C,x}(V_\lambda)\}$$

which is open, dense and connected in $T^*_{C,x}(V_\lambda)$. Moreover, $Z_{H_\lambda}(x)$-orbits in $T^*_{C,x}(V_\lambda)_{\mathrm{sreg}}$ are open, and hence they are also closed in $T^*_{C,x}(V_\lambda)_{\mathrm{sreg}}$. By the connectedness of $T^*_{C,x}(V_\lambda)_{\mathrm{sreg}}$, we can conclude it is a single $Z_{H_\lambda}(x)$-orbit. □

The equivariant fundamental group of $T^*_C(V_\lambda)_{\mathrm{sreg}}$ will be denoted by $A_{T^*_C(V_\lambda)_{\mathrm{sreg}}}$. Since H_λ acts transitively on $T^*_C(V_\lambda)_{\mathrm{sreg}}$,

$$A_{T^*_C(V_\lambda)_{\mathrm{sreg}}} \cong \pi_0(Z_{H_\lambda}(x,\xi)) = Z_{H_\lambda}(x,\xi)/Z_{H_\lambda}(x,\xi)^0, \tag{70}$$

for every $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{sreg}}$. Consequently, each $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{sreg}}$ determines an equivalence

$$\mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}) \to \mathsf{Rep}(A_{T^*_C(V_\lambda)_{\mathrm{sreg}}}).$$

4.6. **From Arthur parameters to strongly regular conormal vectors.** For $\psi \in Q({}^LG)$, define

$$\psi_0 := \psi^0|_{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})} : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$$

and

$$\psi_1 := \psi_0|_{\mathrm{SL}(2,\mathbb{C})\times 1} : \mathrm{SL}(2,\mathbb{C}) \to \widehat{G} \qquad \text{and} \qquad \psi_2 := \psi_0|_{1\times\mathrm{SL}(2,\mathbb{C})} : \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}.$$

Set

$$x_\psi := \mathrm{d}\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \quad y_\psi := \mathrm{d}\psi_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \quad \text{and} \quad \xi_\psi := \mathrm{d}\psi_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}}. \tag{71}$$

It follows easily from these definitions that

$$x_\psi, y_\psi \in V_{\lambda_\psi} \qquad \text{and} \qquad \xi_\psi \in {}^tV_{\lambda_\psi}$$

and

$$(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda).$$

Proposition 4.6.1. *For any $\psi \in Q({}^LG)$,*

$$(x_\psi, \xi_\psi) \in T^*_{H_{\lambda_\psi}}(V_{\lambda_\psi})_{sreg}.$$

Proof. Set $\lambda = \lambda_\psi$. Define $f_\lambda, s_\lambda, t_\lambda \in \widehat{G}$ as in Section 3.3. Then

$$s_\lambda \rtimes 1 = \psi(1, d_{\mathrm{Fr}}, d_{\mathrm{Fr}}) \qquad \text{and} \qquad t_\lambda \times \mathrm{Fr} = \psi(\mathrm{Fr}, 1, 1).$$

Recall $\lambda_{\mathrm{nr}} : W_F \to J^0_\lambda$ from Section 3.3. By Proposition 3.4.1,

$$V_\lambda = V_{\lambda_{\mathrm{nr}}} = \mathfrak{j}_{\lambda,2}.$$

Since the image of $\psi_0 : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$ lies in J^0_λ, we may define

$$\psi_{\mathrm{nr}} : W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to J^0_\lambda$$

such that its restriction to W_F is trivial and its restriction to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is ψ_0. Let

$$\iota_\psi : \mathbb{G}_{\mathrm{m}} \longrightarrow J^0_\lambda$$

be the cocharacter obtained by composing

$$\mathbb{G}_{\mathrm{m}} \to W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}), \qquad z \mapsto 1 \times \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \times \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}$$

with of $\psi_{\mathrm{nr}} : L_F \times \mathrm{SL}(2,\mathbb{C}) \to J^0_\lambda$. Then

$$\iota_\psi(q_F^{1/2}) = \lambda_{\mathrm{nr}}(\mathrm{Fr}).$$

Recall $H_\lambda \subseteq J_\lambda \subseteq K_\lambda \subseteq \widehat{G}$ from Sections 2.2 and 3.3. For the rest of the proof we set $J = J_\lambda$. We must show that the orbit $\mathcal{O}_{Z_{H_\lambda}(x_\psi)}(\xi_\psi)$ is open and dense in $T^*_{C_\psi,x_\psi}(V_\lambda)$, where $C_\psi = \mathcal{O}_{H_\lambda}(x_\psi)$. With Lemma 3.6.1 in hand, it is enough to show the tangent space to the orbit $\mathcal{O}_{Z_{H_\lambda}(x_\psi)}(\xi_\psi)$ at ξ_ψ is isomorphic to $T^*_{C_\psi,x_\psi}(V_\lambda)$; in other words, it is enough to show

$$[\mathrm{Lie} Z_{H_\lambda}(x_\psi), \xi_\psi] = \{\xi \in \mathfrak{j}_{-2} \mid [x_\psi, \xi] = 0\}.$$

The adjoint action of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ on $\mathfrak{j}$ through ψ_{nr} gives two commuting representations of $\mathrm{SL}(2,\mathbb{C})$, which induce the weight decomposition

$$\mathfrak{j}_n = \bigoplus_{r+s=n} \mathfrak{j}_{r,s} \tag{72}$$

where $r, s \in \mathbb{Z}$. Note $\mathrm{Lie}(H_\lambda)) = \mathfrak{j}_0$. So it is enough to show

$$[\mathfrak{j}_0 \cap \mathrm{Lie}(Z_{\widehat{G}}(x_\psi)), \xi_\psi] = \mathfrak{j}_{-2} \cap \mathrm{Lie}(Z_{\widehat{G}}(x_\psi)). \tag{73}$$

For this we can consider the following diagram in case $r + s = 0$.

$$\begin{tikzcd}
\mathfrak{j}_{r,s} \arrow[r, "\mathrm{ad}(x_\psi)"] \arrow[d, "\mathrm{ad}(\xi_\psi)"'] & \mathfrak{j}_{r+2,s} \arrow[d, "\mathrm{ad}(\xi_\psi)"] \\
\mathfrak{j}_{r,s-2} \arrow[r, "\mathrm{ad}(x_\psi)"] & \mathfrak{j}_{r+2,s-2}
\end{tikzcd}$$

It is easy to see

$$\mathrm{LHS}(73) = \bigoplus_{r+s=0} \mathrm{ad}(\xi_\psi)\big(\ker(\mathrm{ad}(x_\psi)|_{\mathfrak{j}_{r,s}})\big)$$
$$\mathrm{RHS}(73) = \bigoplus_{r+s=0} \ker(\mathrm{ad}(x_\psi)|_{\mathfrak{j}_{r,s-2}})$$

By $\mathfrak{sl}_2$-representation theory, $\mathrm{ad}(x_\psi)$ in the diagram are injective for $r < 0$ and surjective for $r \geqslant 0$. So we only need to consider $r \geqslant 0$ and hence $s \leqslant 0$. In this case, the two instances of $\mathrm{ad}(\xi_\psi)$ in the diagram above are surjective by $\mathfrak{sl}_2$-representation theory again.

It is obvious that $\mathrm{LHS}(73) \subseteq \mathrm{RHS}(73)$. For the other direction, let us choose $x \in \mathfrak{j}_{r,s-2}$ such that $[x_\psi, x] = 0$. So x is primitive for the action of the first $\mathfrak{sl}_2$, and it generates an irreducible representation V. Let $\widetilde{x}$ be a preimage of x in $\mathfrak{g}_{r,s}$ and W be the representation of the first $\mathfrak{sl}_2$ generated by $\widetilde{x}$. Then $\mathrm{ad}(\xi_\psi)$ induces a morphism of $\mathfrak{sl}_2$-representations from W to V. By the semisimplicity of W, this morphism admits a splitting and we can denote the image of x by ξ. It is clear that $\xi \in \mathfrak{j}_{r,s}$ and $[x_\psi, \xi] = 0$. This finishes the proof. □

Corollary 4.6.2. *Let $\psi : W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ be an Arthur parameter with infinitesimal parameter λ. If $C_\psi \subseteq V_\lambda$ is the H_λ-orbit of x_ψ, then*

$$\widehat{C_\psi} = C_{\widehat{\psi}},$$

where $\widehat{C_\psi} = {}^tC^*_\psi$ (69) and where the map $\widehat{\psi} : W_F \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ is defined by $\widehat{\psi}(w,x,y) := \psi(w,y,x)$.

4.7. **Arthur component groups are equivariant fundamental groups.** Recall the definition of $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ from Section 4.5 as well as the notation $A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}$ for its equivariant fundamental group. Also recall $A_\psi := \pi_0(Z_{\widehat{G}}(\psi))$ from Section 1.4.

Proposition 4.7.1.

$$A_{T^*_{C_\psi}(V_\lambda)_{sreg}} = A_\psi.$$

Proof. We use the notation from the proof of Proposition 4.6.1. It is clear that $Z_{\widehat{G}}(\psi) = Z_J(\psi_{\mathrm{nr}}) = Z_J(\psi_1) \cap Z_J(\psi_2)$. By Lemma 3.3.1, we also have

$$Z_{\widehat{G}}(\lambda)_{(x_\psi,\xi_\psi)} = Z_J(\lambda_{\mathrm{nr}}) \cap Z_J(x_\psi) \cap Z_J(\xi_\psi).$$

First we would like to compute the right hand side of the above identity. Note

$$Z_J(\lambda_{\mathrm{nr}}) \cap Z_J(x_\psi) = (Z_J(\psi_1) \cap Z_J(\lambda_{\mathrm{nr}})) \cdot U$$

where U is the unipotent radical of the left hand side. Moreover,

$$Z_J(\psi_1) \cap Z_J(\lambda_{\mathrm{nr}}) = Z_J(\psi_1) \cap Z_J(t_\psi)$$

and

$$\mathrm{Lie}(U) \subseteq \bigoplus_{\substack{r+s=0\\ r>0}} \mathfrak{j}_{r,s}$$

in the notation of (72). For $u \in U$, we have

$$\mathrm{Ad}(u)(\xi_\psi) \in \xi_\psi + \bigoplus_{\substack{r+s=-2\\ s<-2}} \mathfrak{j}_{r,s}.$$

Suppose $\mathrm{Ad}(lu)$ stabilises ξ_ψ for $l \in Z_J(\psi_1) \cap Z_J(t_\psi)$ and $u \in U$. Since $\mathrm{Ad}(l)$ preserves $\mathfrak{j}_{r,s}$, we have

$$\xi_\psi = \mathrm{Ad}(lu)(\xi_\psi) \in \mathrm{Ad}(l)(\xi_\psi) + \bigoplus_{\substack{r+s=-2\\ s<-2}} \mathfrak{j}_{r,s}$$

Note $\xi_\psi \in \mathfrak{j}_{0,-2}$. It follows $\xi_\psi = \mathrm{Ad}(l)(\xi_\psi)$. Hence $\xi_\psi = \mathrm{Ad}(u)(\xi_\psi)$. As a result,

$$Z_J(\lambda_{\mathrm{nr}}) \cap Z_J(x_\psi) \cap Z_J(\xi_\psi) = (Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi)) \cdot (U \cap Z_J(\xi_\psi)).$$

Since $U \cap Z_J(\xi_\psi)$ is connected, we only need to show

$$Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi) = Z_J(\psi_1) \cap Z_J(\psi_2).$$

Take any $g \in Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi)$, it suffices to show $\mathrm{Ad}(g)$ stabilises y_ψ. Note

$$[y_\psi, \xi_\psi] = \mathrm{d}\psi_2(\ln(|\mathrm{Fr}|)),$$

and

$$[\mathrm{Ad}(g)(y_\psi), \xi_\psi] = [\mathrm{Ad}(g)(y_\psi), \mathrm{Ad}(g)\xi_\psi] = \mathrm{Ad}(g)(\mathrm{d}\psi_2(\ln(|\mathrm{Fr}|))) = \mathrm{d}\psi_2(\ln(|\mathrm{Fr}|))$$

Since $[\cdot, \xi_\psi]$ is injective on $\mathfrak{j}_{0,2}$ and $\mathrm{Ad}(g)(y_\psi) \in \mathfrak{j}_{0,2}$, it is necessary that $\mathrm{Ad}(g)(y_\psi) = y_\psi$. This finishes the proof. $\square$

4.8. **Proof of Theorem 4.1.1.** Theorem 4.1.1 is now a direct consequence of Propositions 4.5.1, 4.6.1 and 4.7.1.

4.9. **Equivariant Local systems.** We close Section 4 with a practical tool for understanding local systems on strata $C \subseteq V_\lambda$ and on $T^*_C(V_\lambda)_{\mathrm{sreg}}$ and on $C^* \subseteq {}^tV_\lambda$. Pick a base point $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{sreg}}$. Recall $T^*_C(V_\lambda)_{\mathrm{reg}}$ by Lemma 4.4.2 and $T^*_C(V_\lambda)_{\mathrm{sreg}} \subseteq T^*_C(V_\lambda)_{\mathrm{reg}}$ by Proposition 4.5.1. The projections

$$C \longleftarrow T^*_C(V_\lambda)_{\mathrm{sreg}} \longrightarrow C^*$$

induce homomorphisms of fundamental groups:

$$\begin{tikzcd}
A_C \arrow[d, equal] & A_{T^*_C(V_\lambda)_{\mathrm{sreg}}} \arrow[l] \arrow[r] \arrow[d, equal] & A_{C^*} \arrow[d, equal] \\
Z_{H_\lambda}(x)/Z_{H_\lambda}(x)^0 & Z_{H_\lambda}(x,\xi)/Z_{H_\lambda}(x,\xi)^0 \arrow[l] \arrow[r] & Z_{H_\lambda}(\xi)/Z_{H_\lambda}(\xi)^0.
\end{tikzcd}$$

The horizontal homomorphisms are surjective by an application of [1, Lemma 24.6]. This can be used to enumerate all the simple local systems on H_λ-orbits in V_λ and $T^*_{H_\lambda}(V_\lambda)_{\mathrm{sreg}}$ and ${}^tV_\lambda$.

5. Vanishing cycles of perverse sheaves on Vogan's moduli space

We now turn to a study of the vanishing cycles of the equivariant perverse sheaves on V_λ with respect to integral models determined by regular covectors $(x,\xi) \in T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$. In this section, is G is an arbitrary connected reductive algebraic group over a p-adic field F.

Although we will use [14, Exposés XIII, XIV] freely, we begin by recalling a few essential facts and setting some notation. Let $R := \mathbb{C}[[t]]$ and $K := \mathbb{C}((t))$, the fraction field of R. Set $S = \mathrm{Spec}(R)$ and $\eta = \mathrm{Spec}(K)$ and $s = \mathrm{Spec}(\mathbb{C})$. Observe that S is a trait with generic fibre η and special fibre s. Because S is an equal characteristic trait the morphism $s \to S$ admits canonical section, corresponding to $\mathbb{C} \to \mathbb{C}[[t]]$.

$$\begin{tikzcd}
\eta \arrow[r, tail, "j_\eta"] & S \arrow[r, bend left] & s \arrow[l, tail, "i_s"]
\end{tikzcd}$$

For any s-scheme Z, we will use the notation $Z_S := Z \times_s S$. Since $S \to s$ is flat, the functor $Z \mapsto Z_S$ is exact from s-schemes to S-schemes.

Let $\bar\eta$ be a geometric point of S localised at η; thus, $\bar\eta$ is a morphism $\mathrm{Spec}(\bar K) \to \eta \to S$, where $\bar K$ is a separable closure of K. Then $\mathrm{Gal}(\bar\eta/\eta) \cong \hat{\mathbb{Z}}$. Let $\bar R$ be the integral closure of R in $\bar K$; then $\bar R$ has residue field s. Set $\bar S = \mathrm{Spec}(\bar R)$. Then for any morphism $X \to S$ we have the cartesian diagram

$$\begin{tikzcd}
& X_{\bar\eta} \arrow[rr, tail, "j_{X_{\bar\eta}}"] \arrow[dl, "b_{X_\eta}"'] \arrow[dd] & & \bar X \arrow[dl, "b_X"] \arrow[dd] & & X_{\bar s} \arrow[ll, tail, "i_{X_{\bar s}}"'] \arrow[dl, equal] \arrow[dd] \\
X_\eta \arrow[rr, tail, "j_{X_\eta}"] \arrow[dd] & & X \arrow[dd] & & X_s \arrow[ll, tail, "i_{X_s}"'] \arrow[dd] & \\
& \bar\eta \arrow[rr, tail, "j_{\bar\eta}"] \arrow[dl, "b_\eta"'] & & \bar S \arrow[dl, "b_S"'] & & \bar s \arrow[ll, tail, "i_{\bar s}"'] \arrow[dl, equal, "b_s"'] \\
\eta \arrow[rr, tail, "j_\eta"] & & S & & s \arrow[ll, tail, "i_s"'] &
\end{tikzcd}$$

From [14, Exposé XIII] we recall the nearby cycles functor $\mathsf{R}\Psi_{X_\eta} : \mathsf{D}(X_\eta) \to \mathsf{D}(X_s \times_s \eta)$; in particular, we recall that, for any $\mathcal{F} \in \mathsf{D}(X_\eta)$, the object $\mathsf{R}\Psi_{X_\eta}\mathcal{F}$ in $\mathsf{D}(X_s \times_s \eta)$ is the sheaf

$$\mathsf{R}\Psi_{X_{\bar\eta}}\mathcal{F} := (i_{X_{\bar\eta}})^*(j_{X_{\bar\eta}})_*(b_{X_\eta})^*\mathcal{F}$$

on $X_{\bar s}$ equipped with an action of $\mathrm{Gal}(\bar\eta/\eta)$, called the action of inertia, obtained by transport of structure from the canonical action of $\mathrm{Gal}(\bar\eta/\eta)$ on $(b_{X_\eta})^*\mathcal{F}$. From [14, Exposé XIII] we also recall the functor $\mathsf{R}\Psi_X : \mathsf{D}(X) \to \mathsf{D}(X_s \times_s S)$; in particular, recall that when followed by $\mathsf{D}(X_s \times_s S) \to \mathsf{D}(X_{\bar\eta})$, this is given by $\mathsf{R}\Psi_{X_{\bar\eta}}(j_{X_\eta})^*$. Finally, the vanishing cycles functor $\mathsf{R}\Phi_X : \mathsf{D}(X) \to \mathsf{D}(X_s \times_s S)$ is defined by the following distinguished triangle in $\mathsf{D}(X_s \times_s S)$.

$$\begin{tikzcd} & \mathsf{R}\Phi_X \arrow[dl, "(1)"'] & \\ i^*_{X_s} \arrow[rr] & & \mathsf{R}\Psi_X \arrow[ul] \end{tikzcd} \tag{74}$$

5.1. **Functor of vanishing cycles.** Let $(\ |\ ) : T^*(V_\lambda) \to \mathbb{A}^1_{\mathbb{C}}$ be the s-morphism obtained by restriction from the non-degenerate, symmetric J_λ-invariant bilinear form of Section 4.2. Let $f : T^*(V_\lambda) \to S$ be the unique s-morphism so that $(\ |\ ) : T^*(V_\lambda) \to \mathbb{A}^1_{\mathbb{C}}$ is the composition of $f : T^*(V_\lambda) \to S$ and $S \to \mathbb{A}^1_{\mathbb{C}}$. Using f, we view $T^*(V_\lambda)$ as an S-scheme; as such, its ring of global sections is $R[T^*(V_\lambda)] = k[T^*(V_\lambda)] \otimes_k R/(f-t) = \mathbb{C}[[t]][x,\xi]/(f(x,\xi)-t)$.

For any H_λ-orbit $B \subseteq {}^tV_\lambda$, consider the locally closed subvariety $V_\lambda \times B \subseteq T^*(V_\lambda)$ and let $f_B : V_\lambda \times B \to S$ be the restriction of $f : T^*(V_\lambda) \to S$ to $V_\lambda \times B$. Using f_B, we may view $V_\lambda \times B$ as an S-scheme: let

$$f_B : X_B \to S$$

be the S-scheme with structure sheaf

$$\mathcal{O}_{X_B} = R \otimes_{\mathbb{C}} \mathcal{O}_{V_\lambda} \otimes_{\mathbb{C}} \mathcal{O}_B/(f-t).$$

Then the special fibre of X_B is the s-scheme

$$X_{B,s} = f_B^{-1}(s) = f_B^{-1}(0) = \{(x,\xi) \in V_\lambda \times B \mid (\,x\,|\,\xi\,) = 0\}$$

and the generic fibre of X_B is the K-scheme obtained by base change from the generic fibre of $(\ |\ ) : T^*(V_\lambda) \to \mathbb{A}^1_{\mathbb{C}}$:

$$X_{B,\eta} = f_B^{-1}(\eta) = \{(x,\xi) \in V_\lambda \times B \mid (\,x\,|\,\xi\,) \neq 0\} \times_s \eta.$$

In this way, $f_B : X_B \to S$ defines

$$\mathsf{R}\Phi_{f_B} := \mathsf{R}\Phi_{X_{B,\eta}} : \mathsf{D}(f_B^{-1}(\eta)) \to \mathsf{D}(f_B^{-1}(0) \times_s S). \tag{75}$$

and

$$\mathsf{R}\Phi_{X_B} : \mathsf{D}(X_B) \to \mathsf{D}(f_B^{-1}(0) \times_s S). \tag{76}$$

Now, as an s-scheme, $V_\lambda \times B$ comes equipped with an H_λ-action. Applying base change along $S \to s$ gives an action of $H_\lambda \times_s S$ on $(V_\lambda \times B)_S$. Because f_B is H_λ-invariant, this defines an action of $H_\lambda \times_s S$ on $\{(x,\xi,t) \in (V_\lambda \times B)_S \mid f(x,\xi) = t\}$. But this is precisely

$V_\lambda \times B$ as an S-scheme, via $f_B : X_B \to S$. So, $H_\lambda \times_s S$ acts on X_B in the category of S-schemes and we have the exact functor

$$\mathsf{D}_{H_\lambda}(V_\lambda \times B) \to \mathsf{D}_{H_\lambda \times_s S}(X_B). \tag{77}$$

See [9, Section 2] for the equivariant derived category $\mathsf{D}_H(X)$. Combining this with the vanishing cycles functors above defines an exact functor

$$\mathsf{R}\Phi_{X_B} : \mathsf{D}_{H_\lambda \times_s S}(V_\lambda \times B) \to \mathsf{D}_{H_\lambda}(f_B^{-1}(0) \times_s S). \tag{78}$$

Finally we come to the main definition for Section 5: For any H_λ-orbit $C \subseteq V_\lambda$, let

$$\mathsf{Ev}_C : \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s \eta) \tag{79}$$

be the functor defined by the diagram

$$\begin{tikzcd}
\mathsf{D}_{H_\lambda}(V_\lambda) \arrow[rrr, "\mathsf{Ev}_C"] \arrow[d, "\cdot\, \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}"] & & & \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S) \\
\mathsf{D}_{H_\lambda}(V_\lambda \times C^*) \arrow[r, "\text{base change}"'] & \mathsf{D}_{H_\lambda \times_s S}(X_{C^*}) \arrow[rr, "\mathsf{R}\Phi_{X_{C^*}}"] & & \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S), \arrow[u, "\text{restriction}"']
\end{tikzcd}$$

where:

(1) $\cdot \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*} : \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(V_\lambda \times C^*)$ is pullback along the projection $V_\lambda \times C^* \to V_\lambda$;
(2) $\mathsf{D}_{H_\lambda}(V_\lambda \times C^*)) \to \mathsf{D}_{H_\lambda \times_s S}(X_{C^*})$ is (77) in the case $B = C^*$;
(3) $\mathsf{R}\Phi_{X_{C^*}}[-1] : \mathsf{D}_{H_\lambda \times_s S}(X_{C^*}) \to \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S)$ is (76) in the case $B = C^*$;
(4) $\mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S)$, is obtained by pullback along the inclusion $T^*_C(V_\lambda)_{\mathrm{reg}} \hookrightarrow f_{C^*}^{-1}(0)$, using Proposition 4.4.3.

When we wish to ignore the action of inertia, we write

$$\mathsf{Ev}_{C,\bar{\eta}} : \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}}) \tag{80}$$

for Ev_C followed by the forgetful functor $\mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}})$.

The main properties of Ev_C are given in Theorem 5.3.1.

We has used notation Ev_C to make oblique reference to [8, Notation 1.14], where one finds a sheaf on $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$ with the same stalks, after shift, as our Ev_C. That sheaf is described in [8, Proposition 1.15] and [8, Remarque 1.13]. From [8, Théorème 1.9] we also see that the sheaf in [8, Notation 1.14] is produced by a functor. Both of these results rely on [8, Théorème 1.9], which is attributed to [20, Théorème 3.2.5]. Sadly, [20, Théorème 3.2.5] does not exist in the published version of the original notes, and we have not been able to procure the original notes, so we have been obliged to rebuild this result – as far as we need it – in Theorem 5.3.1.

5.2. **Proper base change.**

Lemma 5.2.1. *Suppose $\pi : W \to V_\lambda$ is proper with fibres of dimension n. Suppose H_λ acts on W and $\pi : W \to V_\lambda$ is equivariant. Then*

$$\mathsf{Ev}_C\, \pi_! \mathcal{E} = (\pi''_s)_! \left((\mathsf{R}\Phi_{g_{C^*}}(\mathcal{E} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}))|_{(W \times C^*)_{\pi\text{-}reg}} \right),$$

where $\pi' := \pi \times \mathrm{id}_{C^*}$, π'_s *is its restriction to special fibres,* $g_{C^*} := f_{C^*} \circ \pi'$, *and* π''_s *and* $(W \times C^*)_{\pi\text{-}reg}$ *are defined by the cartesian diagrams below.*

$$\begin{tikzcd}
W \arrow[d, "\pi"'] & W \times C^* \arrow[l, "p'_{C^*}"] \arrow[d, "\pi'"'] \arrow[dd, bend left=60, "g_{C^*}"] \\
V_\lambda & V_\lambda \times C^* \arrow[l, "p_{C^*}"] \arrow[d, "f_{C^*}"'] \\
& S
\end{tikzcd}
\qquad
\begin{tikzcd}
(g_{C^*})^{-1}(0) \arrow[d, "\pi'_s"'] & (W \times C^*)_{\pi\text{-}reg} \arrow[l] \arrow[d, "\pi''_s"] \\
f_{C^*}^{-1}(0) \arrow[d] & T^*_C(V_\lambda)_{reg} \arrow[l] \\
s &
\end{tikzcd}$$

Proof. Suppose $\mathcal{E} \in \mathsf{D}_{H_\lambda}(W)$. Then $\pi_! \mathcal{E} \in \mathsf{D}_{H_\lambda}(V_\lambda)$. Let $p_{C^*} : V_\lambda \times C^* \to V_\lambda$ be projection. Then, by repeated application of proper base change,

$$\begin{aligned}
\mathsf{Ev}_C\, \pi_! \mathcal{E} &= (\mathsf{R}\Phi_{f_{C^*}} p_{C^*}^* \pi_! \mathcal{E})|_{T^*_C(V_\lambda)_{\mathrm{reg}}} \\
&= (\mathsf{R}\Phi_{f_{C^*}} (\pi')_! (p'_{C^*})^* \mathcal{E})|_{T^*_C(V_\lambda)_{\mathrm{reg}}} \\
&= ((\pi'_s)_! \mathsf{R}\Phi_{g_{C^*}} (p'_{C^*})^* \mathcal{E})|_{T^*_C(V_\lambda)_{\mathrm{reg}}} \\
&= (\pi''_s)_! \left((\mathsf{R}\Phi_{g_{C^*}} (\mathcal{E} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}))|_{(W \times C^*)_{\pi\text{-reg}}} \right).
\end{aligned}$$

$\square$

5.3. **Main properties of vanishing cycles.** Using the s-morphism $S \to \mathbb{A}^1_{\mathbb{C}}$ corresponding to $\mathbb{C}[t] \hookrightarrow \mathbb{C}[[t]]$, every S-scheme is a scheme over $\mathbb{A}^1_{\mathbb{C}}$. Using this, we will consider schemes over S as schemes over $\mathbb{A}^1_{\mathbb{C}}$, also.

$$\begin{tikzcd}
\eta \arrow[r, tail] \arrow[d] & S \arrow[d] & s \arrow[l, tail] \arrow[d, equal] \\
\mathbb{A}^{1,\times}_{\mathbb{C}} \arrow[r, tail] & \mathbb{A}^1_{\mathbb{C}} & 0 \arrow[l, tail]
\end{tikzcd}$$

For any $\xi_0 \in {}^tV_\lambda$, define $f_{\xi_0} : V_\lambda \to S$ by $f_{\xi_0}(x) := f(x, \xi_0)$. This allows us to view V_λ as an S-scheme; when we wish to emphasise this perspective, we denote this scheme by X_{ξ_0}, with structure sheaf

$$\mathcal{O}_{X_{\xi_0}} = R \otimes_{\mathbb{C}} \mathcal{O}_{V_\lambda} / (f_{\xi_0} - t).$$

Thus, the special fibre of $X_{\xi_0} \to S$ is

$$X_{\xi_0, s} = f_{\xi_0}^{-1}(0) = (\,-\,|\,\xi_0\,)^{-1}(0) = \{x \in V_\lambda \mid (\,x\,|\,\xi_0\,) = 0\}$$

and the generic fibre of X_{ξ_0} is the base change of the generic fibre of $(\,-\,|\,\xi_0\,)$:

$$X_{\xi_0, \eta} = f_{\xi_0}^{-1}(\eta) = \{x \in V_\lambda \mid (\,x\,|\,\xi_0\,) \neq 0\} \times_s \eta.$$

Using this, we define

$$\mathsf{R}\Phi_{f_{\xi_0}} : \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(f_\xi^{-1}(0) \times_s \eta)$$

by

$$\begin{tikzcd}
\mathsf{D}_{H_\lambda}(V_\lambda) \arrow[r, "\mathsf{R}\Phi_{f_{\xi_0}}"] \arrow[d, "\text{forget}"] & \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(f_\xi^{-1}(0) \times_s \eta) \\
\mathsf{D}_{Z_{H_\lambda}(\xi_0)}(V_\lambda) \arrow[r, "\text{base change}"] & \mathsf{D}_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}). \arrow[u, "{\mathsf{R}\Phi_{X_{\xi_0}}[-1]}"']
\end{tikzcd}$$

We are now ready to state the main properties of Ev_C.

Theorem 5.3.1. *Let $C \subseteq V_\lambda$ be an H_λ-orbit.*

(a) *The functor*
$$\mathsf{Ev}_{C,\bar\eta} : \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{reg})$$
is exact.

(b) *If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ then $\mathsf{R\Phi}_{f_{C^*}}[-1]\left(\mathcal{P} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}[d_{C^*}]\right)$ is an equivariant perverse sheaf on $f_{C^*}^{-1}(0)$ and $\mathsf{Ev}_{C,\bar\eta}\, \mathcal{P}[-1+\dim C^*]$ is its restriction to $T^*_C(V_\lambda)_{reg}$.*

(c) *If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ then $\mathsf{Ev}_{C,\bar\eta}\, \mathcal{P}$ is cohomologically concentrated in one degree.*

(d) *If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ then*
$$\mathsf{Ev}_C\, \mathcal{P} = 0 \qquad \textit{unless} \qquad C \subseteq \operatorname{supp} \mathcal{P}.$$

(e) *For every $\mathcal{F} \in \mathsf{D}_{H_\lambda}(V_\lambda)$ and every $(x_0, \xi_0) \in T^*_C(V_\lambda)_{reg}$, there is a canonical isomorphism*
$$\left(\mathsf{Ev}_{C,\bar\eta}\, \mathcal{F}\right)_{(x_0,\xi_0)} \cong (\mathsf{R\Phi}_{f_{\xi_0}} \mathcal{F})_{x_0}.$$

(f) *If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ then $\mathsf{Ev}_{C,\bar\eta}\, \mathcal{P}$ is an H_λ-equivariant local system concentrated in one degree.*

(g) *For every local system $\mathcal{L}$ on C,*
$$\mathsf{Ev}_{C,\bar\eta}\ \mathcal{IC}(C,\mathcal{L}) = \left(\mathcal{L}[\dim C] \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}\right)|_{T^*_C(V_\lambda)_{reg}}.$$

Theorem 5.3.1 will be proved in Section 5.5.

Using Theorem 5.3.1, let
$$\mathsf{Ev}^0_C : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s \eta) \tag{81}$$
be the exact functor so that
$$\mathsf{Ev}_C = \mathsf{Ev}^0_C[\dim V_\lambda].$$
Recall that $\dim T^*_C(V_\lambda) = \dim V_\lambda$. Thus, the exact functor
$$\mathsf{Ev}_C = \mathsf{Ev}^0_C[\dim T^*_C(V_\lambda)] : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s \eta) \tag{82}$$
produces only H_λ-equivariant local systems shifted to degree $\dim T^*_C(V_\lambda)$, and is given by
$$\mathsf{Ev}_C\, \mathcal{P} = \left(\mathsf{R\Phi}_{f_{C^*}}\left(\mathcal{P} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}\right)\right)|_{T^*_C(V_\lambda)_{\mathrm{reg}}}.$$

Since $T^*_C(V_\lambda)_{\mathrm{reg}}$ is a component of $T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$ (Section 4), the exact functors Ev_C uniquely determine an exact functor
$$\mathsf{Ev}_\lambda : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}} \times_s \eta) \tag{83}$$
so that
$$\mathsf{Ev}_C\, \mathcal{P} = (\mathsf{Ev}_\lambda\, \mathcal{P})|_{T^*_C(V_\lambda)_{\mathrm{reg}}}$$
for every $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$. The functor $\mathsf{Ev}_{\lambda,\bar\eta}$
$$\mathsf{Ev}_{\lambda,\bar\eta} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}). \tag{84}$$
appeared in the Introduction (10).

5.4. **Descent.**

Lemma 5.4.1. *For every $\xi_0 \in B$*

$$X_B \cong (H_\lambda \times_s S) \times_{\left(Z_{H_\lambda}(\xi_0) \times_s S\right)} X_{\xi_0}$$

in S-schemes.

Proof. First we must show that $(H_\lambda \times_s S) \times_{\left(Z_{H_\lambda}(\xi_0) \times_s S\right)} X_{\xi_0}$ exists in S-schemes. To do that, it will be helpful to prove: for every $\delta \in \mathbb{A}^1_{\mathbb{C}}$ and $\xi_0 \in B$ there is an H_λ-isomorphism

$$f_B^{-1}(\delta) \cong H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$$

in s-schemes, where $H_\lambda \times f_{\xi_0}^{-1}(\delta) \to H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$ is an $Z_{H_\lambda}(\xi_0)$-torsor in $\mathbb{C}$-varieties. Since $Z_{H_\lambda}(\xi_0)$ is a closed subgroup of H_λ, the quotient $H_\lambda \to H_\lambda/Z_{H_\lambda}(\xi_0)$ exists in $\mathbb{C}$-varieties. Consider the monomorphism

$$H_\lambda \times f_{\xi_0}^{-1}(\delta) \to (H_\lambda/Z_{H_\lambda}(\xi_0)) \times T^*(V_\lambda)$$

given by $(h,x) \mapsto (hZ_{H_\lambda}(\xi_0), h\cdot(x,\xi_0))$. Note that $f_{\xi_0}^{-1}(\delta)$ is a closed subvariety of V_λ. The promised $Z_{H_\lambda}(\xi_0)$-quotient $H_\lambda \times_{Z_{H_\lambda}(x_0)} f_{\xi_0}^{-1}(\delta)$ is this morphism restricted to the image:

$$H \times f_{\xi_0}^{-1}(\delta) \to \{(hZ_{H_\lambda}(\xi_0), h\cdot(x,\xi_0)) \in (H_\lambda/Z_{H_\lambda}(\xi_0)) \times V_\lambda \times B \mid h^{-1}\cdot x \in f_{\xi_0}^{-1}(\delta)\}.$$

Following standard practice, we use the notation $(h,x) \mapsto [h,x]_{Z_{H_\lambda}(\xi_0)}$ for this map. Now, projection to the second coordinate

$$H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta) \to f_B^{-1}(\delta)$$

is given by $[h,x]_{Z_{H_\lambda}(\xi_0)} \mapsto h\cdot(x,\xi_0)$, which is the promised isomorphism. This shows that $Z_{H_\lambda}(\xi_0)$-torsor $H_\lambda \times f_{\xi_0}^{-1}(\delta) \to H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$ exists in $\mathbb{C}$-varieties and also that the map

$$H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta) \to T^*(V_\lambda), \qquad [h,x]_{Z_{H_\lambda}(\xi_0)} \mapsto h\cdot(x,\xi_0),$$

is an H_λ-isomorphism onto $f_B^{-1}(\delta) \subseteq T^*(V_\lambda)$.

Applying pull-back along the flat morphism $S \to s$ to $Z_{H_\lambda}(\xi_0) \to H_\lambda \to H_\lambda/Z_{H_\lambda}(\xi_0)$ determines the cokernel of $Z_{H_\lambda}(\xi_0) \times_s S \to H_\lambda \times_s S$ and also shows that the local trivialisation of $H_\lambda \to H_\lambda/Z_{H_\lambda}(\xi_0)$ determines a local trivialisation of $H_\lambda \times_s S \to (H_\lambda \times_s S)/(Z_{H_\lambda}(\xi_0)\times_s S)$. Now we may argue as above to see that $(H_\lambda \times_s S)\times_{(Z_{H_\lambda}(\xi_0)\times_s S)} X_{\xi_0} \to T^*(V_\lambda) \times_s S$, defined by $[h,x]_{Z_{H_\lambda}(\xi_0)\times_s S} \mapsto h\cdot(x,\xi_0)$, is an isomorphism onto X_B over S. $\square$

For each $\xi_0 \in B$, the map $x \mapsto (x,\xi_0)$ is a section of projection $V_\lambda \times B \to V_\lambda$ over s. We have now seen how to view both $V_\lambda \times B$ and V_λ as S-schemes using f_B and f_{ξ_0}, respectively. While $V_\lambda \times B \to V_\lambda$ does not extend to a map of these S-schemes, the section $V_\lambda \times B \to V_\lambda$ above, does. The following lemma shows why this is important.

Lemma 5.4.2. *For every H_λ-orbit $B \subseteq {}^tV_\lambda$ and every $\xi_0 \in B$, the S-morphism*

$$\begin{aligned} i_{\xi_0} : X_{\xi_0} &\to X_B \\ x &\mapsto (x,\xi_0) \end{aligned}$$

is equivariant for the $Z_{H_\lambda}(\xi_0)\times_s S$-action on X_{ξ_0} and the $H_\lambda \times_s S$-action on X_B. Using equivariant pullback, the induced functor

$$i^*_{\xi_0}[-\dim B] : \mathsf{D}_{H_\lambda\times_s S}(X_B) \to \mathsf{D}_{Z_{H_\lambda}(\xi_0)\times_s S}(X_{\xi_0})$$

is an equivalence of categories, taking equivariant perverse sheaves to equivariant perverse sheaves.

Proof. This follows directly from Lemma 5.4.1 by using equivariant descent [9, Section 2.6.2]. The shift by $-\dim B$ is needed to preserve perversity. □

5.5. **Proof of Theorem 5.3.1.** With reference to the diagram below (79), we see that Ev_C is exact since it is defined as the composition of four exact functors. This gives Theorem 5.3.1, Part (a).

We now prove Theorem 5.3.1, Part (b). From the definition of Ev_C we have

$$\mathsf{Ev}_{C,\bar\eta}\,\mathcal{P} = \left(\mathsf{R}\Phi_{f_{C^*}}(\mathcal{P}\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*})\right)|_{T^*_C(V)_{\mathrm{reg}}}.$$

Since C^* is smooth, $(\bar{\mathbb{Q}}_\ell)_{C^*}[d_{C^*}]$ is perverse, and it follows that $\mathcal{P}\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*}[d_{C^*}]$ is a perverse sheaf on $V_\lambda\times C^*$; or argue using [6, 4.2.4]. It follows from [6, Proposition 4.4.2] (see also [8, Théorème 1.2]) that $\mathsf{R}\Phi_{f_{C^*}}[-1](\mathcal{P}\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*}[\dim C^*])$ is perverse. It is also H_λ-equivariant by transport of structure. Thus, the exact functor

$$\mathsf{R}\Phi_{f_{C^*}}(-\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*})[-1+\dim C^*] : \mathsf{D}_{H_\lambda}(V_\lambda)\to \mathsf{D}_H(f_{C^*}^{-1}(0))$$

takes equivariant perverse sheaves to equivariant perverse sheaves. This proves Theorem 5.3.1, Part (b).

Theorem 5.3.1, Part (c) is a consequence of [8, Théorème 1.9] which is attributed there to [20]. Alternatively, using [14, Exposé XIV, Théorème 2.8] we may pass from the algebraic description of $\mathsf{R}\Phi$ given above, which is based on [14, Exposé XII], to the analytic version of $\mathsf{R}\Phi$, given in [14, Exposé XIV]. Then the fact that the restriction of the perverse sheaf $\mathsf{R}\Phi_{f_{C^*}}[-1](\mathcal{P}\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*}[\dim C^*])$ to $T^*_C(V_\lambda)_{\mathrm{reg}})$ is concentrated in one degree follows from [16, Section II.6.4] and [16, Section II.6.A.3]. In fact, that degree is $\dim V_\lambda + \dim C^* - 1$. Equivalently, $\mathsf{Ev}_C\,\mathcal{P}$ is concentrated in degree $\dim V_\lambda = \dim T^*_C(V_\lambda)_{\mathrm{reg}}$. This proves Theorem 5.3.1, Part (c).

We now prove Theorem 5.3.1, Part (d). Without loss of generality, we may assume $\mathcal{P} = \mathcal{IC}(C_1,\mathcal{L}_1)$. Let $i_{\bar C_1} : \bar C_1 \hookrightarrow V_\lambda$ be inclusion. Then

$$\mathcal{IC}(C_1,\mathcal{L}_1) = (i_{\bar C_1})_!(i_{\bar C_1})^*\mathcal{IC}(C_1,\mathcal{L}_1).$$

Since $i_{\bar C_1}$ is proper, we may apply Lemma 5.2.1 to this case with $W=\bar C_1$ and $\pi = i_{\bar C_1}$ and $g_{C^*} = f|_{\bar C_1\times C^*}$. Then $\pi' = i_{\bar C_1}\times \mathrm{id}_{C^*}$ and

$$g_{C^*}^{-1}(0) = \{(x,\xi)\in \bar C_1\times C^* \mid (\,x\,|\,\xi\,) = 0\}$$

and

$$(W\times C^*)_{\pi\text{-reg}} = (\bar C_1\times C^*)\cap T^*_C(V)_{\mathrm{reg}} = T^*_C(V)_{\mathrm{reg}}.$$

Thus,

$$\begin{array}{rcl}\mathsf{Ev}_C\,\mathcal{IC}(C_1,\mathcal{L}_1) &=& \mathsf{Ev}_C(i_{\bar C_1})_!(i_{\bar C_1})^*\mathcal{IC}(C_1,\mathcal{L}_1)\\ &=& (\mathsf{R}\Phi_{g_{C^*}}((i_{\bar C_1})^*\mathcal{IC}(C_1,\mathcal{L}_1)\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*}))|_{T^*_C(V_\lambda)_{\mathrm{reg}}},\end{array}$$

by Lemma 5.2.1. The support of $(i_{\bar C_1})^*\mathcal{IC}(C_1,\mathcal{L}_1)\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*}$ is contained in $\bar C_1\times C^*$, so the support of

$$\mathsf{R}\Phi_{g_{C^*}}((i_{\bar C_1})^*\mathcal{IC}(C_1,\mathcal{L}_1)\boxtimes(\bar{\mathbb{Q}}_\ell)_{C^*})$$

is contained in $g_{C^*}^{-1}(0) \cap (\bar{C}_1 \times C^*)$ so the support of $\mathsf{Ev}_C\, \mathcal{IC}(C_1, \mathcal{L}_1)$ is contained in

$$T^*_C(V_\lambda)_{\mathrm{reg}} \cap (\bar{C}_1 \times C^*).$$

Since $T^*_C(V_\lambda)_{\mathrm{reg}} \subseteq C \times C^*$, this is empty unless $C \subseteq \bar{C}_1$. This concludes the proof of Theorem 5.3.1, Part (d).

By the definition of Ev_C, Theorem 5.3.1, Part (e) is equivalent to the following statement: for all $\mathcal{F} \in \mathsf{D}_{H_\lambda}(V_\lambda)$,

$$\left(\mathsf{R\Phi}_{f_{C^*}} \left(\mathcal{F} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}\right)\right)_{(x_0,\xi_0)} \cong (\mathsf{R\Phi}_{f_{\xi_0}} \mathcal{F})_{x_0}, \tag{85}$$

compatible with the natural $Z_{H_\lambda}(x_0, \xi_0)$-action. First, note that $(x_0, \xi_0) \in X_{C^*,s}$ by Proposition 4.4.3, since $(x_0, \xi_0) \in T^*_C(V_\lambda)_{\mathrm{reg}}$. Thus,

$$\left(i^*_{X_s^{C^*}} \left(\mathcal{F} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}\right)\right)_{(x_0,\xi_0)} = \mathcal{F}_{x_0} = (i^*_{\xi_0}\mathcal{F})_{x_0},$$

as $Z_{H_\lambda}(x_0, \xi_0)$-spaces. So, using (74) with $B = C^*$, it suffices to show

$$\left(\mathsf{R\Psi}_{f_{C^*}} \left(\mathcal{F} \boxtimes (\bar{\mathbb{Q}}_\ell)_{C^*}\right)\right)_{(x_0,\xi_0)} \cong (\mathsf{R\Psi}_{f_{\xi_0}} \mathcal{F})_{x_0}. \tag{86}$$

Lemma 5.4.2 determines the equivalences in the commuting diagram below.

$$\begin{tikzcd}
\mathsf{D}_{H_\lambda \times_s \eta}(f_{C^*}^{-1}(\eta)) \arrow[r, "(j_{C^*,S})_*"] \arrow[d, "i^*_{\xi_0,\eta}[-\dim C^*]", "\text{equiv.}"'] & \mathsf{D}_{H_\lambda \times_s S}(X_{C^*}) \arrow[r, "(i_{C^*,S})^*"] \arrow[d, "i^*_{\xi_0,S}[-\dim C^*]", "\text{equiv.}"'] & \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0)) \arrow[d, "i^*_{\xi_0,s}[-\dim C^*]", "\text{equiv.}"'] \\
\mathsf{D}_{Z_{H_\lambda}(\xi_0) \times_s \eta}(f_{\xi_0}^{-1}(\eta)) \arrow[r, "(j_{\xi_0,S})_*"] & \mathsf{D}_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}) \arrow[r, "(i_{\xi_0,S})^*"] & \mathsf{D}_{Z_{H_\lambda}(\xi_0)_s}(f_{\xi_0}^{-1}(0))
\end{tikzcd}$$

Thus,

$$\mathsf{R\Psi}_{f_{C^*}}\ i^*_{\xi_0,\eta} = i^*_{\xi_0,s}\ \mathsf{R\Psi}_{f_{\xi_0}}.$$

We find this equation at the heart of the following commuting diagram.

$$\begin{tikzcd}
\mathsf{D}_{H_\lambda}(V_\lambda) \arrow[rr, "p^*_{C^*}[\dim C^*]"] \arrow[d] & & \mathsf{D}_{H_\lambda}(V_\lambda \times C^*) \arrow[d] \\
\mathsf{D}_{Z_{H_\lambda}(\xi_0) \times_s \eta}(f_{\xi_0}^{-1}(\eta)) \arrow[d, "\mathsf{R\Psi}_{f_{\xi_0}}"] & & \mathsf{D}_{H_\lambda \times_s \eta}(f_{C^*}^{-1}(\eta)) \arrow[ll, "\text{equiv.}"', "i^*_{\xi_0,\bar{\eta}}[-\dim C^*]"] \arrow[d, "\mathsf{R\Psi}_{f_{C^*}}"] \\
\mathsf{D}_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0)) \arrow[d] & & \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0)) \arrow[ll, "\text{equiv.}"', "i^*_{\xi_0,s}[-\dim C^*]"] \arrow[d] \\
\mathsf{D}_{Z_{H_\lambda}(x_0,\xi_0)}(\{(x_0,\xi_0)\}) & & \mathsf{D}_H(T^*_C(V_\lambda)_{\mathrm{sreg}}) \arrow[ll, "{[-\dim C^*]}"]
\end{tikzcd}$$

This proves (86). Since the isomorphism in (86) comes from this commuting diagram of functors, it is compatible with the $Z_{H_\lambda}(x_0, \xi_0)$-actions. This concludes the proof of Theorem 5.3.1, Part (e).

To prove Theorem 5.3.1, Part (f) we again use [14, Exposé XIV, Théorème 2.8] to pass from the algebraic description of the vanishing cycles functor to the analytic version of the vanishing cycles functor. It now follows from [16, Section II.6.A.2] that for fixed $x \in C$, the stalks of $\mathsf{Ev}_{C,\eta}\, \mathcal{P}$ at $(x, \xi) \in T^*_C(V_\lambda)_{\mathrm{reg}}$ are canonically identified with the Morse groups $\mathcal{A}^i_\xi(\mathcal{P})$, after shifting to where these are non-trivial. It now follows from [16, Section II.6.A.1] that these are the stalks of a local system, giving Theorem 5.3.1, Part (f). We note too that by [1, Definition 24.11] these stalks are given by $(Q^{\mathrm{mic}})^i(\mathcal{P})_{(x,\xi)} = H^{i-\dim C}(J,K;\mathcal{P})$, where J and K are as defined in [1, (24.10)(a)]. So Theorem 5.3.1, Part (f) may also be deduced from [1, Theorem 24.8].

Arguing as above, Theorem 5.3.1, Part (g) may be deduced from [1, Theorem 24.8 (b)].

5.6. **Vanishing cycles and Arthur parameters.** Suppose the stratum $C \subseteq V_\lambda$ is of Arthur type, so $C = C_\psi$ for an Arthur parameter ψ, unique up to H_λ-conjugacy. Then $T^*_C(V_\lambda)_{\mathrm{sreg}} \subseteq T^*_C(V_\lambda)_{\mathrm{reg}}$ is a non-empty open H_λ-stable subvariety. With reference to (81), define

$$\mathsf{Evs}_C : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}) \tag{87}$$

by

$$\mathsf{Evs}_C = \mathsf{Ev}^0_C |_{T^*_C(V_\lambda)_{\mathrm{sreg}}}.$$

It follows from [6, Lemme 4.3.2] that

$$\mathsf{Ev}_C\, \mathcal{P} = \mathcal{IC}(T^*_C(V_\lambda)_{\mathrm{sreg}}, \mathsf{Evs}_C\, \mathcal{P}). \tag{88}$$

Now, the choice of an Arthur parameter ψ with $C = C_\psi$ determines an equivalence

$$\mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}) \to \mathsf{Rep}(A_\psi).$$

Define

$$\mathsf{Ev}_\psi : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Rep}(A_\psi) \tag{89}$$

by composing these two functors. This is the functor appearing in the Introduction (12).

If the stratum $C \subseteq V_\lambda$ is not of Arthur type, we do not know if $T^*_C(V_\lambda)_{\mathrm{sreg}}$ is non-empty. So in this case we use [1, Lemma 24.3 (f)] and choose a non-empty open H_λ-stable subvariety $U \subseteq T^*_C(V_\lambda)_{\mathrm{reg}}$ and define

$$\mathsf{Evs}_C : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(U)$$

by

$$\mathsf{Evs}_C = \mathsf{Ev}^0_C |_U.$$

We do not know if such U is unique, but regardless of the choice of U, we again have

$$\mathsf{Ev}_C\, \mathcal{P} = \mathcal{IC}(U, \mathsf{Evs}_C\, \mathcal{P})$$

by [6, Lemme 4.3.2]. By [1, Lemma 24.3 (f)], each $(x,\xi) \in U$ determines an equivalence

$$\mathsf{Loc}_{H_\lambda}(U) \to \mathsf{Rep}(\pi_0(Z_{H_\lambda}(x,\xi))).$$

By [1, Lemma 24.3 (g)], the isomorphism type of $\pi_0(Z_{H_\lambda}(x,\xi))$ is independent of the choice of $(x,\xi) \in U$. Indeed, by [1, Definition 24.7] $\pi_0(Z_{H_\lambda}(x,\xi))$ is the microlocal fundamental group A^{mic}_C of $T^*_C(V_\lambda)_{\mathrm{reg}}$, given up to isomorphism by

$$A_{(x,\xi)} := \pi_1(U,(x,\xi))_{H_\lambda} = \pi_0(Z_{H_\lambda}(x,\xi)). \tag{90}$$

Define

$$\mathsf{Ev}_{(x,\xi)} : \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Rep}(A_{(x,\xi)}) \tag{91}$$

by composing these two functors.

5.7. **Relation to microlocalisation.** As we saw in the proof of Theorem 5.3.1, for $(x,\xi) \in T^*_{H_\lambda}(V_\lambda)_{\mathrm{reg}}$,

$$Q^{\mathrm{mic}}_C \mathcal{P} = \mathsf{Ev}^0_{C,\bar\eta}\,\mathcal{P} = (\mathsf{R}\Phi_{f_\xi}\mathcal{P})_x[-\dim V_\lambda]. \tag{92}$$

The functor Q^{mic}_C is ultimately defined by [16, Proposition II.6.A.1] but, as the discussion following [1, Theorem 24.8] makes plain, it coincides with the microlocalisation functor as defined in [8, Théorème 1.9]. Consequently, functors Ev^0_λ and Q^{mic} may both be understood as perspectives on the microlocalistion functor.

We found it quite difficult to calculate Q^{mic} in examples using the tools outlined in [1], even drawing on [16]. By contrast, and as the examples presented in [10] show, we found that the vanishing cycles perspective is amenable to making calculations.

6. Arthur packets and Adams-Barbasch-Vogan packets

In this section we review the main ideas of this paper and articulate the conjectures which, taken together, lie at the heart of the concept of p-adic ABV packets. In this section, G is a quasi-split connected reductive linear algebraic group over F. When referring to work of Arthur, we will further assume G is a symplectic or special orthogonal group.

6.1. **Adams-Barbasch-Vogan packets.** We fix an admissible homomorphism $\lambda : W_F \to {}^LG$ and recall the Vogan variety V_λ from Section 2. As above, set $H_\lambda := Z_{\widehat{G}}(\lambda)$. From Proposition 2.6.2 recall that the local Langlands correspondence for pure rational forms determines a canonical bijection between isomorphism classes of simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ and $\Pi_{\mathrm{pure},\lambda}(G/F)$:

$$\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}} \leftrightarrow \Pi_{\mathrm{pure},\lambda}(G/F).$$

We use the notation $\mathcal{P}(\pi,\delta)$ for a simple H_λ-equivariant perverse sheaf on V_λ matching a representation (π,δ) of a pure rational form of G under this correspondence.

For any $\lambda \in R({}^LG)$, and any H_λ-orbit C in V_λ, the *ABV packet* for C is

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},C}(G/F) := \{[\pi,\delta] \in \Pi_{\mathrm{pure},\lambda}(G/F) \mid \mathsf{Ev}_C\,\mathcal{P}(\pi,\delta) \neq 0\}. \tag{93}$$

If $C = C_\phi$ for a Langlands parameter ϕ, we may use the notation

$$\Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi}(G/F) := \Pi^{\mathrm{ABV}}_{\mathrm{pure},C_\phi}(G/F),$$

as in (35).

6.2. **Arthur perverse sheaves.** For any H_λ-orbit C in V_λ, consider the *Arthur perverse sheaf* $\mathcal{A}_C \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ defined (up to isomorphism) by

$$\mathcal{A}_C := \bigoplus_{\mathcal{P}\in \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}} \operatorname{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}\right)\ \mathcal{P}. \tag{94}$$

By Theorem 5.3.1, Part (d), the summation can be taken the over simple $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$ supported by $\bar C$:

$$\mathcal{A}_C = \bigoplus_{\mathcal{P}\in \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}},\ \mathrm{supp}(\mathcal{P})\subseteq \bar C} \operatorname{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}\right)\ \mathcal{P}.$$

Taking the cases when $\mathcal{P} = \mathcal{IC}(C,\mathcal{L})$, consider the summand *pure packet perverse sheaf*

$$\mathcal{B}_C := \bigoplus_{\mathcal{L}\in \mathsf{Loc}_{H_\lambda}(C)^{\text{simple}}_{/\text{iso}}} \operatorname{rank}\left(\mathsf{Ev}^0_C\, \mathcal{IC}(C,\mathcal{L})\right)\ \mathcal{IC}(C,\mathcal{L}) \tag{95}$$

where the sum runs over all simple H_λ-equivariant local systems $\mathcal{L}$ on C. By Theorem 5.3.1, Part (g), $\operatorname{rank}(\mathsf{Ev}^0_C\, \mathcal{IC}(C,\mathcal{L})) = \operatorname{rank}(\mathcal{L})$, so

$$\mathcal{B}_C = \bigoplus_{\mathcal{L}\in \mathsf{Loc}_{H_\lambda}(C)^{\text{simple}}_{/\text{iso}}} \operatorname{rank}(\mathcal{L})\ \ \mathcal{IC}(C,\mathcal{L}).$$

The simple perverse sheaves appearing in $\mathcal{B}$ correspond exactly to the irreducible admissible representations in the pure Langlands packet $\Pi_{\text{pure},\phi_C}(G/F)$, where ϕ_C is the Langlands parameter matching C under Proposition 2.2.2. The perverse sheaf

$$\mathcal{C}_C := \bigoplus_{\mathcal{IC}(C_1,\mathcal{L}_1)\in \mathsf{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}},\ C_1 \lneq C} \operatorname{rank}\left(\mathsf{Ev}^0_C\, \mathcal{IC}(C_1,\mathcal{L}_1)\right)\ \mathcal{IC}(C_1,\mathcal{L}_1) \tag{96}$$

is called the *coronal perverse sheaf* for C, where the sum is taken over all $C_1 \subset \bar{C}$ with $C_1 \neq C$ and over all simple H_λ-equivariant local systems $\mathcal{L}_1$ on C_1. So

$$\mathcal{A}_C = \mathcal{B}_C \oplus \mathcal{C}_C. \tag{97}$$

6.3. **Pairing of Grothendieck groups.** Consider the pairing

$$\langle\,\cdot\,,\cdot\,\rangle : \mathsf{K}\Pi_{\text{pure},\lambda}(G/F) \times \mathsf{KPer}_{H_\lambda}(V_\lambda) \to \mathbb{Z}$$

deduced from [1] and defined on $\Pi_{\text{pure},\lambda}(G/F)$ and isomorphism classes of simple objects in $\mathsf{Per}_{H_\lambda}(V_\lambda)$ by

$$\langle [\pi,\delta],\mathcal{P}\rangle = \begin{cases} e(\mathcal{P})(-1)^{\dim \operatorname{supp}(\mathcal{P})}, & \text{if } \mathcal{P} = \mathcal{P}(\pi,\delta) \\ 0, & \text{otherwise,} \end{cases}$$

where $e(\mathcal{P})$ is the Kottwitz sign [22] of the group $G_{\delta_\mathcal{P}}$ for the pure rational form $\delta_\mathcal{P}$ of G determined by $\mathcal{P}$.

6.4. **Virtual representations from stable perverse sheaves.** The Arthur perverse sheaf $\mathcal{A}_C$ now determines a virtual representation $\eta^{ABV}_C \in \mathsf{K}\Pi_{\text{pure},\lambda}(G/F)$ by

$$\eta^{\text{ABV}}_C := (-1)^{\dim C} \sum_{[\pi,\delta]\in \Pi_{\text{pure},\lambda}(G/F)} \langle [\pi,\delta],\mathcal{A}_C\rangle\ [\pi,\delta]. \tag{98}$$

Then

$$\begin{aligned}
\eta_C^{\mathrm{ABV}} &= (-1)^{\dim C} \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\lambda}(G/F)} \langle [\pi,\delta], \mathcal{A}_C\rangle\ [\pi,\delta] \\
&= (-1)^{\dim C} \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\lambda}(G/F)} \sum_{\mathcal{P}\in \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}} \mathrm{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}\right)\ \langle [\pi,\delta], \mathcal{P}\rangle\ [\pi,\delta] \\
&= (-1)^{\dim C} \sum_{[\pi,\delta]\in\Pi^{\mathrm{ABV}}_{\mathrm{pure},C}(G/F)} \mathrm{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}(\pi,\delta)\right)\ \langle [\pi,\delta], \mathcal{P}(\pi,\delta)\rangle\ [\pi,\delta] \\
&= (-1)^{\dim C} \sum_{[\pi,\delta]\in\Pi^{\mathrm{ABV}}_{\mathrm{pure},C}(G/F)} \mathrm{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}(\pi,\delta)\right) e(\delta)(-1)^{\dim \mathrm{supp}(\mathcal{P}(\pi,\delta))}\ [\pi,\delta] \\
&= (-1)^{\dim C} \sum_{[\pi,\delta]\in\Pi^{\mathrm{ABV}}_{\mathrm{pure},C}(G/F)} \mathrm{rank}\left(\mathsf{Ev}^0_C\,\mathcal{P}(\pi,\delta)\right)\ e(\delta)\ (-1)^{\dim C_{[\pi,\delta]}}\ [\pi,\delta],
\end{aligned}$$

where $C_{[\pi,\delta]}$ is the unique H_λ-orbit in V_λ determined by the Langlands parameter of $[\pi,\delta]$. When $C = C_\psi$ for an Arthur parameter ψ, we will use the notation $\eta_\psi^{\mathrm{ABV}} := \eta_{C_\psi}^{\mathrm{ABV}}$.

6.5. **Pure Arthur packets are ABV packets.** From Section 1.11 recall the definition

$$\eta_\psi = \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\psi}(G/F)} \langle a_\psi, [\pi,\delta]\rangle_\psi\ e(\delta)\ [\pi,\delta]$$

based on Arthur's work. From Section 6.4 recall the definition

$$\eta_\psi^{\mathrm{ABV}} = (-1)^{\dim C_\psi} \sum_{[\pi,\delta]\in\Pi^{\mathrm{ABV}}_{\mathrm{pure},C}(G/F)} \mathrm{rank}\left(\mathsf{Ev}_\psi\,\mathcal{P}(\pi,\delta)\right)\ e(\delta)\ (-1)^{\dim C_{[\pi,\delta]}}\ [\pi,\delta],$$

where $C_{[\pi,\delta]}$ is the unique H_λ-orbit in V_λ determined by the Langlands parameter of $[\pi,\delta]$.

Conjecture 1. Let G be a quasi-split symplectic or special orthogonal algebraic group over a p-adic field F. Let $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ be an Arthur parameter for G. Then

$$\Pi_{\mathrm{pure},\psi}(G/F) = \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_\psi}(G/F)$$

and

$$\eta_\psi = \eta_\psi^{\mathrm{ABV}}.$$

In particular, for every $[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)$,

$$\langle a_\psi, [\pi,\delta]\rangle_\psi = (-1)^{\dim C_\psi - \dim C_{[\pi,\delta]}}\,\mathrm{rank}(\mathsf{Ev}_\psi\,\mathcal{P}(\pi,\delta)),$$

where $a_\psi \in A_\psi$ is defined in Section 1.11.

We will also find another perspective useful regarding Conjecture 1. Using the pairing of Section 6.3, it is easy to check

$$\langle \eta_\psi^{\mathrm{ABV}}, \mathcal{P}(\pi,\delta)\rangle = (-1)^{\dim C_\psi}\,\mathrm{rank}\left(\mathsf{Ev}_\psi\,\mathcal{P}(\pi,\delta)\right).$$

for $[\pi,\delta] \in \Pi_{\mathrm{pure},\lambda_\psi}(G/F)$. Extend to $\mathcal{P} \in \mathsf{KPer}_{H_{\lambda_\psi}}(V_{\lambda_\psi})$ by linearity, we have

$$\langle \eta_\psi^{\mathrm{ABV}}, \mathcal{P}\rangle = (-1)^{\dim C_\psi}\,\mathrm{rank}\left(\mathsf{Ev}_\psi\,\mathcal{P}\right).$$

Thus, Conjecture 1 is equivalent to:

$$\langle \eta_\psi, \mathcal{P}\rangle = (-1)^{\dim C_\psi}\,\mathrm{rank}(\mathsf{Ev}_\psi\,\mathcal{P}), \tag{99}$$

for all $\mathcal{P} \in \mathsf{KPer}_{H_{\lambda_\psi}}(V_{\lambda_\psi})$.

6.6. **Representations of the component group of an Arthur parameter.** Conjecture 1 is itself a consequence of Conjecture 2, below, which claims that Ev_ψ gives a way to calculate the functions (26).

From Section 1.11, recall the definition

$$\eta_{\psi,s} = \sum_{[\pi,\delta]\in\Pi_{\mathrm{pure},\psi}(G/F)} e(\delta)\ \langle a_\psi a_s, [\pi,\delta]\rangle_\psi\ [\pi,\delta].$$

for $s \in Z_{\widehat{G}}(\psi)$, where a_s is the image of s in A_ψ. We define

$$\eta^{\mathrm{ABV}}_{\psi,s} := \sum_{[\pi,\delta]\in\Pi^{\mathrm{ABV}}_{\mathrm{pure},\psi}(G/F)} (-1)^{\dim C_\psi - \dim C_{[\pi,\delta]}} \operatorname{trace}\left(\mathsf{Ev}_\psi \mathcal{P}(\pi,\delta)\right)(a_s)\ e(\delta)\ [\pi,\delta]. \tag{100}$$

Conjecture 2. Let G be a quasi-split symplectic or special orthogonal p-adic group. Let $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ be an Arthur parameter. Then

$$\Pi_{\mathrm{pure},\psi}(G/F) = \Pi^{\mathrm{ABV}}_{\mathrm{pure},\phi_\psi}(G/F)$$

and

$$\eta_{\psi,s} = \eta^{\mathrm{ABV}}_{\psi,s}, \tag{101}$$

for every $s \in Z_{\widehat{G}}(\psi)$. Equivalently, for every $[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)$ and for every $s \in Z_{\widehat{G}}(\psi)$,

$$\langle a_s a_\psi, [\pi,\delta]\rangle_\psi = (-1)^{\dim C_\psi - \dim C_{[\pi,\delta]}} \operatorname{trace}\left(\mathsf{Ev}_\psi \mathcal{P}(\pi,\delta)\right)(a_s), \tag{102}$$

where $a_\psi \in A_\psi$ is defined in Section 1.11 and a_s is the image of s in A_ψ.

In [11] we use the following version of Conjecture 2. Using the pairing of Grothendieck groups from Section 6.3, Conjecture 2 is equivalent to:

$$\langle \eta_{\psi,s}, \mathcal{P}\rangle = (-1)^{\dim(C_\psi)} \operatorname{trace}(\mathsf{Ev}_\psi \mathcal{P})(a_s), \tag{103}$$

for every $s \in Z_{\widehat{G}}(\psi)$ and for every $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$.

Conjecture 2 gives a new way to calculate the character $\langle a_s, [\pi,\delta]\rangle_\psi$ when π is an admissible representation of $G_\delta(F)$ for a pure rational form δ of G, and when the complete Langlands parameter for (π,δ) is known; this fact is illustrated with examples in [10]. Conjecture 2 also suggests how to define the character for Langlands parameters that are not of Arthur type. We also show several examples of this strategy in [10].

6.7. **A basis for strongly stable virtual representations.** Conjecture 3, below, is an adaptation of [32, Conjecture 8.15′]. It suggests how to extend the definition of Arthur packets from Langlands parameters of Arthur type to all Langlands parameters and also how to find the associated stable distributions.

Conjecture 3. Let G be a quasi-split connected reductive linear algebraic group over F. For any $\lambda \in \Lambda({}^LG)$ (Section 2.1), the virtual representations η^{ABV}_C are strongly stable in the sense of [32, 1.6] and

$$\{\eta^{\mathrm{ABV}}_C \mid\ H_\lambda\text{-orbits } C \subseteq V_\lambda\}$$

is a basis for the Grothendieck group of strongly stable virtual representations with infinitesimal character λ.

It should be noted that strongly stable virtual representations of G produce stable virtual representations, and thus stable distributions, of all the groups $G_\delta(F)$ as δ ranges over pure rational forms of G. It should also be noted that here we dropped the hypothesis that G is a quasi-split symplectic or special orthogonal p-adic group, which appeared in Conjectures 1 and 2, and replaced it with the hypothesis that G is any quasi-split connected reductive algebraic group over F. The scope of Conjecture 3 is therefore very broad, as it refers to all pure inner forms of all quasi-split connected reductive p-adic groups.

In [10] we gather evidence for Conjectures 1, 2 and 3 by verifying them for 38 admissible representations of 12 p-adic groups.

References

[1] Jeffrey Adams, Dan Barbasch, and David A. Vogan Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston, Inc., Boston, MA, 1992.

[2] James Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[3] ———, *On the transfer of distributions: weighted orbital integrals*, Duke Math. J. **99** (1999), no. 2, 209–283.

[4] ———, *A note on L-packets*, Pure Appl. Math. Q. **2** (2006), no. 1, Special Issue: In honor of John H. Coates., 199–217.

[5] ———, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71. Orbites unipotentes et représentations, II.

[6] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analyse et topologie sur les espaces singuliers, I (CIRM, Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[7] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[8] Jean-Luc Brylinski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque **140-141** (1986), 3–134, 251. Géométrie et analyse microlocales.

[9] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994.

[10] Clifton Cunningham, Andrew Fiori, Ahmed Moussaoui, James Mracek, and Bin Xu, *Arthur packets and Adams-Barbasch-Vogan packets for p-adic groups, 2: Examples.* In preparation.

[11] Clifton Cunningham, Andrew Fiori, Ahmed Moussaoui, and Bin Xu, *Arthur packets and Adams-Barbasch-Vogan packets for p-adic groups: Unipotent representations of odd orthogonal groups*. In preparation.

[12] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome I. Propriétés générales des schémas en groupes*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 7, Société Mathématique de France, Paris, 2011.

[13] M. Demazure and A. Grothendieck, *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin-New York, 1970.

[14] Pierre Deligne and Nicholas Katz, *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II); Dirigé par P. Deligne et N. Katz.

[15] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Invent. Math. **84** (1986), no. 2, 327–402.

[16] Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 14, Springer-Verlag, Berlin, 1988.

[17] Benedict H. Gross and Mark Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), no. no. 3, 431–508.

[18] Luc Illusie (ed.), *Cohomologie l-adique et fonctions L*, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5); Edité par Luc Illusie.

[19] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Mathematics, Vol. 114, Springer-Verlag, Berlin-New York, 1970.

[20] Masaki Kashiwara, *Systems of microdifferential equations*, Progress in Mathematics, vol. 34, Birkhäuser Boston, Inc., Boston, MA, 1983. Based on lecture notes by Teresa Monteiro Fernandes translated from the French; With an introduction by Jean-Luc Brylinski.

[21] Robert E. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650.

[22] ———, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297.

[23] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque **255** (1999).

[24] George Lusztig, *Cuspidal local systems and graded Hecke algebras. II*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 217–275. With errata for Part I [Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 145–202].

[25] ———, *Study of Perverse Sheaves Arising from Graded Lie Algebras*, Advances in Mathematics **112** (1995), no. 2, 147 - 217.

[26] ———, *Quivers, perverse sheaves, and quantized enveloping algebras*, Journal of the american mathematical society **4** (1991), no. 2, 365–421.

[27] I. Mirković and K. Vilonen, *Characteristic varieties of character sheaves*, Invent. Math. **93** (1988), no. 2, 405–418.

[28] C. Mœglin, *Multiplicité 1 dans les paquets d'Arthur aux places p-adiques*, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011.

[29] Dipendra Prasad, *A 'relative' local Langlands correspondence.* arXiv:1512.04347v1 [math.NT].

[30] V. S. Pjasenkiĭ, *Linear Lie groups that act with a finite number of orbits*, Functional Anal. Appl. **9** (1975), no. 4, 351–353.

[31] Angelo Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[32] David A. Vogan Jr., *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.

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