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COMMUTATIVE CHARACTER SHEAVES AND GEOMETRIC TYPES FOR SUPERCUSPIDAL REPRESENTATIONS

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Abstract. We show that the types for supercuspidal representations of tamely ramified p-adic groups that appear in Jiu-Kang Yu's work are geometrizable, subject to a mild hypothesis. To do this we must find the function-sheaf dictionary for one-dimensional characters of arbitrary smooth group schemes over finite fields. In previous work we considered the case of commutative smooth group schemes and found that the standard definition of character sheaves produced a dictionary with a nontrivial kernel. In this paper we give a modification of the category of character sheaves that remedies this defect, and is also extensible to non-commutative groups. We then use these *commutative character sheaves* to geometrize the linear characters that appear in the types introduced by Jiu-Kang Yu. We combine these sheaves with Lusztig's character sheaves on reductive algebraic groups over finite fields and the geometrization of the Weil representation found by Gurevich and Hadani, to define *geometric types* for supercuspidal representations of tamely ramified p-adic groups.

Introduction

As proved by Ju-Lee Kim in [13], all irreducible supercuspidal representations of tamely ramified p-adic groups can be built from "data" introduced by Jiu-Kang Yu in [17, §15]. While the type, in the sense of Bushnell & Kutzko [4], of a supercuspidal representation built from Yu data can be constructed directly from the datum, it is convenient to consider an intermediate object, introduced in [17, Remark 15.4], which we call a *Yu type datum*. Yu type data are studied in [18], which concludes with the following observation.

> *Therefore, up to some linear characters, all the ingredient representations are on groups of the form $\underline{H}(\mathcal{O})$, where $\underline{H}$ is a smooth group scheme over [a henselian discrete valuation ring with finite residue field κ] $\mathcal{O}$, and the representations are inflated from $\underline{H}(\kappa)$. These results suggest that algebraic geometry and group schemes should play an important role in the representation theory of p-adic groups.*

In this paper we follow the suggestion above by showing that Yu type data are geometrizable, in the following sense. A Yu type datum determines a sequence of representations ${}^\circ\rho_i$ of compact p-adic groups ${}^\circ K^i$, for $i = 0, \dots, d$, such that $({}^\circ K^d, \rho_d)$ is a type for a supercuspidal representation of a p-adic group. Let R be the ring of integers of a local field with finite residue field k. The main result of [18] shows how to find, for each $i = 0, \dots, d$, a smooth group scheme $\underline{G}^i$ over

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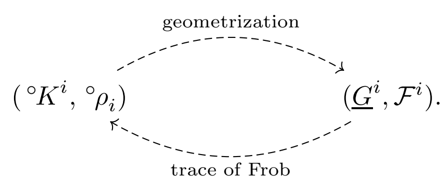
the ring R with $\underline{G}^i(R) = {}^\circ K^i$. The geometrization of the Yu type datum uses Lusztig's theory of character sheaves on reductive groups over finite fields, so it is necessary to assume that the geometric component group of the reductive quotient of the special fibre of the group scheme $\underline{G}^0$ is cyclic, in order to bring his work to bear. Under this assumption we show how each representation ${}^\circ\rho_i$ can be replaced by a pair $(\underline{G}^i, \mathcal{F}^i)$, where $\mathcal{F}^i$ is a *rational virtual* sheaf complex on the Greenberg transform G^i of $\underline{G}^i$, by which we mean $\mathcal{F}^i$ is an element in the group obtained by tensoring the Grothendieck group of such sheaves with $\mathbb{Q}$. Writing $t_{\mathcal{F}^i}$ for the function on $G^i(k) = \underline{G}^i(R) = {}^\circ K^i$ obtained by evaluating the trace of the action of Frobenius on the rational virtual complex $\mathcal{F}^i$, we show in Theorem 4.2 that

$$t_{\mathcal{F}^i} = \mathrm{Tr}({}^\circ\rho_i). \tag{1}$$

By this theorem, then, we obtain geometric avatars for each type in a Yu datum:



We refer to the pair $(\underline{G}^d, \mathcal{F}^d)$ as a *geometric type*.

To prove Theorem 4.2, we must find a way to geometrize linear characters of groups of the form $\underline{H}(R)$, where $\underline{H}$ is a smooth group scheme over R. In order to do so in a systematic manner, we begin this paper by describing the function-sheaf dictionary for characters of arbitrary smooth group schemes over finite fields. When coupled with the Greenberg transform, this dictionary will allow for the geometrization of linear characters of $\underline{H}(R)$.

The function-sheaf dictionary over a finite field k [6, Sommes trig.] provides a way of encoding functions on the k-rational points of an algebraic group G as ℓ-adic local systems on G. More specifically, if G is a connected, commutative, algebraic group then there is a certain category $\mathcal{CS}(G)$ of rank-one local systems on G and an explicit isomorphism between isomorphism classes of objects in $\mathcal{CS}(G)$ and $G(k)^* := \mathrm{Hom}(G(k), \bar{\mathbb{Q}}_\ell^\times)$; the isomorphism is given by mapping $\mathcal{L}$ to the function $\mathrm{Tr}_G : g \mapsto \mathrm{Tr}(\mathrm{Fr}\,|\mathcal{L}_g)$.

In previous work [5], we generalized the function-sheaf dictionary to smooth commutative group schemes G, allowing for non-connected groups. We gave a description of the category $\mathcal{CS}(G)$ in this context, as well as an epimorphism $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$. In contrast to the connected case, Tr_G may have nontrivial kernel; we gave an explicit description of its kernel as $\mathrm{H}^2(\pi_0(\bar{G}), \bar{\mathbb{Q}}_\ell^\times)^{\mathrm{Fr}}$ [5, Theorem 3.6].

We repair this defect in the function-sheaf dictionary by describing a full subcategory $\mathcal{CCS}(G)$ of $\mathcal{CS}(G)$ so that Tr_G restricts to an isomorphism $\mathcal{CCS}(G)_{/\mathrm{iso}} \to G(k)^*$. We refer to objects of $\mathcal{CS}(G)$ as character sheaves and objects in $\mathcal{CCS}(G)$ as *commutative character sheaves*, since the passage from $\mathcal{CS}(G)$ to $\mathcal{CCS}(G)$ involves a condition that exchanges the inputs to the multiplication morphism on G (see Definition 2.1). When G is connected, all character sheaves on G are commutative.

Category $\mathcal{CCS}(G)$ clarifies several questions about $\mathcal{CS}(G)$. Invisible character sheaves [5, Def. 2.8] are precisely those $\mathcal{L}$ with $\mathrm{Tr}_G(\mathcal{L}) = 1$ that are not commutative. Moreover, $\mathrm{Tr}_G^{-1} : G(k)^* \to \mathcal{CCS}(G)_{/\mathrm{iso}}$ provides a canonical splitting of $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ [5, Rem. 3.7].

Next, we broaden our scope further to encompass smooth group schemes G over k that are not necessarily commutative. We assume G is smooth, but not that it is connected, reductive or commutative. The category $\mathcal{CS}(G)$ has a straightforward generalization to this case, but again there are more character sheaves than there are characters, as pointed out by Kamgarpour [12, (1.1)]. We then define category $\mathcal{CCS}(G)$ for such G and a forgetful functor to $\mathcal{CS}(G)$ so that $\mathrm{Tr}_G : \mathcal{CCS}(G)_{/\mathrm{iso}} \to G_{\mathrm{ab}}(k)^*$ is an isomorphism. Since $G_{\mathrm{ab}}(k)^*$ surjects onto $G(k)^*$, it follows that for each character $\chi \in G(k)^*$ there is a commutative character sheaf $\mathcal{L}$ on G with $\mathrm{Tr}_G(\mathcal{L}) = \chi$. Moreover, we find that pullback along the quotient $q : G \to G_{\mathrm{ab}}$ defines an equivalence of categories $\mathcal{CCS}(G_{\mathrm{ab}}) \to \mathcal{CCS}(G)$. The functor $\mathcal{CCS}(G) \to \mathcal{CS}(G)$ is not essentially surjective, missing the kinds of linear character sheaves highlighted by Kamgarpour.

In order to provide further justification for referring to objects in $\mathcal{CCS}(G)$ as commutative character sheaves, suppose for the moment that G is a connected, reductive algebraic group over k. Let $\bar{\mathcal{L}}$ be the geometric part of an object in $\mathcal{CCS}(G)$; see Section 1. Let T be a maximal torus in $\bar{G}$ and let $\bar{\mathcal{L}}_T$ be the restriction of $\bar{\mathcal{L}}$ to T. Then the perverse sheaf $\bar{\mathcal{L}}[\dim G]$ appears in the semisimple complex $\mathrm{ind}_{B,T}^{\bar{G}}(\bar{\mathcal{L}}_T)$ produced by parabolic induction. It follows that every object in $\mathcal{CCS}(G)$ determines a Frobenius-stable character sheaf on G, in the sense of [14, Def. 2.10]. Of course, the sheaves arising in this way represent a small part of Lusztig's geometrization of characters of representations of connected, reductive groups over finite fields, but they are precisely those needed to describe one-dimensional characters of such groups.

Armed with the function-sheaf dictionary for smooth group schemes over finite fields, we return to the task of geometrizing Yu type data. The proof of Theorem 4.2 requires: Yu's work on smooth integral models [18]; the geometrization of the character of the Heisenberg-Weil representation over finite fields by Gurevich & Hadani [9]; Lusztig's character sheaves on reductive groups over finite fields; and finally, the function-sheaf dictionary for characters of smooth group schemes over finite fields, now at our disposal in Theorem 3.12. These pieces are assembled in Section 4.4, where we prove Theorem 4.2. With this theorem, we provide all of the ingredients needed to parametrize supercuspidal representations of arbitrary depth in the same category: rational virtual Weil perverse sheaves on group schemes over finite fields.

The hypothesis in Theorem 4.2 – that the geometric component group of the reductive quotient of the special fibre of the smooth group scheme $\underline{G}^0$ appearing in the Yu type datum is cyclic – is required only because Lusztig's theory of character sheaves has the same hypothesis. If Lusztig's theory of character sheaves can be generalized to all disconnected reductive algebraic groups, then the hypothesis in Theorem 4.2 can be removed.

We now summarize the sections of the paper in more detail. In Section 1, we recall the category $\mathcal{CS}(G)$ from [5] and note that it still makes sense when G is not commutative. We focus on the case of commutative G in Section 2, giving the definition of a commutative character sheaf and proving our first main theorem, that $\operatorname{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ induces an isomorphism on $\mathcal{CCS}(G)_{/\mathrm{iso}}$. Passing to the case that G is non-commutative, we give the definition of and main results about commutative character sheaves in Section 3. We note that we should only consider character sheaves that arise via pullback from G_{ab} in order to eliminate those that have nontrivial restriction to the derived subgroup. This observation underlies the definition of commutative character sheaves for non-commutative G. We state our second main result, Theorem 3.12, that pullback along the abelianization map defines an equivalence of categories $\mathcal{CCS}(G) \to \mathcal{CCS}(G_{\mathrm{ab}})$. In Section 3.3, we use Galois cohomology to describe the relationship between $G(k)^*$ and $G_{\mathrm{ab}}(k)^*$. We also compute the automorphism groups in $\mathcal{CCS}(G)$. In Section 4 we use Theorem 3.12 to geometrize types for supercuspidal representations of p-adic groups, in a sense made precise in Theorem 4.2. As preparation for the proof, we review some facts about the Heisenberg-Weil representation and its geometrization, in Section 4.2. Then, in Section 4.3, we review Yu's theory of types and his study of smooth integral models. These elements are pulled together in Section 4.4, where the proof Theorem 4.2 is given.

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1. Recollections and definitions

Let G be a smooth group scheme over a finite field k; that is, let G be a group scheme over k for which the structure morphism $G \to \operatorname{Spec}(k)$ is smooth in the sense of [8, Def 17.3.1]. This implies $G \to \operatorname{Spec}(k)$ is locally of finite type, but not that it is of finite type. We remark that the identity component G^0 of G is of finite type over k, while the component group scheme $\pi_0(G)$ of G is an étale group scheme over k, and both are smooth over k.

In this paper we use a common formalism for Weil sheaves, writing $\mathcal{L}$ for the pair $(\bar{\mathcal{L}}, \phi)$, where $\bar{\mathcal{L}}$ is an ℓ-adic sheaf on $\bar{G} := G \otimes_k \bar{k}$ and where $\phi : \operatorname{Fr}^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of ℓ-adic sheaves. We also follow convention by referring to $\mathcal{L}$ as a Weil sheaf on G. If $\mathcal{L}$ and $\mathcal{L}' := (\bar{\mathcal{L}}', \phi')$ are Weil sheaves, we write $\alpha : \mathcal{L} \to \mathcal{L}'$ for a morphism $\alpha : \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ such that

$$\begin{array}{ccc} \operatorname{Fr}^* \bar{\mathcal{L}} & \xrightarrow{\operatorname{Fr}^* \alpha} & \operatorname{Fr}^* \bar{\mathcal{L}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ \bar{\mathcal{L}} & \xrightarrow{\quad\alpha\quad} & \bar{\mathcal{L}} \end{array}$$

commutes. These conventions simplify notation considerably, but they were not employed in [5].

We write $m : G \times G \to G$ for the multiplication morphism, and $G(k)^*$ for $\operatorname{Hom}(G(k), \bar{\mathbb{Q}}_\ell^\times)$. Define $\theta : G \times G \to G \times G$ by $\theta(g,h) = (h,g)$.

When G is commutative, a *character sheaf* on G is a triple $(\bar{\mathcal{L}}, \mu, \phi)$, where $\bar{\mathcal{L}}$ is a rank-one ℓ-adic local system on $\bar{G}$, $\mu : \bar{m}^*\bar{\mathcal{L}} \to \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$, and $\phi : \operatorname{Fr}_G^* \bar{\mathcal{L}} \to \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G}$; the triple $(\bar{\mathcal{L}}, \mu, \phi)$ is required to satisfy certain conditions [5, Def. 1.1]. Write $\mathcal{CS}(G)$ for the category of character sheaves on G.

Even when G is not commutative, the category $\mathcal{CS}(G)$, defined as in [5, Def. 1.1], still makes sense. In order to distinguish the resulting objects from the character sheaves of Lusztig, we will refer to the former as *linear character sheaves* (to evoke the one-dimensional character sheaves of [12]).

2. Commutative character sheaves on commutative groups

We consider first the case that G is commutative, which we will later apply to the case of general smooth G. Let $\mathcal{L}$ be a character sheaf on G. Since $m = m \circ \theta$ in this case, there is a canonical isomorphism $\xi : m^*\mathcal{L} \to \theta^* m^*\mathcal{L}$. There is also an isomorphism $\vartheta : \mathcal{L} \boxtimes \mathcal{L} \to \theta^*(\mathcal{L} \boxtimes \mathcal{L})$ given on stalks by the canonical map $\bar{\mathcal{L}}_g \otimes \bar{\mathcal{L}}_h \to \bar{\mathcal{L}}_h \otimes \bar{\mathcal{L}}_g$.

Definition 2.1. A character sheaf $(\mathcal{L}, \mu)$ on a smooth commutative group scheme G is *commutative* if the following diagram of Weil sheaves on $G \times G$ commutes.

$$\begin{array}{ccc} m^*\mathcal{L} & \xrightarrow{\mu} & \mathcal{L} \boxtimes \mathcal{L} \\ {\scriptstyle \xi}\downarrow{\scriptstyle m = m\circ\theta} & & \downarrow{\scriptstyle \vartheta} \\ \theta^*(m^*\mathcal{L}) & \xrightarrow{\theta^*\mu} & \theta^*(\mathcal{L} \boxtimes \mathcal{L}) \end{array}$$

We write $\mathcal{CCS}(G)$ for the full subcategory of $\mathcal{CS}(G)$ consisting of commutative character sheaves.

In [5, Theorem 3.6], we showed that $\mathrm{Tr}_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to G(k)^*$ is surjective and explicitly computed its kernel. In this section, we show that the corresponding map $\mathrm{Tr}_G : \mathcal{CCS}(G)_{/\mathrm{iso}} \to G(k)^*$ for commutative character sheaves is an isomorphism. We begin by reinterpreting Definition 2.1 in terms of cocycles.

Let G be a commutative étale group scheme over k. For a character sheaf $\mathcal{L}$ on G, recall [5, §2.3] that $S_G : \mathcal{CS}(G)_{/\mathrm{iso}} \to \mathrm{H}^2(E_G^\bullet)$ is an isomorphism mapping $[\mathcal{L}]$ to $[\alpha \oplus \beta]$, where $E_G^\bullet$ is the total space of the zeroth page of the Hochschild-Serre spectral sequence, $\alpha \in {}^\circ K^0(\mathcal{W}, {}^\circ K^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ is obtained from μ and $\beta \in {}^\circ K^1(\mathcal{W}, {}^\circ K^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ is obtained from ϕ.

Let $a \in Z^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ correspond to α. We say that $[\alpha \oplus \beta] \in \mathrm{H}^2(E_G^\bullet)$ is *symmetric* if $a(x, y) = a(y, x)$ for all $x, y \in \bar{G}$. This condition is well defined, since every coboundary in $B^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ is symmetric. The connection between commutative character sheaves and symmetric classes is given in the following lemma.

Lemma 2.2. *Suppose G is a smooth commutative group scheme, and let $\mathcal{L}$ be a character sheaf on G. Then $\mathcal{L}$ is commutative if and only if $S_G(\mathcal{L})$ is symmetric.*

Proof. The symmetry of $S_G(\mathcal{L})$ is a direct consequence of the commutativity of the diagram in Definition 2.1 after choosing bases for each stalk. □

We may similarly define a symmetric class in $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ to be one represented by a symmetric 2-cocycle. The following lemma will allow us to show that there are no invisible commutative character sheaves.

Lemma 2.3. *Let $\bar{G}$ be a commutative group. Then the only symmetric class in $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ is the trivial class.*

Proof. By the universal coefficient theorem,

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(\bar{G}, \mathbb{Z}), \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{H}^n(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(\mathrm{H}_n(\bar{G}, \mathbb{Z}), \bar{\mathbb{Q}}_\ell^\times) \to 0$$

is exact for all $n > 0$. When $n = 2$, using the fact that $\bar{G}$ is commutative, we have that $\mathrm{H}_1(\bar{G}, \mathbb{Z}) \cong \bar{G}$ and that $\mathrm{H}_2(\bar{G}, \mathbb{Z}) \cong \wedge^2 \bar{G}$. We get

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(\wedge^2 \bar{G}, \bar{\mathbb{Q}}_\ell^\times) \to 0.$$

The map $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(\wedge^2 \bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ maps a 2-cocycle f to the alternating function

$$(x, y) \mapsto \frac{f(x, y)}{f(y, x)}.$$

Thus the cohomology classes represented by symmetric cocycles are precisely those in the image of $\mathrm{Ext}^1_{\mathbb{Z}}(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$. But $\mathrm{Ext}^1_{\mathbb{Z}}(-, \bar{\mathbb{Q}}_\ell^\times)$ vanishes because $\bar{\mathbb{Q}}_\ell^\times$ is divisible. □

Lemma 2.4. *If G is a connected commutative algebraic group over k then every character sheaf on G is commutative.*

Proof. Suppose $S_G(\mathcal{L}) = [\alpha \oplus \beta] \in \mathrm{H}^2(E_G^\bullet)$. We can use étale descent to see that pullback by the Lang isogeny defines an equivalence of categories between local systems on G and $G(k)$-equivariant local systems on G. Thus every character sheaf $\mathcal{L}$ on G arises through the Lang isogeny, together with a character $G(k) \to \bar{\mathbb{Q}}_\ell^\times$. Pushing forward the Lang isogeny along this character defines an extension of $\bar{G}$ by $\bar{\mathbb{Q}}_\ell^\times$ whose class is fixed by Frobenius; let $a \in Z^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)$ be a representative 2-cocycle. Then a corresponds to the $\alpha \in {}^\circ K^0(\mathcal{W}, {}^\circ K^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$, above. Since the covering group of the Lang isogeny is $G(k)$, which is commutative, the class of this extension satisfies $a(x, y) = a(y, x)$ for all $x, y \in \bar{G}$. This shows that $S_G(\mathcal{L})$ is symmetric. It follows from Lemma 2.2 that $\mathcal{L}$ is a commutative character sheaf. □

Theorem 2.5. *If G is a smooth commutative group scheme over k then $\mathrm{Tr}_G : \mathcal{CCS}(G)_{/iso} \to G(k)^*$ is an isomorphism.*

Proof. Suppose first that G is étale. Consider the isomorphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \mathrm{Tr}_G & \longrightarrow & \mathcal{CS}(G)_{/\mathrm{iso}} & \xrightarrow{\mathrm{Tr}_G} & G(k)^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle S_G} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & \mathrm{H}^2(E_G^\bullet) & \longrightarrow & \mathrm{H}^1(\mathcal{W}, \mathrm{H}^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)) & \longrightarrow & 0
\end{array}$$

from [5, Prop. 2.7].

Suppose that $\mathcal{L}$ is a commutative character sheaf with $t_{\mathcal{L}} = 1$, and set $[\alpha, \beta] = S_G([\mathcal{L}])$. Then $S_G([\mathcal{L}])$ is in the image of $\mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times)^{\mathcal{W}}$, so is cohomologous to $[\alpha', 0]$. Since α is symmetric and coboundaries are symmetric, α' is symmetric as well. So by Lemma 2.3, α' is cohomologically trivial, and thus $[\mathcal{L}]$ is trivial as well.

To see that Tr_G is still surjective on $\mathcal{CCS}(G)_{/\mathrm{iso}}$, note that the character sheaf constructed in the proof of [5, Prop. 2.6] has trivial α, and is thus commutative.

For general smooth commutative group schemes, we use Lemma 2.4 and the snake lemma, as in the proof of [5, Theorem 3.6] □

Remark 2.6. Since $\mathrm{H}^0(\mathcal{W}, \mathrm{H}^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times))$ is not necessarily trivial [5, Ex. 2.10], the functor $\mathcal{CCS}(G) \to \mathcal{CS}(G)$ is not necessarily essentially surjective. Indeed, the invisible character sheaves [5, Def. 2.8] defined in our previous paper are precisely those non-commutative character sheaves with trivial trace of Frobenius.

3. Commutative character sheaves on non-commutative groups

We now consider the case of a smooth group scheme without the commutativity assumption. We start by relating character sheaves on G to character sheaves on its abelianization.

If $\chi \in G(k)^*$ is a character, it must vanish on the derived subgroup $G_{\mathrm{der}}(k)$. Kamgarpour gives an example [12, (1.1)] of a character sheaf that does not vanish on G_{der}, defined by the extension

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1.$$

In order to obtain a relationship between character sheaves on G and characters of $G(k)$, he opts to give a different definition of commutator and, in doing so, introduces a 'stacky abelianization' of G in order to geometrize characters of $G(k)$. Since we have already seen the need to adapt the notion of character sheaf, even in the commutative case, we instead add restrictions to the definition of commutative character sheaf and leave the definition of G_{der} unchanged, allowing us to use the schematic abelianization of G in the geometrization of characters of $G(k)$.

We begin this section with the main definition in this paper - the category $\mathcal{CCS}(G)$ of commutative character sheaves, Definition 3.2. This definition is delicate and somewhat technical, but it is vindicated in Theorem 3.10 which shows that $\mathcal{CCS}(G)$ is equivalent to the category of commutative character sheaves on the abelianization G_{ab} of G. To prove Theorem 3.10 we use descent theory in Section 3.2, in the process giving insight into Definition 3.2. Section 3 concludes with Theorem 3.12, showing that the dictionary from $\mathcal{CCS}(G)$ to characters of $G(k)$ is as precise as possible.

3.1. Main definition.

In order to get character sheaves that correspond to characters in $G(k)^*$, we must discard those character sheaves whose restriction to the derived subgroup is nontrivial. Recall from Section 1 that we refer to objects in category $\mathcal{CS}(G)$, defined as in [5, Def. 1.1], as linear character sheaves when G is smooth but not necessarily commutative. We define the following category to track the trivialization on the derived subgroup; commutative character sheaves will then be defined as a subcategory.

Definition 3.1. Let $\mathcal{CS}_{\mathrm{ab}}(G)$ denote the category of triples $(\mathcal{L}, \mu, \beta)$ where $(\mathcal{L}, \mu) \in \mathcal{CS}(G)$ and $\beta : \mathcal{L}|_{G_{\mathrm{der}}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ is an isomorphism in $\mathcal{CS}(G_{\mathrm{der}})$. A morphism $(\mathcal{L}, \mu, \beta) \to (\mathcal{L}', \mu', \beta')$ is a morphism $\alpha : (\mathcal{L}, \mu) \to (\mathcal{L}', \mu')$ in $\mathcal{CS}(G)$ such that $\beta = \beta' \circ \alpha|_{G_{\mathrm{der}}}$.

The reason for tracking β is that it determines an isomorphism $\gamma : m^*\mathcal{L} \to \theta^* m^* \mathcal{L}$ as follows, which will replace the ξ of Definition 2.1. Let $i : G \to G$ be inversion and $c : G \times G \to G_{\mathrm{der}}$ be the commutator map, defined by $c(x, y) = xyx^{-1}y^{-1}$. Both are smooth morphisms of k-schemes. Set $m' = i \circ m \circ \theta$ and let $j_{\mathrm{der}} : G_{\mathrm{der}} \to G$ be inclusion; then $j_{\mathrm{der}} \circ c = m \circ (m \times m')$. Then $\beta : \mathcal{L}|_{G_{\mathrm{der}}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ determines the isomorphism $\gamma' : m^*\mathcal{L} \otimes \theta^* m^* i^* \mathcal{L} \to (\bar{\mathbb{Q}}_\ell)_{G\times G}$ by the diagram of isomorphisms below.

$$\begin{array}{ccc}
c^*(\mathcal{L}|_{G_{\mathrm{der}}}) & \xrightarrow{c^*(\beta)} & c^*((\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}) \\
\| & & \| \\
c^* j_{\mathrm{der}}^* \mathcal{L} & & (\bar{\mathbb{Q}}_\ell)_{G\times G} \\
{\scriptstyle j_{\mathrm{der}}\circ c = m\circ(m\times m')}\downarrow & & \uparrow{\scriptstyle \gamma'} \\
(m\times m')^* m^*\mathcal{L} & & m^*\mathcal{L}\otimes\theta^* m^* i^*\mathcal{L} \\
{\scriptstyle (m\times m')^*(\mu)}\downarrow & & \uparrow{\scriptstyle m'=i\circ m\circ\theta} \\
(m\times m')^*(\mathcal{L}\boxtimes\mathcal{L}) & = & m^*\mathcal{L}\otimes(m')^*\mathcal{L}
\end{array} \tag{2}$$

In the diagram above, the arrows labeled with equations come from canonical isomorphisms of functors on Weil sheaves derived from the equations; so, for example, the middle left isomorphism comes from $(m \times m')^* m^* \cong c^* j_{\mathrm{der}}^*$ since $j_{\mathrm{der}} \circ c = m \circ (m \times m')$. Using the monoidal structure of the category of Weil local systems on $G \times G$, the isomorphism $\gamma' : m^*\mathcal{L} \otimes \theta^* m^* i^* \mathcal{L} \to (\bar{\mathbb{Q}}_\ell)_{G\times G}$ defines an isomorphism

$$m^*\mathcal{L} \to (\theta^* m^* i^* \mathcal{L})^\vee.$$

Applying the canonical isomorphisms $(\theta^* m^* i^* \mathcal{L})^\vee \cong \theta^* m^* i^* (\mathcal{L}^\vee)$ and $i^*(\mathcal{L}^\vee) \cong \mathcal{L}$, this map provides the promised isomorphism

$$\gamma : m^*\mathcal{L} \longrightarrow \theta^* m^* \mathcal{L}.$$

Definition 3.2. The category $\mathcal{CCS}(G)$ of commutative character sheaves on G is the full subcategory of $\mathcal{CS}_{\mathrm{ab}}(G)$ consisting of triples $(\mathcal{L}, \mu, \beta)$ such that the following diagram of Weil sheaves on $G \times G$ commutes:

$$\begin{array}{ccc}
m^*\mathcal{L} & \xrightarrow{\mu} & \mathcal{L}\boxtimes\mathcal{L} \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\vartheta} \\
\theta^*(m^*\mathcal{L}) & \xrightarrow{\theta^*\mu} & \theta^*(\mathcal{L}\boxtimes\mathcal{L}).
\end{array}$$

Here $\gamma : m^*\mathcal{L} \to \theta^* m^* \mathcal{L}$ is the isomorphism built from $\beta : \mathcal{L}|_{G_{\mathrm{der}}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ as above.

3.2. **Descent.** In this section we give an equivalence of categories between $\mathcal{CS}(G_{\mathrm{ab}})$ and $\mathcal{CS}_{\mathrm{ab}}(G)$ and use it to describe the pullback functor $q^* : \mathcal{CS}(G_{\mathrm{ab}}) \to \mathcal{CS}(G)$ in terms of the forgetful functor $\mathcal{CS}_{\mathrm{ab}}(G) \to \mathcal{CS}(G)$, where $q : G \to G_{\mathrm{ab}}$ is the abelianization quotient with kernel G_{der}. But first, in order to study commutative character sheaves, we need some auxiliary categories.

3.2.1. *Equivariant Weil local systems.* Let $\mathrm{Loc}(G)$ and $\mathrm{Loc}(G_{\mathrm{ab}})$ be the categories of Weil local systems on G and G_{ab}, respectively. Let $\mathrm{Loc}_{\mathrm{der}}(G)$ be the category of G_{der}-equivariant Weil local systems on G, whose definition we now recall. Let $n : G_{\mathrm{der}} \times G \to G$ be the restriction of $m : G \times G \to G$ to $G_{\mathrm{der}} \times G$, let $p : G_{\mathrm{der}} \times G \to G$ be projection to the second component, and let $s : G \to G_{\mathrm{der}} \times G$ be given by $s(g) = (1, g)$. Then the quotient $q : G \to G_{\mathrm{ab}}$ is a regular epimorphism of smooth group schemes with kernel pair (n, p).

$$G_{\mathrm{der}} \times G \underset{p}{\overset{n}{\rightrightarrows}} G \xrightarrow{q} G_{\mathrm{ab}}$$

Consider the morphisms

$$G_{\mathrm{der}} \times G_{\mathrm{der}} \times G \overset{b_1,b_2,b_3}{\rightrightarrows} G_{\mathrm{der}} \times G \underset{p}{\overset{n}{\rightrightarrows}} G$$

defined by

$$\begin{aligned} b_1(h_1, h_2, g) &= (h_1 h_2, g) \\ b_2(h_1, h_2, g) &= (h_1, h_2 g) \\ b_3(h_1, h_2, g) &= (h_2, g). \end{aligned}$$

Note that

$$\begin{aligned} n \circ b_1 &= n \circ b_2 \\ n \circ b_3 &= p \circ b_2 \\ p \circ b_1 &= p \circ b_3. \end{aligned} \tag{3}$$

A G_{der}-equivariant Weil local system on G is a Weil local system $\mathcal{L}$ on G together with an isomorphism

$$\nu : n^*\mathcal{L} \to p^*\mathcal{L}$$

of Weil local systems on $G_{\mathrm{der}} \times G$ such that

$$s^*(\nu) = \mathrm{id}_{\mathcal{L}} \tag{4}$$

and such that the following diagram of isomorphisms of local systems on $G_{\mathrm{der}} \times G_{\mathrm{der}} \times G$ commutes.

$$\begin{array}{ccccccc} & & b_2^* n^* \mathcal{L} & \xrightarrow{b_2^*(\nu)} & b_2^* p^* \mathcal{L} & & \\ & \swarrow^{n\circ b_1 = n\circ b_2} & & & & \searrow^{p\circ b_2 = n\circ b_3} & \\ b_1^* n^* \mathcal{L} & & & & & & b_3^* n^* \mathcal{L} \\ & \searrow_{b_1^*(\nu)} & & & & \swarrow_{b_3^*(\nu)} & \\ & & b_1^* p^* \mathcal{L} & \xleftarrow{p\circ b_3 = p\circ b_1} & b_3^* p^* \mathcal{L} & & \end{array} \tag{5}$$

Morphisms of H-equivariant Weil local systems $(\mathcal{L}, \nu) \to (\mathcal{L}', \nu')$ are morphisms of Weil local systems $\alpha : \mathcal{L} \to \mathcal{L}'$ for which the diagram

$$\begin{array}{ccc} n^*\mathcal{L} & \xrightarrow{n^*(\alpha)} & n^*\mathcal{L}' \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu'} \\ p^*\mathcal{L} & \xrightarrow{p^*(\alpha)} & p^*\mathcal{L}' \end{array} \tag{6}$$

commutes. This defines $\mathrm{Loc}_{\mathrm{der}}(G)$, the category of G_{der}-equivariant Weil local systems on G. The reader will recognize this notion as the Weil local system version of equivariant sheaves for the action n of G_{der} on G, as can be found, for example, in [2, 0.2].

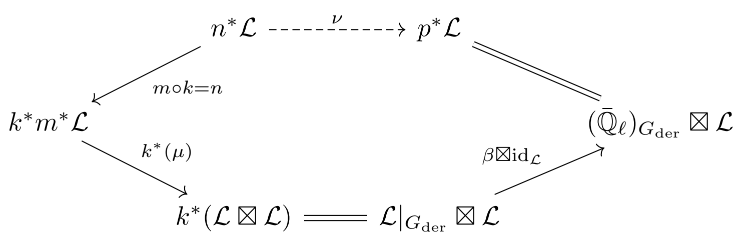
3.2.2. *Equivariant linear character sheaves.* With reference to Section 3.2.1, we define a G_{der}-equivariant linear character sheaf on G to be a triple $(\mathcal{L}, \mu, \nu)$, where $(\mathcal{L}, \mu)$ is a linear character sheaf and $(\mathcal{L}, \nu)$ is an G_{der}-equivariant Weil local system. A morphism of G_{der}-equivariant linear character sheaves $(\mathcal{L}, \mu, \nu) \to (\mathcal{L}', \mu', \nu')$ is a morphism of G_{der}-equivariant Weil local systems $\alpha : \mathcal{L} \to \mathcal{L}'$ which is also a morphism of linear character sheaves. Let $\mathcal{CS}_{\rm der}(G)$ be the category of $G_{\rm der}$-equivariant linear character sheaves on G.

Lemma 3.3. *Categories $\mathcal{CS}_{\rm der}(G)$ and $\mathcal{CS}_{\rm ab}(G)$ are equivalent.*

Proof. Let $i_{\rm der} : G_{\rm der} \to G$ be the kernel of $q : G \to G_{\rm ab}$ and define $j : G_{\rm der} \to G_{\rm der} \times G$ by $j(h) = (h, 1)$. If $(\mathcal{L}, \mu, \nu) \in \mathcal{CS}_{\rm der}(G)$ then $j^*(\nu) : \mathcal{L}|_{G_{\rm der}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\rm der}}$ is an isomorphism. This defines a functor by

$$\begin{array}{rcl} \mathcal{CS}_{\rm der}(G) & \to & \mathcal{CS}_{\rm ab}(G) \\ (\mathcal{L}, \mu, \nu) & \mapsto & (\mathcal{L}, \mu, j^*(\nu)) \end{array}$$

on objects and trivially on morphisms. It is easy to verify that morphisms that commute with μ and ν also commute with μ and $j^*(\nu)$. This functor is an equivalence; its adjoint is given as follows. Define $k : G_{\rm der} \times G \to G \times G$ by $k(h, g) = (i_{\rm der}(h), g)$. Then for $(\mathcal{L}, \mu, \beta) \in \mathcal{CS}_{\rm ab}(G)$, define $\nu : n^*\mathcal{L} \to p^*\mathcal{L}$ by the following diagram.



This defines the functor $\mathcal{CS}_{\rm ab}(G) \to \mathcal{CS}_{\rm der}(G)$, after confirming that morphisms that commute with μ and β also commute with μ and ν. ☐

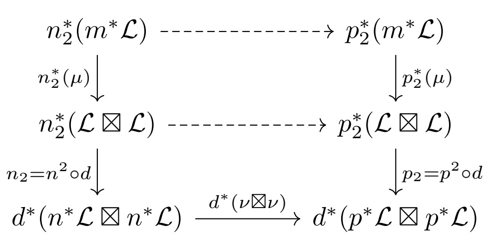
Set $G^2 = G \times G$, so $G^2_{\rm der} = G_{\rm der} \times G_{\rm der}$ and $G^2_{\rm ab} = G_{\rm ab} \times G_{\rm ab}$. Likewise define $n^2 : G^2_{\rm der} \times G^2 \to G^2$ and $p^2 : G^2_{\rm der} \times G \to G$.

Lemma 3.4. *If $(\mathcal{L}, \mu, \nu)$ is a $G_{\rm der}$-equivariant linear character sheaf on G then $\mu : m^*\mathcal{L} \to \mathcal{L} \boxtimes \mathcal{L}$ is a morphism of $G^2_{\rm der}$-equivariant Weil local systems on G^2.*

Proof. Define

$$\begin{aligned} d : G_{\rm der} \times G_{\rm der} \times G \times G &\to G_{\rm der} \times G \times G_{\rm der} \times G \\ (h_1, h_2, g_1, g_2) &\mapsto (h_1, g_1, h_2, g_2) \\ n_2 : G_{\rm der} \times G \times G_{\rm der} \times G &\to G \times G \\ (h_1, g_1, h_2, g_2) &\mapsto (h_1 g_1, h_2 g_2) \\ p_2 : G_{\rm der} \times G \times G_{\rm der} \times G &\to G \times G \\ (h_1, g_1, h_2, g_2) &\mapsto (g_1, g_2). \end{aligned}$$

The following diagram defines the isomorphisms needed to see that both $m^*\mathcal{L}$ and $\mathcal{L} \boxtimes \mathcal{L}$ are $G^2_{\rm der}$-equivariant Weil local systems.



The dashed arrows both satisfy (4) and (5) as they apply here. This diagram also shows that $\mu : m^*\mathcal{L} \to \mathcal{L} \boxtimes \mathcal{L}$ is a morphism of G^2_{der}-equivariant local systems, since it satisfies (6) as it applies here. $\square$

3.2.3. *Descent.* We may now relate $\mathcal{CS}_{\text{ab}}(G)$ to $\mathcal{CS}(G_{\text{ab}})$. To do so, we use decent along $q : G \to G_{\text{ab}}$.

If $\mathcal{L}_{\text{ab}} \in \text{Loc}(G_{\text{ab}})$ then $q^*\mathcal{L}_{\text{ab}} \in \text{Loc}(G)$ comes equipped with a canonical isomorphism $\nu(\mathcal{L}_{\text{ab}}) : n^*\mathcal{L} \to p^*\mathcal{L}$ defined by the following diagram of isomorphisms.

$$\begin{array}{ccc} n^*\mathcal{L} & \xrightarrow{\nu(\mathcal{L}_{\text{ab}})} & p^*\mathcal{L} \\ \| & & \| \\ n^*(q^*\mathcal{L}_{\text{ab}}) & \xrightarrow{q\circ n = q\circ p} & p^*(q^*\mathcal{L}_{\text{ab}}) \end{array}$$

Then $(q^*\mathcal{L}_{\text{ab}}, \nu(\mathcal{L}_{\text{ab}}))$ satisfies (4) and (5), so $(q^*\mathcal{L}_{\text{ab}}, \nu(\mathcal{L}_{\text{ab}})) \in \text{Loc}_{\text{der}}(G)$. Moreover, if $\alpha_{\text{ab}} : \mathcal{L}_{\text{ab}} \to \mathcal{L}_{\text{ab}}$ is a morphism in $\text{Loc}(G_{\text{ab}})$ then $q^*(\alpha_{\text{ab}})$ satisfies the condition in (6), so $q^*(\alpha_{\text{ab}})$ is a morphism in $\text{Loc}_{\text{der}}(G)$. This defines the functor

$$L : \text{Loc}(G_{\text{ab}}) \to \text{Loc}_{\text{der}}(G)$$

Lemma 3.5. *The functor $L : \text{Loc}(G_{\text{ab}}) \to \text{Loc}_{\text{der}}(G)$ is an equivalence.*

Proof. The quotient $q : G \to G_{\text{ab}}$ is an G_{der}-torsor in the fppf topology by [7, Thm. 3.2], and thus a G_{der}-torsor in the fpqc topology. The lemma is now a result from descent theory, arguing as in [16, Theorem 4.46] for example. $\square$

Consider the functor

$$q^* : \mathcal{CS}(G_{\text{ab}}) \to \mathcal{CS}(G)$$

given on objects by $(\mathcal{L}_{\text{ab}}, \mu_{\text{ab}}) \mapsto (q^*\mathcal{L}_{\text{ab}}, (q^2)^*\mu_{\text{ab}})$; this is an instance of [5, Lem. 1.4]. To see that $(q^*\mathcal{L}_{\text{ab}}, (q^2)^*\mu_{\text{ab}})$ is indeed a linear character sheaf on G, verify [5, CS.3]. Now set $L(\mathcal{L}_{\text{ab}}) = (\mathcal{L}, \nu)$, where $L : \text{Loc}(G_{\text{ab}}) \to \text{Loc}_{\text{der}}(G)$ is the comparison functor above, so $\mathcal{L} = q^*\mathcal{L}_{\text{ab}}$ and $\nu = \nu(\mathcal{L}_{\text{ab}})$. Then $(\mathcal{L}, \mu, \nu)$ is an object in $\mathcal{CS}_{\text{der}}(G)$. If $\alpha_{\text{ab}} : (\mathcal{L}_{\text{ab}}, \mu_{\text{ab}}) \to (\mathcal{L}'_{\text{ab}}, \mu'_{\text{ab}})$ is a morphism in $\mathcal{CS}(G_{\text{ab}})$, then $q^*(\alpha_{\text{ab}}) : (\mathcal{L}, \mu) \to (\mathcal{L}', \mu')$ satisfies [5, CS4], so $\alpha = q^*(\alpha_{\text{ab}})$ is a morphism in $\mathcal{CS}(G)$. These simple observations define the comparison functor

$$q^*_{\text{ab}} : \mathcal{CS}(G_{\text{ab}}) \to \mathcal{CS}_{\text{der}}(G)$$

and also show that the functor $q^* : \mathcal{CS}(G_{\text{ab}}) \to \mathcal{CS}(G)$ factors according to the following commuting diagram of functors

$$\begin{array}{ccc} \mathcal{CS}(G) & \xleftarrow{q^*} & \mathcal{CS}(G_{\text{ab}}) \\ {\scriptstyle\text{forget}}\uparrow & \swarrow {\scriptstyle q^*_{\text{ab}}} & \\ \mathcal{CS}_{\text{der}}(G). & & \end{array} \tag{7}$$

The definition of $q^*_{\text{ab}} : \mathcal{CS}(G_{\text{ab}}) \to \mathcal{CS}_{\text{der}}(G)$ will be revisited in the proof of the following result.

Proposition 3.6. *Suppose G is a smooth group scheme. Then pullback along $q : G \to G_{\text{ab}}$ defines an equivalence $\mathcal{CS}(G_{\text{ab}}) \to \mathcal{CS}_{\text{ab}}(G)$.*

Proof. In light of Lemma 3.3, it suffices to prove that the comparison functor $q_{\mathrm{ab}}^* : \mathcal{CS}(G_{\mathrm{ab}}) \to \mathcal{CS}_{\mathrm{der}}(G)$ is an equivalence. Let $L^2 : \mathrm{Loc}(G_{\mathrm{ab}}^2) \to \mathrm{Loc}_{\mathrm{der}}(G^2)$ be the comparison functor for the quotient $q^2 : G^2 \to G_{\mathrm{ab}}^2$. Then L^2 is also an equivalence by Lemma 3.5. Using Lemma 3.4, we may rewrite the comparison functor q_{ab}^* on objects by

$$\begin{array}{ccc} \mathcal{CS}(G_{\mathrm{ab}}) & \to & \mathcal{CS}_{\mathrm{der}}(G) \\ (\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) & \mapsto & (L(\mathcal{L}_{\mathrm{ab}}), L^2(\mu_{\mathrm{ab}})) \end{array}$$

and on morphisms by $\alpha \mapsto L(\alpha)$. The proposition now follows from the fact that both L and L^2 are equivalences. $\square$

Corollary 3.7. *If G is a smooth group scheme and $(\mathcal{L}, \mu) \in \mathcal{CS}(G)$, then the restriction of $\mathcal{L}$ to G_{der} is trivial if and only if $(\mathcal{L}, \mu) \cong q^*(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$ in $\mathcal{CS}(G)$, for some $(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) \in \mathcal{CS}(G_{\mathrm{ab}})$.*

Proof. Notation as in the proof of Proposition 3.6. Consider the following diagram.

$$\begin{array}{ccccc} \mathcal{CS}(G_{\mathrm{der}}) & \xleftarrow{\;i_{\mathrm{der}}^*\;} & \mathcal{CS}(G) & \xleftarrow{\;q^*\;} & \mathcal{CS}(G_{\mathrm{ab}}) \\ & & \uparrow{\scriptstyle \mathrm{forget}} & \swarrow{\scriptstyle q_{\mathrm{ab}}^*} & \\ & & \mathcal{CS}_{\mathrm{ab}}(G) & & \end{array}$$

Now, suppose $(\mathcal{L}, \mu) \in \mathcal{CS}(G)$ and there is an isomorphism $\beta : \mathcal{L}|_{G_{\mathrm{der}}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ in $\mathcal{CS}(G_{\mathrm{der}})$, so that $(\mathcal{L}, \mu, \beta) \in \mathcal{CS}_{\mathrm{ab}}(G)$. By Proposition 3.6, there is some $(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) \in \mathcal{CS}(G_{\mathrm{ab}})$ with $(\mathcal{L}, \mu, \beta) \cong q_{\mathrm{ab}}^*(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$. Applying the forgetful functor $\mathcal{CS}_{G_{\mathrm{der}}}(G) \to \mathcal{CS}(G)$ to this isomorphism, it follows that $(\mathcal{L}, \mu) \cong q^*(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$ in $\mathcal{CS}(G)$, as desired.

Conversely, suppose $(\mathcal{L}, \mu) \in \mathcal{CS}(G)$ and $(\mathcal{L}, \mu) \cong q^*(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$ in $\mathcal{CS}(G)$. Then

$$i_{\mathrm{der}}^*(\mathcal{L}, \mu) \cong i_{\mathrm{der}}^* q^*(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$$

in $\mathcal{CS}(G_{\mathrm{der}})$. Since $q \circ i_{\mathrm{der}} = 1$, it follows that $\mathcal{L}|_{G_{\mathrm{der}}} \cong (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ in $\mathcal{CS}(G_{\mathrm{der}})$. $\square$

We may interpret this corollary as measuring how far q^* is from being essentially surjective. The next result shows that it is also not full. Let C denote the cokernel of the natural map

$$\mathrm{Hom}(\pi_0(\bar{G})_{\mathrm{Fr}}, \bar{\mathbb{Q}}_\ell^\times) \to \mathrm{Hom}(\pi_0(\bar{G}_{\mathrm{der}})_{\mathrm{Fr}}, \bar{\mathbb{Q}}_\ell^\times),$$

where $\pi_0(\bar{G})_{\mathrm{Fr}}$ denotes the covariants of the action of Frobenius on the component group of $\bar{G}$

Corollary 3.8. *If G is a smooth group scheme and $(\mathcal{L}, \mu)$ is a character sheaf on G with trivial restriction to G_{der}, then the set of isomorphism classes of objects in $\mathcal{CS}(G_{\mathrm{ab}})$ mapping to $(\mathcal{L}, \mu)$ under q^* is a principal homogeneous space for C.*

Proof. By Proposition 3.6, it suffices to find the set of isomorphism classes in $\mathcal{CS}_{\mathrm{ab}}(G)$ mapping to $(\mathcal{L}, \mu)$ under the forgetful functor. By the previous corollary this set is nonempty. If $(\mathcal{L}, \mu, \beta)$ and $(\mathcal{L}, \mu, \beta')$ both map to $(\mathcal{L}, \mu)$, then $\beta' \circ \beta^{-1}$ is an automorphism of the constant sheaf on G_{der}. Conversely, if φ is an automorphism of $(\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ and $(\mathcal{L}, \mu, \beta) \in \mathcal{CS}_{\mathrm{ab}}(G)$ then $(\mathcal{L}, \mu, \varphi \circ \beta) \in \mathcal{CS}_{\mathrm{ab}}(G)$. By [5, Theorem 3.9], the automorphism group is isomorphic to $\mathrm{Hom}(\pi_0(\bar{G}_{\mathrm{der}})_{\mathrm{Fr}}, \bar{\mathbb{Q}}_\ell^\times)$. Finally, we note that any automorphism α of $(\mathcal{L}, \mu) \in \mathcal{CS}(G)$ defines an isomorphism $(\mathcal{L}, \mu, \beta \circ \alpha|_{G_{\mathrm{der}}}) \to (\mathcal{L}, \mu, \beta)$. Applying [5, Theorem 3.9] again yields the desired result. $\square$

3.3. **Objects and maps in commutative character sheaves.** We are now in a position to prove that commutative character sheaves on G match perfectly with commutative character sheaves on G_{ab}. We start with a method that will allow us to situate the diagram in Definition 3.2 within $\mathcal{CS}_{\mathrm{ab}}(G^2)$.

Lemma 3.9. *If $(\mathcal{L},\mu,\beta)\in\mathcal{CS}_{\mathrm{ab}}(G)$ then $\mu : m^*\mathcal{L}\to\mathcal{L}\boxtimes\mathcal{L}$, $\gamma : m^*\mathcal{L}\to\theta^*(m^*\mathcal{L})$ and $\vartheta : \mathcal{L}\boxtimes\mathcal{L}\to\theta^*(\mathcal{L}\boxtimes\mathcal{L})$ are morphisms in $\mathcal{CS}_{\mathrm{ab}}(G\times G)$.*

Proof. Define $m^2 : G^2\times G^2\to G^2$ by $m^2(g_1,g_2,g_1',g_2') = (g_1g_1', g_2g_2')$. Also define $p_i^2 : G^2\times G^2\to G^2$ by $p_i^2(g_1,g_2,g_1,'g_2') = (g_i,g_i')$. First we show that $m^*\mathcal{L}$ is an object in $\mathcal{CS}(G^2)$ by equipping it with an isomorphism $\mu_m^2 : (m^2)^*(m^*\mathcal{L})\to m^*\mathcal{L}\boxtimes m^*\mathcal{L}$ defined by the diagram below.

$$\begin{array}{ccccc}
(m^2)^*(m^*\mathcal{L}) & & \overset{\mu_m^2}{\dashrightarrow} & & m^*\mathcal{L}\boxtimes m^*\mathcal{L} \\
\big\downarrow{\scriptstyle (m^2)^*\mu} & & & & \big\| \\
(m^2)^*(\mathcal{L}\boxtimes\mathcal{L}) & = & (m^2)^*(p_1)^*\mathcal{L}\otimes(m^2)^*(p_2)^*\mathcal{L} & \longrightarrow & (p_1^2)^*m^*\mathcal{L}\otimes(p_2^2)^*m^*\mathcal{L}
\end{array}$$

The pair $(m^*\mathcal{L},\mu_m^2)$ satisfies the conditions appearing in [5, Def. 1.1]. The restriction of $m^*\mathcal{L}$ to $G_{\mathrm{der}}^2 = G_{\mathrm{der}}\times G_{\mathrm{der}}$ is canonically isomorphic to $(\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}^2}$ by

$$\begin{array}{ccc}
(m^*\mathcal{L})|_{G_{\mathrm{der}}^2} & \overset{\beta_m^2}{\dashrightarrow} & (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}^2} \\
\big\downarrow{\scriptstyle \mu|_{G_{\mathrm{der}}^2}} & & \big\uparrow{\scriptstyle \beta\boxtimes\beta} \\
(\mathcal{L}\boxtimes\mathcal{L})|_{G_{\mathrm{der}}^2} & = & (\mathcal{L}|_{G_{\mathrm{der}}})\boxtimes(\mathcal{L}|_{G_{\mathrm{der}}}).
\end{array}$$

This shows that $(m^*\mathcal{L},\mu_m^2,\beta_m^2)\in\mathcal{CS}_{\mathrm{ab}}(G^2)$. Similar work defines $(\mathcal{L}\boxtimes\mathcal{L},\mu_{\boxtimes}^2,\beta_{\boxtimes}^2)\in\mathcal{CS}_{\mathrm{ab}}(G^2)$. By construction, $\mu : m^*\mathcal{L}\to\mathcal{L}\boxtimes\mathcal{L}$ is a morphism in $\mathcal{CS}_{\mathrm{ab}}(G^2)$. Similar work shows that $\gamma : m^*\mathcal{L}\to\theta^*(m^*\mathcal{L})$ and $\vartheta : \mathcal{L}\boxtimes\mathcal{L}\to\theta^*(\mathcal{L}\boxtimes\mathcal{L})$ are also morphisms in $\mathcal{CS}_{\mathrm{ab}}(G^2)$. □

Suppose G is commutative, so $G_{\mathrm{der}} = 1$. Suppose $(\mathcal{L},\mu,\beta)$ is an object in $\mathcal{CS}_{\mathrm{ab}}(G)$. Then $\beta : \mathcal{L}_1\to\bar{\mathbb{Q}}_\ell$ is an isomorphism in $\mathcal{CS}(1)$, which is unique by [5, Theorem 3.9]. Tracing through the construction of $\gamma : m^*\mathcal{L}\to\theta^*m^*\mathcal{L}$ from $\beta : \mathcal{L}_1\to\bar{\mathbb{Q}}_\ell$, we find that $\gamma : m^*\mathcal{L}\to\theta^*m^*\mathcal{L}$ is the canonical isomorphism coming from the equation $m = m\circ\theta$. Thus, when G is commutative, Definition 3.2 agrees with Definition 2.1. The next result generalizes this observation.

Theorem 3.10. *Pull-back along the abelianization $q : G\to G_{\mathrm{ab}}$ defines an equivalence of categories*

$$\mathcal{CCS}(G_{\mathrm{ab}})\to\mathcal{CCS}(G).$$

Proof. By definition, $\mathcal{CCS}(G)$ is a full subcategory of $\mathcal{CS}_{\mathrm{ab}}(G)$; likewise, $\mathcal{CCS}(G_{\mathrm{ab}})$ is a full subcategory of $\mathcal{CS}_{\mathrm{ab}}(G_{\mathrm{ab}})$. We have just seen that $\mathcal{CS}_{\mathrm{ab}}(G_{\mathrm{ab}})$ is equivalent to $\mathcal{CS}(G_{\mathrm{ab}})$. By Proposition 3.6, pullback along the abelianization $q : G\to G_{\mathrm{ab}}$ induces an equivalence $q_{\mathrm{ab}}^* : \mathcal{CS}(G_{\mathrm{ab}})\to\mathcal{CS}_{\mathrm{ab}}(G)$. Thus, the functor $\mathcal{CS}_{\mathrm{ab}}(G_{\mathrm{ab}})\to\mathcal{CS}_{\mathrm{ab}}(G)$ induced by pullback along q is an equivalence. The functor $\mathcal{CCS}(G_{\mathrm{ab}})\to$

$\mathcal{CCS}(G)$ under consideration is the restriction of $\mathcal{CS}_{\mathrm{ab}}(G_{\mathrm{ab}}) \to \mathcal{CS}_{\mathrm{ab}}(G)$ to the subcategory $\mathcal{CCS}(G_{\mathrm{ab}})$.

$$\begin{array}{ccc} & & \mathcal{CS}(G_{\mathrm{ab}}) \\ & \swarrow^{q_{\mathrm{ab}}^*} & \uparrow \scriptstyle{\text{equiv.}} \\ \mathcal{CS}_{\mathrm{ab}}(G) & \longleftarrow & \mathcal{CS}_{\mathrm{ab}}(G_{\mathrm{ab}}) \\ \uparrow & & \uparrow \\ \mathcal{CCS}(G) & \longleftarrow & \mathcal{CCS}(G_{\mathrm{ab}}) \end{array}$$

To prove the theorem, it is now sufficient to show that $\mathcal{CCS}(G_{\mathrm{ab}}) \to \mathcal{CCS}(G)$ is essentially surjective. Suppose $(\mathcal{L}, \nu, \beta) \in \mathcal{CCS}(G)$. Then $(\mathcal{L}, \nu, \beta) \in \mathcal{CS}_{\mathrm{ab}}(G)$. Let $(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) \in \mathcal{CS}(G_{\mathrm{ab}})$ be given by the equivalences above. Let $\xi : m_{\mathrm{ab}}^* \mathcal{L}_{\mathrm{ab}} \to \theta^* m_{\mathrm{ab}}^* \mathcal{L}_{\mathrm{ab}}$ be the isomorphism attached to $(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) \in \mathcal{CS}(G_{\mathrm{ab}})$ as in Section 2. Let $\gamma : m^*\mathcal{L} \to \theta^* m^* \mathcal{L}$ be the isomorphism attached to $\beta : \mathcal{L}|_{G_{\mathrm{der}}} \to (\bar{\mathbb{Q}}_\ell)_{G_{\mathrm{der}}}$ as in Section 3.1. By Lemma 3.9, the diagrams below are in $\mathcal{CS}(G_{\mathrm{ab}})$ (right) and $\mathcal{CS}_{\mathrm{ab}}(G)$ (left).

$$\begin{array}{ccc} m^*\mathcal{L} & \xrightarrow{\mu} & \mathcal{L} \boxtimes \mathcal{L} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\vartheta} \\ \theta^*(m^*\mathcal{L}) & \xrightarrow{\theta^*\mu} & \theta^*(\mathcal{L} \boxtimes \mathcal{L}) \end{array} \quad \xleftarrow{(q^2)_{\mathrm{ab}}^*} \quad \begin{array}{ccc} m_{\mathrm{ab}}^*\mathcal{L}_{\mathrm{ab}} & \xrightarrow{\mu_{\mathrm{ab}}} & \mathcal{L}_{\mathrm{ab}} \boxtimes \mathcal{L}_{\mathrm{ab}} \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\vartheta} \\ \theta^*(m_{\mathrm{ab}}^*\mathcal{L}_{\mathrm{ab}}) & \xrightarrow{\theta^*\mu_{\mathrm{ab}}} & \theta^*(\mathcal{L}_{\mathrm{ab}} \boxtimes \mathcal{L}_{\mathrm{ab}}) \end{array}$$

The diagram on the left is the result of applying the functor $(q^2)_{\mathrm{ab}}^*$ to the one on the right; in particular $\gamma = (q^2)_{\mathrm{ab}}^* \xi$. Since $(q^2)_{\mathrm{ab}}^*$ is an equivalence by Proposition 3.6, it follows that the diagram in Definition 3.2 commutes if and only if the diagram in Definition 2.1 commutes. In other words, $(\mathcal{L}, \mu, \beta) \in \mathcal{CCS}(G)$ if and only if $(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) \in \mathcal{CCS}(G_{\mathrm{ab}})$.

□

Theorem 3.10 shows that $\mathcal{CCS}(G)$ is a categorical solution to the problem that linear character sheaves on G need not be trivial on G_{der}, as discussed at the beginning of Section 3. At the same time, changing $\mathcal{CS}(G)$ to $\mathcal{CCS}(G)$ resolves the lack of bijectivity in [5, Theorem 3.6]. We may also use Theorem 3.10 to give a description of the morphisms and the isomorphism classes of objects in $\mathcal{CCS}(G)$.

Corollary 3.11. *The category $\mathcal{CCS}(G)$ is monoidal and there is a canonical isomorphism*

$$\mathcal{CCS}(G)_{/iso} \cong \mathrm{Hom}(G_{\mathrm{ab}}(k), \bar{\mathbb{Q}}_\ell^\times).$$

Every map in $\mathcal{CCS}(G)$ is either trivial or an isomorphism, and the automorphism group of any object in $\mathcal{CCS}(G)$ is canonically isomorphic to $\mathrm{Hom}(\pi_0(\bar{G}_{\mathrm{ab}})_{\mathrm{Fr}}, \bar{\mathbb{Q}}_\ell^\times)$.

Proof. The first claim follows from Theorems 2.5 and 3.10. Let us write $(\mathcal{L}, \mu, \beta) \mapsto (\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}})$ to indicate the equivalence appearing in Theorem 3.10; then

$$\mathrm{Aut}_{\mathcal{CCS}(G)}(\mathcal{L}, \mu, \beta) = \mathrm{Aut}_{\mathcal{CCS}(G_{\mathrm{ab}})}(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}).$$

By [5, Theorem 3.9], $\mathrm{Aut}_{\mathcal{CCS}(G_{\mathrm{ab}})}(\mathcal{L}_{\mathrm{ab}}, \mu_{\mathrm{ab}}) = \mathrm{Hom}(\pi_0(\bar{G}_{\mathrm{ab}})_{\mathrm{Fr}}, \bar{\mathbb{Q}}_\ell^\times)$. □

3.4. **Geometrization of characters.** Corollary 3.11 shows that commutative character sheaves on G provide a natural geometrization of characters of $G_{\mathrm{ab}}(k)$. In Theorem 3.12 we take this one small step further by exploring the relation between characters of $G(k)$ and objects in $\mathcal{CCS}(G)$.

Theorem 3.12. *The trace of Frobenius* $\mathrm{Tr} : \mathcal{CCS}(G)_{/iso} \to G(k)^*$ *fits into the following diagram,*

$$\begin{array}{ccccccccc} & & & & \mathcal{CCS}(G_{\mathrm{ab}})_{/iso} & \xrightarrow{\cong} & \mathcal{CCS}(G)_{/iso} & & \\ & & & & \cong \downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} & & \\ 1 & \longrightarrow & \Delta_G^* & \longrightarrow & G_{\mathrm{ab}}(k)^* & \longrightarrow & G(k)^* & \longrightarrow & 1, \end{array}$$

where Δ_G *is the image of the connecting homomorphism* $G_{\mathrm{ab}}(k) \to \mathrm{H}^1(k, G_{\mathrm{der}})$. *Thus the category* $\mathcal{CCS}(G)$ *geometrizes characters of* $G(k)$ *in the following sense: for every group homomorphism* $\chi : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ *there is an object* $(\mathcal{L}, \mu, \beta)$ *in* $\mathcal{CCS}(G)$ *such that* $t_{\mathcal{L}} = \chi$. *While the geometrization of* $\chi : G(k) \to \bar{\mathbb{Q}}_\ell^\times$ *is not unique, the group of isomorphism classes of possibilities are enumerated by* Δ_G^*.

Proof. By the definition of Δ_G, we have a short exact sequence

$$1 \to G(k)/G_{\mathrm{der}}(k) \to G_{\mathrm{ab}}(k) \to \Delta_G \to 1.$$

Applying $\mathrm{Hom}(-, \bar{\mathbb{Q}}_\ell^\times)$ and using the fact that every homomorphism $G(k) \to \bar{\mathbb{Q}}_\ell^\times$ vanishes on $G_{\mathrm{der}}(k)$, we get

$$1 \to \Delta_G^* \to G_{\mathrm{ab}}(k)^* \to G(k)^* \to 1.$$

By Theorem 3.10, the map $\mathcal{CCS}(G_{\mathrm{ab}})_{/\mathrm{iso}} \to \mathcal{CCS}(G)_{/\mathrm{iso}}$ is an isomorphism. Moreover, since both $\mathcal{CCS}(G_{\mathrm{ab}})_{/\mathrm{iso}} \to \mathcal{CCS}(G)_{/\mathrm{iso}}$ and $G_{\mathrm{ab}}(k)^* \to G(k)^*$ are defined by pullback along q, the square in the statement of the theorem commutes. Finally, $\mathrm{Tr} : \mathcal{CCS}(G_{\mathrm{ab}})_{/\mathrm{iso}} \to G_{\mathrm{ab}}(k)^*$ is an isomorphism by Corollary 3.11. ☐

Remark 3.13. Note that when $\mathrm{H}^1(k, G_{\mathrm{der}}) = 0$ then $\mathcal{CCS}(G)_{/\mathrm{iso}} \cong G(k)^*$, so we succeed in geometrizing characters of $G(k)$ on the nose.

4. Application to type theory for p-adic groups

We now show how to use Theorem 3.12 to geometrize Yu type data and how to geometrize types for supercuspidal representations of tamely ramified p-adic groups.

4.1. **Quasicharacters of smooth group schemes over certain henselian traits.** Recall that R is the ring of integers of a local field with finite residue field k. The maximal idea of R will be denoted by $\mathfrak{m}$. Let $\underline{G}$ be a smooth group scheme over R. Here we shall use [3] for the definition and fundamental properties of the Greenberg transform. Let G be the Greenberg transform of $\underline{G}$; then G is a group scheme over k and there is a canonical isomorphism

$$G(k) = \underline{G}(R).$$

Proposition 4.1. *For every quasicharacter character* $\varphi : \underline{G}(R) \to \bar{\mathbb{Q}}_\ell^\times$ *there is a Weil sheaf* $\mathcal{L}$ *on* G *such that*

$$t_{\mathcal{L}} = \varphi.$$

Proof. By continuity of $\varphi : \underline{G}(R) \to \bar{\mathbb{Q}}_\ell^\times$, there is some $m \in \mathbb{N}$ and a factorization

$$\begin{array}{ccc} \underline{G}(R) & \xrightarrow{\quad\varphi\quad} & \bar{\mathbb{Q}}_\ell^\times \\ & \searrow \quad \nearrow_{\varphi_m} & \\ & \underline{G}(R/\mathfrak{p}^{m+1}) & \end{array}$$

Set $R_m = R/\mathfrak{p}^{m+1}$ and set $G_m = \mathrm{Gr}_m^R(\underline{G})$, the Greenberg transform of $\underline{G} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_m)$. Then G_m is a smooth group scheme over k and $G_m(k) = \underline{G}(R_m)$. Using Theorem 3.12, let $\mathcal{L}_m$ be a geometrization of the character $\varphi_m : G_m(k) \to \bar{\mathbb{Q}}_\ell^\times$; so

$$t_{\mathcal{L}_m} = \varphi_m$$

on $G_m(k)$. Recall that the full Greenberg transform $G := \mathrm{Gr}^R(\underline{G})$ is a group scheme over k such that $G(k) = \underline{G}(R)$; it comes equipped with a morphism $G \to G_m$. Let $\mathcal{L}$ be the Weil sheaf on G obtained from $\mathcal{L}_m$ by pullback along $G \to G_m$. Then $\mathcal{L}$ is a quasicharacter sheaf on G, in the sense of [5, Def 4.3], such that $t_{\mathcal{L}} = \varphi$. □

4.2. **Jacobi theory over finite fields.** For use below, we recall some facts about the Heisenberg-Weil representation.

Let V be a finite-dimensional vector space over a finite field k equipped with a symplectic paring $\langle\ ,\ \rangle : V \times V \to Z$, where Z is a one-dimensional vector space over k. Let $V^\sharp$ be the Heisenberg group determined by $(Z, \langle\ ,\ \rangle)$ [9, §1.1]. Let $\mathrm{Sp}(V)$ be the symplectic group determined by the symplectic pairing $\langle\ ,\ \rangle$; this group acts on $V^\sharp$. The group $\mathrm{Sp}(V) \ltimes V^\sharp$ is called the Jacobi group. From the construction above, it is clear that the Jacobi group may be viewed as the k-points of an algebraic group over k; we will refer to that algebraic group as the Jacobi group.

Let $\psi : Z \to \bar{\mathbb{Q}}_\ell^\times$ be an additive character and let ω_ψ be the Heisenberg representation on $V^\sharp$ with central character ψ [9, §1.1]. The Heisenberg representation determines a representation π_ψ of $\mathrm{Sp}(V)$ with the same representation space as ω_ψ and with the defining property: for each $g \in \mathrm{Sp}(V)$, $\pi_\psi(g)$ determines an isomorphism of representations $\omega_\psi^g \to \omega_\psi$. Let $W_\psi = \pi_\psi \ltimes \omega_\psi$ be the Heisenberg-Weil representation of the Jacobi group $\mathrm{Sp}(V) \ltimes V^\sharp$ given by ω_ψ and π_ψ [9, §2.2].

There is a Weil sheaf complex $\mathcal{K}_\psi$ on $\mathrm{Sp}(V) \ltimes V^\sharp$ [9, Theorem 3.2.2.1] (see also [10, Theorem 4.5]) such that

$$t_{\mathcal{K}_\psi} = \mathrm{Tr}(W_\psi). \tag{8}$$

Since $\mathcal{K}_\psi$ is an object in Deligne's category $D_c^b(\mathrm{Sp}(V) \ltimes V^\sharp, \bar{\mathbb{Q}}_\ell)$, the left hand side of this equality must be interpreted accordingly.

4.3. **Review of Yu's types and associated models.** For the rest of Section 4, K is a p-adic field and R is the ring of integers of K. A Yu type datum consists of the following:

- **Y0** a sequence of compact groups ${}^\circ K^0 \subseteq {}^\circ K^1 \subseteq \cdots \subseteq {}^\circ K^d = {}^\circ K$;
- **Y1** a continuous representation ${}^\circ\rho^0$ of ${}^\circ K^0$;
- **Y2** quasicharacters $\varphi^i : {}^\circ K^i \to \mathbb{C}^\times$, for $i = 0, \ldots d$.

The representation ${}^\circ\rho^0$ and the quasicharacters $(\varphi^0, \ldots, \varphi^d)$ enjoy certain properties which allow Yu to construct the representations ${}^\circ\rho_i$ of ${}^\circ K^i$ that form the sequence of types $({}^\circ K^i, {}^\circ\rho_i)$, for $i = 1, \ldots, d$. In order to prepare for the construction of the geometric types of Theorem 4.2 we review some further detail here. In

Table 1 we explain how to convert the constructions appearing in this section into the notation of [17].

First, Yu introduces

Y3 compact groups $J_i \subset {}^\circ K$, for $i = 0, \ldots d$, such that ${}^\circ K^i = J_0 \cdots J_i$ and, for $i = 0, \ldots d-1$, a natural action of ${}^\circ K^i$ on J_{i+1} defining the groups ${}^\circ K^i \ltimes J_{i+1}$.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & {}^\circ K^i \cap J_{i+1} & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & J_{i+1} & \longrightarrow & {}^\circ K^i \ltimes J_{i+1} & \xrightarrow{p_i} & {}^\circ K^i & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \pi_{i+1}} & & & & \\
 & & & & {}^\circ K^{i+1} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}
\tag{9}
$$

Next, Yu defines a group homomorphism (in fact, a quotient) $J_{i+1} \to V_{i+1}$ where V_{i+1} is a finite abelian group, the latter also given the structure of a k-vector space. The vector space V_{i+1} is then equipped with a symplectic pairing $\langle\ ,\ \rangle_{i+1} : V_{i+1} \times V_{i+1} \to Z_{i+1}$, where Z_{i+1} is a one-dimensional vector space over k, itself equipped with an additive character $\psi_{i+1} : Z_{i+1} \to \mathbb{C}^\times$. This, in turn, is used to define a map $J_{i+1} \to V_{i+1}^\sharp$, where $V_{i+1}^\sharp$ is the Heisenberg group determined by V_{i+1}, Z_{i+1}, $\langle\ ,\ \rangle_{i+1}$ and ψ_{i+1}, as in Section 4.2. In fact, the quotient $J_{i+1} \to V_{i+1}^\sharp$ factors through a quotient $J_{i+1} \to H_{i+1}$ and an isomorphism $j_{i+1} : H_{i+1} \to V_{i+1}^\sharp$, where H_{i+1} is a Heisenberg p-group in the sense of [17]. Finally, Yu constructs a group homomorphism $f_{i+1} : {}^\circ K^i \to \mathrm{Sp}(V_{i+1})$ such that the pair (f_{i+1}, j_{i+1}) is a symplectic action of ${}^\circ K^i$ on H_{i+1} in the sense of [17]. Taken together, this defines

Y4 a group homomorphism $h_{i+1} : {}^\circ K^i \ltimes J_{i+1} \to \mathrm{Sp}(V_{i+1}) \ltimes V_{i+1}^\sharp$ making the following diagram commute.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & J_{i+1} & \longrightarrow & {}^\circ K^i \ltimes J_{i+1} & \xrightarrow{p_i} & {}^\circ K^i & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle h_{i+1}} & & \downarrow{\scriptstyle f_i} & & \\
1 & \longrightarrow & V_{i+1}^\sharp & \longrightarrow & \mathrm{Sp}(V_{i+1}) \ltimes V_{i+1}^\sharp & \longrightarrow & \mathrm{Sp}(V_{i+1}) & \longrightarrow & 1
\end{array}
$$

We can now recall how Yu uses all this to construct representations ${}^\circ\rho^i$ of ${}^\circ K^i$, for $i = 1, \ldots, d$ and the types $({}^\circ K^i, {}^\circ\rho_i)$; see [17, §§4, 15]. The representations ${}^\circ\rho^i$ and ${}^\circ\rho_i$ are defined recursively. For the base case $i = 0$, set ${}^\circ\rho_0 := {}^\circ\rho^0 \otimes \varphi^0$; see Y1 above. Now fix i. Let W_{i+1} be the Heisenberg-Weil representation of the Jacobi group $\mathrm{Sp}(V_{i+1}) \ltimes V_{i+1}^\sharp$, whose restriction to $V_{i+1}^\sharp$ has central character ψ_{i+1}. Pull-back along h_{i+1} to form $h_{i+1}^*(W_{i+1})$, a representation of ${}^\circ K^i \ltimes J_{i+1}$. Write $\inf({}^\circ\rho_i)$ for the representation of ${}^\circ K^i \ltimes J_{i+1}$ obtained by pulling back ${}^\circ\rho^i$ along

Table 1. Notation conversion chart.

this paper	Jiu-Kang Yu, *Construction of tame supercuspidal representations*	[17]
$^\circ K^0$	$^\circ K^0 = G^0(F)_y$	[17, §15]
$^\circ K^{i+1}$	$^\circ K^{i+1} = (\,^\circ K^0)\vec{G}^{(i+1)}(F)_{y,(0,s_0,\dots,s_i)}$	[17, §15]
$^\circ\rho^0$	$^\circ\rho^0$	[17, §15]
$^\circ\rho^{i+1}$	$^\circ\rho^{i+1}$	[17, §15]
φ^i	$\phi_i\vert_{^\circ K^i}$	[17, §3]
J_{i+1}	$J^{i+1} = (G^i, G^{i+1})(F)_{y,(r_i,s_i)}$	[17, §3]
V_{i+1}	$J^{i+1}/J^{i+1}_+ = (G^i, G^{i+1})(F)_{y,(r_i,s_i)}/(G^i, G^{i+1})(F)_{y,(r_i,s_i^+)}$	[17, §3]
$V^\sharp_{i+1}$	$(G^i, G^{i+1})(F)_{y,(r_i,s_i)}/\ker(\widehat{\phi}_i\vert_{(G^i,G^{i+1})(F)_{y,(r_i,s_i^+)}})$	[17, §4]
Z_{i+1}	$\ker(V^\sharp_{i+1} \to V_{i+1})$	[17, §11]
(f_{j+1}, j_{i+1})	(f, j)	[17, §11]
$\langle\ ,\ \rangle_{i+1}$	$\langle\ ,\ \rangle$	[17, §11]

$^\circ K^i \ltimes J_{i+1} \to {}^\circ K^i$. Consider the representation

$$^\circ\rho^{i+1} := h^*_{i+1}(W_{i+1}) \otimes \inf(\,^\circ\rho_i) \tag{10}$$

of $^\circ K^i \ltimes J_{i+1}$. By [17], the representation $^\circ\rho^{i+1}$ of $^\circ K^i \ltimes J_{i+1}$ is trivial on $^\circ K^i \cap J_{i+1}$ so $^\circ\rho^{i+1}$ descends to $^\circ K^{i+1}$. Set $^\circ\rho_{i+1} = {}^\circ\rho^{i+1} \otimes \varphi^{i+1}$. This completes the recursive definition of the Yu $(\,^\circ K^i, {}^\circ\rho_i)$ for $i = 0, \dots, d$. By [18, Prop 10.2] there is a sequence

$$\underline{G}^0 \to \underline{G}^1 \to \cdots \to \underline{G}^d = \underline{G}$$

of morphisms of affine smooth group schemes of finite type over R such that, on R-points it gives the sequence $^\circ K^0 \subseteq {}^\circ K^1 \subseteq \cdots \subseteq {}^\circ K^d$ above. Indeed, this is the main result of [18].

As explained in [18, §10.4], there is morphism of affine smooth group schemes of finite type over R

$$\underline{J}^i \to \underline{G},$$

for each $i = 0, \dots d$, such that $\underline{J}^i(R) = J_i$ as a subgroup of C and such that the image of the R-points under the multiplication map $\underline{J}^0 \times \cdots \times \underline{J}^i \to \underline{G}$ is $^\circ K^i$, for $i = 0, \dots, d$. There is a natural action of $\underline{G}^i$ on $\underline{J}^{i+1}$ in the category of smooth affine group schemes over R so that the group scheme

$$\underline{G}^i \ltimes \underline{J}^{i+1}$$

gives $(\underline{G}^i \ltimes \underline{J}^{i+1})(R) = {}^\circ K^i \ltimes J_{i+1}$

Write $\underline{J}^{i+1}_k$ for the special fibre $\underline{J}^{i+1} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ of $\underline{J}^{i+1}$. The vector space V_{i+1} may realized as the k-points on a variety V^{i+1} over k, where V^{i+1}, appears as a quotient $\underline{J}^{i+1}_k \to V^{i+1}$ of algebraic groups over k. Then the quotient $J_{i+1} \to V_{i+1}$ is realized as the composition

$$\underline{J}^{i+1}(R) \to \underline{J}^{i+1}(k) = \underline{J}^{i+1}_k(k) \to V^{i+1}(k) = V_{i+1}.$$

Likewise, the Heisenberg p-group H_{i+1}, appearing in 4.3, may be realized as a quotient of algebraic groups, and $\underline{J}^{i+1}_k \to H^{i+1}$ as the composition

$$\underline{J}^{i+1}(R) \to \underline{J}^{i+1}(k) = \underline{J}^{i+1}_k(k) \to H^{i+1}_k(k) = H_{i+1}.$$

Finally, the group homomorphism $f_i : J_0 \cdots J_i \to \mathrm{Sp}(V_{i+1})$ may be made geometric in much the same way. Writing $\underline{G}^i_k$ for the special fibre $\underline{G}^i \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ of $\underline{G}^i$, and writing $\underline{G}^{i,\mathrm{red}}_k$ for the reductive quotient of $\underline{G}^i_k$, there is a quotient of algebraic groups $\underline{G}^{i,\mathrm{red}}_k \to W^{i+1}_k$ so that $f_i : J_0 \cdots J_i \to \mathrm{Sp}(V_{i+1})$ is realized as the composition

$$\underline{G}^i(R) \to \underline{G}^i(k) = \underline{G}^i_k(k) \to \underline{G}^{i,\mathrm{red}}_k(k) \to W^{i+1}_k(k) = \mathrm{Sp}(V_{i+1}).$$

With all this, we may revisit the quotients appearing in Section 4.3:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \underline{J}^{i+1} & \longrightarrow & \underline{G}^i \ltimes \underline{J}^{i+1} & \longrightarrow & \underline{G}^i & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \underline{J}^{i+1}_k & \longrightarrow & \underline{G}^i_k \ltimes \underline{J}^{i+1}_k & \longrightarrow & \underline{G}^i_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & V^\sharp_{i+1} & \longrightarrow & \mathrm{Sp}(V_{i+1}) \ltimes V^\sharp_{i+1} & \longrightarrow & \mathrm{Sp}(V_{i+1}) & \longrightarrow & 1,
\end{array}$$

where the last two rows are now understood as forming a diagram in the category of algebraic groups over k. This realizes the Jacobi group $\mathrm{Sp}(V_{i+1}) \ltimes V^\sharp_{i+1}$ as a quotient of the special fibre of the smooth group scheme $\underline{G}^i \ltimes \underline{J}^{i+1}$ over R.

We may now revisit the ingredients in the construction of the representation ρ of $\underline{G}(R)$ along the lines indicated by Yu and recalled in Section 4.3.

M0 The compact groups ${}^\circ K^i$ have been replaced by the smooth group schemes $\underline{G}^i$.

M1 The continuous representation ${}^\circ\rho^0$ of ${}^\circ K^0$ is a representation of $\underline{G}^0(R)$ obtained by inflation along $\underline{G}^0(R) \to \underline{G}^0(k)$ from a representation ϱ_0 of $\underline{G}^0(k) = \underline{G}^0_k(k)$. In fact, ϱ_0 is itself obtained by pulling back a representation ϱ_0^{red} along the k-points of the quotient $\underline{G}^0_k \to (\underline{G}^0)^{\mathrm{red}}_k$.

M2 The quasicharacters φ^i are quasicharacters of $\underline{G}^i(R)$, for $i = 0, \ldots, d$.

M3 Diagram (9) is now replaced by the following diagram of smooth group schemes over R.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \underline{G}^i \times_{\underline{G}} \underline{J}^{i+1} & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & \underline{J}^{i+1} & \longrightarrow & \underline{G}^i \ltimes \underline{J}^{i+1} & \longrightarrow & \underline{G}^i & \longrightarrow & 1 \\
 & & & & \downarrow & & & & \\
 & & & & \underline{G}^{i+1} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array} \tag{11}$$

M4 The representation $h_{i+1}^*(W_{i+1})$ appearing in Y4 is now obtained by pulling back a representation along

$$(\underline{G}^i \ltimes \underline{J}^{i+1})(R) \to (\underline{G}^i \ltimes \underline{J}^{i+1})(k).$$

Let w_{i+1} be that representation of $(\underline{G}^i \ltimes \underline{J}^{i+1})(k) = (\underline{G}^i_k \ltimes \underline{J}^{i+1}_k)(k)$. Then w_{i+1} is itself obtained by pulling back the representation W_{i+1} along the k-points of the quotient

$$\underline{G}^i_k \ltimes \underline{J}^{i+1}_k \to \mathrm{Sp}(V_{i+1}) \ltimes V_{i+1}^\sharp.$$

This brings us back to [18, §10.5] as quoted in the Introduction to this paper.

4.4. **Geometrization of characters of types.** Finally, we come to the main result of Section 4. Since Yu's theory refers to complex representations, and since our geometrization uses ℓ-adic sheaves, we grit our teeth and fix an isomorphism $\mathbb{C} \approx \bar{\mathbb{Q}}_\ell$.

Theorem 4.2. *Let* $^\circ K^i$ $^\circ\rho^0$, φ^i, *for* $i = 0, \dots d$, *be a Yu type datum as in Section 4.3, Y0, Y1 and Y2. Let* $\underline{G}^i$ *and* ϱ_0^{red} *be as in Section 4.3, M0 and M1. Assume* $\pi_0((\underline{G}^0)_k^{\mathrm{red}})$ *is cyclic. For* $i = 0, \dots, d$, *let* $G^i = \mathrm{Gr}_R(\underline{G}^i)$ *be the Greenberg transform of the smooth group scheme* $\underline{G}^i$ *appearing in Section 4.3. Then there is a rational virtual Weil sheaf complex* $\mathcal{F}_i$ *on* G^i *such that* $t_{\mathcal{F}_i} = \mathrm{Tr}(^\circ\rho_i)$, *for* $i = 0, \dots, d$.

Proof. For every $i = 0, \dots, d$, set $G^i = \mathrm{Gr}_R(\underline{G}^i)$. Recall that $G^i(k) = \underline{G}^i(R) = {}^\circ K^i$, canonically. We begin with an argument already employed in the proof of Proposition 4.1. By continuity of the quasicharacters $\varphi^i : G^i(k) \to \mathbb{C}^\times$, there is some $m \in \mathbb{N}$ and a factorization

$$\begin{array}{ccc} G^i(k) & \xrightarrow{\quad\varphi^i\quad} & \bar{\mathbb{Q}}_\ell^\times \\ & \searrow \quad\quad \nearrow_{\varphi^i_m} & \\ & \underline{G}^i(R_m) & \end{array}$$

for all $i = 0, \dots, d$. Set $G^i_m = \mathrm{Gr}^R_m(\underline{G}^i)$. Then G^i_m is a smooth group scheme over k and $G^i_m(k) = \underline{G}^i(R_m)$, canonically. Using Theorem 3.12, let $\mathcal{L}^i_m$ be a geometrization of the linear character $\varphi^i_m : G^i_m(k) \to \mathbb{C}^\times$; so

$$t_{\mathcal{L}^i_m} = \varphi^i_m.$$

By [15], there is a rational virtual Weil character sheaf $A = (\bar{A}, \phi)$ on $(\underline{G}^0)_k^{\mathrm{red}}$ such that $\bar{A}$ is a rational virtual character sheaf on $(\underline{G}^0)_{\bar{k}}^{\mathrm{red}}$ and

$$t_A = \mathrm{Tr}\, \varrho_0^{\mathrm{red}}.$$

(This uses the hypothesis that $\pi_0((\underline{G}^0)_k^{\mathrm{red}})$ is cyclic.) Let A^0 be the Weil sheaf on $(\underline{G}^0)_k$ obtained by pullback along the quotient $(\underline{G}^0)_k \to (\underline{G}^0)_k^{\mathrm{red}}$. Then

$$t_{A^0} = \mathrm{Tr}\, \varrho_0.$$

The special fibre $(\underline{G}^0)_k$ of the smooth group scheme $\underline{G}^0$ is itself a smooth group scheme, and may be identified with the Greenberg transform $Q^0 = \mathrm{Gr}^R_0(\underline{G}^0)$ [5,

§4.3]. With $m \in \mathbb{N}$ as above, let A^0_m be the Weil sheaf on the algebraic group G^i_m obtained by pull-back from A^0 along the affine morphism $G^i_m \to Q^0$. Factor

$$\begin{array}{ccc} G^0(k) & \xrightarrow{\mathrm{Tr}({}^\circ\rho^0)} & \bar{\mathbb{Q}}_\ell \\ & \searrow \quad \nearrow_{\mathrm{Tr}({}^\circ\rho^0)_m} & \\ & G^0_m(k) & \end{array} \tag{12}$$

Observe that $\mathrm{Tr}({}^\circ\rho^0)_m$ may be recovered from A^0_m:

$$t_{A^0_m} = \mathrm{Tr}({}^\circ\rho^0)_m$$

Consider the Jacobi group $\mathrm{Sp}(V_{i+1}) \ltimes V^\sharp_{i+1}$ and the Heisenberg-Weil representation W_{i+1} appearing in Section 4.3. Let $\mathcal{K}^{i+1}$ be the Weil sheaf on the Jacobi group, recalled in Section 4.2, such that

$$t_{\mathcal{K}^{i+1}} = \mathrm{Tr}(W_{i+1}).$$

Recall from Section 4.3 that $\mathrm{Sp}(V_{i+1}) \ltimes V^\sharp_{i+1}$ is a quotient of the special fibre of the smooth group scheme $\underline{G}^i \ltimes \underline{J}^{i+1}$. Let $\mathcal{K}^{i+1}_0$ be the Weil sheaf on the special fibre of $\underline{G}^i \ltimes \underline{J}^{i+1}$ obtained from W_{i+1} by pullback. Let $\mathcal{K}^{i+1}_m$ be the Weil sheaf on $\mathrm{Gr}^R_m(\underline{G}^i \ltimes \underline{J}^{i+1})$ obtained from $\mathcal{K}^{i+1}_0$ by pullback along the affine morphism $\mathrm{Gr}^R_m(\underline{G}^i \ltimes \underline{J}^{i+1}) \to \mathrm{Gr}^R_0(\underline{G}^i \ltimes \underline{J}^{i+1})$.

We now define Weil sheaves $\mathcal{A}^i_m$ on $G^i_m := \mathrm{Gr}^R_m(\underline{G}^i)$, for $i = 0, \dots, d$, recursively, following the construction of the representations ${}^\circ\rho^i$, as reviewed in Section 4.3. First, set $\mathcal{A}^0_m = A^0_m$ and note that (12) commutes with $\mathrm{Tr}({}^\circ\rho^0)_m$ replaced by $t_{\mathcal{A}^0_m}$. Now, suppose $\mathcal{A}^i_m$ on G^i_m is defined such that

$$\begin{array}{ccc} G^i(k) & \xrightarrow{\mathrm{Tr}({}^\circ\rho^i)} & \bar{\mathbb{Q}}_\ell \\ & \searrow \quad \nearrow_{t_{\mathcal{A}^i_m}} & \\ & G^i_m(k) & \end{array}$$

commutes. Applying the Greenberg functor Gr^R_m, to (11) gives

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & G^i_m \times_{G_m} J^{i+1}_m & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & J^{i+1}_m & \longrightarrow & G^i_m \ltimes J^{i+1}_m & \xrightarrow{p^i_m} & G^i_m & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \pi^{i+1}_m} & & & & \\ & & & & G^{i+1}_m & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array} \tag{13}$$

where $J^{i+1}_m := \mathrm{Gr}^R_m(\underline{J}^{i+1})$ and $G^i_m := \mathrm{Gr}^R_m(\underline{G}^i)$. By [3, Prop 7.1], the sequences are exact. Consider the sheaf

$$\mathcal{B}^{i+1}_m := \mathcal{K}^{i+1}_m \otimes (p^i_m)^*(\mathcal{A}^i_m \otimes \mathcal{L}^i_m)$$

on $G^i_m \ltimes J^{i+1}_m$. Comparing with (10), we see that $t_{\mathcal{B}^{i+1}_m}$ is precisely the function obtained by factoring the character of ${}^\circ\rho^{i+1}$ through $(\underline{G}^i \ltimes \underline{J}^{i+1})(R) \to (\underline{G}^i \ltimes \underline{J}^{i+1})(R_m)$ using the canonical identification $(G^i_m \times_{G_m} J^{i+1}_m)(k) = (\underline{G}^i \ltimes \underline{J}^{i+1})(R_m)$. In particular, $t_{\mathcal{B}^{i+1}_m}$ is constant on $(G^i_m \times_{G_m} J^{i+1}_m)(k)$, taking the value $\dim {}^\circ\rho^{i+1}$. With reference to the morphism $\pi^{i+1}_m : G^i_m \ltimes J^{i+1}_m \to G^{i+1}_m$ from (13), define

$$\mathcal{C}^{i+1}_m := (\pi^{i+1}_m)_!(\mathcal{B}^{i+1}_m).$$

Then

$$t_{\mathcal{C}^{i+1}_m}(x) \quad = \sum_{y \in (\pi^{i+1}_m)^{-1}(x)} t_{\mathcal{B}^{i+1}_m}(y).$$

Since $t_{\mathcal{B}^{i+1}_m}$ is constant on $(G^i_m \times_{G_m} J^{i+1}_m)(k)$, it follows that

$$t_{\mathcal{C}^{i+1}_m} = n t_{\mathcal{B}^{i+1}_m}$$

on $G^{i+1}_m(k)$ for $n = \#(G^i_m \times_{G_m} J^{i+1}_m)(k) \times \dim {}^\circ\rho^{i+1}$. Let $\mathcal{A}^{i+1}_m$ be the *rational virtual* Weil sheaf on G^i_m given by $\mathcal{A}^{i+1}_m = \frac{1}{n}\mathcal{C}^{i+1}_m$. This completes the inductive definition of $\mathcal{A}^i_m$ so that the following diagram commutes.

$$\begin{array}{ccc} G^{i+1}(k) & \xrightarrow{\mathrm{Tr}({}^\circ\rho^{i+1})} & \bar{\mathbb{Q}}_\ell \\ & \searrow \quad \nearrow_{t_{\mathcal{A}^{i+1}_m}} & \\ & G^{i+1}_m(k) & \end{array}$$

Now set $\mathcal{F}^i_m = \mathcal{A}^i_m \otimes \mathcal{L}^i_m$, for $i = 0, \ldots, d$. Then $\mathcal{F}^i_m$ is a rational virtual Weil sheaf on $G^i_m = \mathrm{Gr}^R_m(\underline{G}^i)$ such that

$$\begin{array}{ccc} G^{i}(k) & \xrightarrow{\mathrm{Tr}({}^\circ\rho_i)} & \bar{\mathbb{Q}}_\ell \\ & \searrow \quad \nearrow_{t_{\mathcal{F}^{i}_m}} & \\ & G_m(k) & \end{array}$$

commutes. Let $\mathcal{F}^i$ be the rational virtual Weil sheaf on the group scheme $G^i = \mathrm{Gr}_R(\underline{G}^i)$ obtained by pulling back $\mathcal{F}^i_m$ along $G^i \to G^i_m$. Then

$$t_{\mathcal{F}^i} = \mathrm{Tr}({}^\circ\rho_i),$$

as desired. □

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