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GEOMETRIZATION OF CONTINUOUS CHARACTERS OF $\mathbb{Z}_p^\times$

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We define the p-adic trace of certain rank-one local systems on the multiplicative group over p-adic numbers, using Sekiguchi and Suwa's unification of Kummer and Artin–Schreier–Witt theories. Our main observation is that, for every nonnegative integer n, the p-adic trace defines an isomorphism of abelian groups between local systems whose order divides $(p-1)p^n$ and ℓ-adic characters of the multiplicative group of p-adic integers of depth less than or equal to n.

Introduction. Let p and ℓ be distinct primes and let q be a power of p. Let G be a connected commutative algebraic group over $\mathbb{F}_q$; that is, a smooth commutative group scheme of finite type over a field. To geometrize a character $\psi : G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ one pushes forward the Lang central extension

$$0 \to G(\mathbb{F}_q) \to G \xrightarrow{\text{Lang}} G \to 0, \quad \text{Lang}(x) = \text{Fr}(x) - x,$$

by ψ^{-1} and obtains a local system $\mathscr{L}_\psi$ on G. The trace of Frobenius of $\mathscr{L}_\psi$ equals ψ; which is to say that $\mathscr{L}_\psi$ and ψ correspond under the functions–sheaves dictionary. Thus, we think of $\mathscr{L}_\psi$ as *the geometrization of* ψ. Let $\mathsf{C}(G)$ be the abelian group (under tensor product) consisting of $\mathscr{L}_\psi$ as ψ ranges over $\text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times)$; in other words, $\mathsf{C}(G)$ is the group of irreducible summands of $\text{Lang}_! \overline{\mathbb{Q}}_\ell$. Trace of Frobenius defines an isomorphism of abelian groups

$$t_{\text{Fr}} : \mathsf{C}(G) \xrightarrow{\simeq} \text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times); \tag{1}$$

see [Deligne 1977, Sommes Trig.] and [Laumon 1987, Example 1.1.3].

Here we obtain an analogue of this isomorphism for $\mathbb{G}_m$ over p-adic numbers.

Theorem. *The work of Sekiguchi and Suwa, on unification of Kummer with Artin–Schreier theories, provides an isomorphism between the abelian group of rank-one*

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local systems on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and the abelian group of characters of $\mathbb{Z}_p^\times$ of depth less than or equal to n, for every nonnegative integer n.

Motivation and relation to character sheaves. Before proving the theorem, we take a moment to explain our motivation. Deligne used the local systems $\mathcal{L}_\psi$, appearing above, to prove bounds on the trigonometric sums over finite fields. A key fact used by Deligne in his computation is Grothendieck trace formula. An analogue of this trace formula is missing over p-adic fields. This is the main hurdle for pursuing an analogue of Deligne's results. We hope that the local systems we study here will be of use in obtaining bounds for corresponding sums over p-adic fields.

According to [Lusztig 1985, Section 2], character sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ are perverse sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ (cohomologically) concentrated in degree 1 where they are rank-one Kummer local systems. We restrict our attention to those character sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and find that these are precisely those that admit a $\mathbb{Q}_p(\mu_{p^n})$-rational structure; that is, they can be defined on $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$. In this language, the above theorem states the following: *The p-adic trace (defined below) of every $\mathbb{Q}_p(\mu_{p^\infty})$-rational character sheaf on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ is a continuous character $\mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_\ell^\times$ and, moreover, every continuous ℓ-adic character of $\mathbb{Z}_p^\times$ is obtained in this manner, each one from a unique character sheaf of $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$.*

Our idea for defining a function from a $\mathbb{Q}_p(\mu_{p^n})$-rational character sheaf $\mathcal{K}$ on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ is to consider $\mathbb{Z}_p[\mu_{p^n}]$-models for $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$ such that $\mathcal{K}$ extends to a local system on the model; then, after restriction to the special fibre of the model, we recover a local system to which we may apply the trace of Frobenius function, as above. Using the work of Sekiguchi and Suwa we find that this idea can be realized if one additional step is introduced: we must consider $\mathbb{Z}_p[\mu_{p^n}]$-models for $\mathbb{G}^{n+1}_{m,\mathbb{Q}_p(\mu_{p^n})}$, rather than $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$. We believe that this strategy for passing from character sheaves on p-adic groups with rational structure to smooth characters by judicious use of integral models may be of wider applicability in establishing a relationship between character sheaves on p-adic groups and admissible characters. This note is meant to illustrate a case of this strategy.

Unification of Kummer with Artin–Schreier–Witt. Henceforth, we assume that p is an *odd* prime. Fix a nonnegative integer n and a primitive p^n-th root of unity $\zeta \in \overline{\mathbb{Q}}_p$. Set $R = \mathbb{Z}_p[\zeta]$, $K = \mathbb{Q}_p(\zeta)$. The main theorem of Sekiguchi and Suwa on the unification of Kummer and Artin–Schreier–Witt theories provides us with

- an exact sequence

$$0 \to \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \to \mathcal{Y} \xrightarrow{f} \mathcal{X} \to 0$$

  of commutative group schemes over R,

- isomorphisms $\mathcal{Y}_K := \mathcal{Y} \otimes_R K \xrightarrow{\simeq} \mathbb{G}_{m,K}^{n+1}$ and $\mathcal{X}_K \to \mathbb{G}_{m,K}^{n+1}$,
- isomorphisms $\mathcal{Y}_{\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$ and $\mathcal{X}_{\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$,

where $\mathbb{W}_{n,\mathbb{F}_p}$ is the Witt ring scheme of dimension n over $\mathbb{F}_p$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{G}_{m,K} \arrow[d, "\theta"'] & \mathbb{G}_{m,K}^{n+1} \arrow[l, "m"'] \arrow[d, "\gamma"] & \mathcal{Y}_K \arrow[l] \arrow[r] \arrow[d, "f_K"] & \mathcal{Y} \arrow[d, "f"'] & \mathcal{Y}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \arrow[l] \arrow[d, "f_{\mathbb{F}_p}"] \arrow[d, shift left=20, "\mathrm{Lang}"] \\
\mathbb{G}_{m,K} & \mathbb{G}_{m,K}^{n+1} \arrow[l, "\alpha"] & \mathcal{X}_K \arrow[l] \arrow[r] & \mathcal{X} & \mathcal{X}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}. \arrow[l]
\end{tikzcd}$$

Here, $\theta(x) = x^{(p-1)p^n}$, m denotes the multiplication map, γ and α are defined by

$$\gamma(x_0, \ldots, x_n) = \left(x_0^{p-1}, \frac{x_1^p}{x_2}, \frac{x_2^p}{x_3}, \ldots, \frac{x_n^p}{x_{n-1}}\right),$$

$$\alpha(x_0, x_1, \ldots, x_n) = \frac{(x_0 x_1 x_2 x_3 \cdots x_n)^{p^n}}{x_1 x_2^p x_3^{p^2} \cdots x_n^{p^{n-1}}},$$

and f_K and $f_{\mathbb{F}_p}$ are the restrictions of f to the generic and special fibre, respectively. The theorem in question was announced in [Suwa and Sekiguchi 1995] and a proof appeared in the preprint [Sekiguchi and Suwa 1999]. According to Sekiguchi, the main tools of this preprint have been published in [Sekiguchi and Suwa 2003]. For a general overview see [Tsuchiya 2003].

The p-adic trace function. Let $\mathsf{K}(\mathbb{G}_{m,K})$ denote the group (under tensor product) of local systems that are irreducible summands of $\theta_! \overline{\mathbb{Q}}_\ell$. One can easily check that all the squares in the above diagram are Cartesian; moreover, it is clear that all the vertical arrows are Galois covers of order $(p-1)p^n$. It follows that the diagram above determines a canonical isomorphism of groups

$$S : \mathsf{K}(\mathbb{G}_{m,K}) \xrightarrow{\simeq} \mathsf{C}(\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}). \tag{2}$$

We define the *p-adic trace function* by

$$\begin{aligned} \mathfrak{Tr}_n : \mathsf{K}(\mathbb{G}_{m,K}) &\longrightarrow \mathrm{Hom}\big(\mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p), \overline{\mathbb{Q}}_\ell^\times\big) \\ \mathcal{K} &\mapsto t_{\mathrm{Fr}}(S(\mathcal{K})). \end{aligned} \tag{3}$$

It follows at once from (1) and (2) that $\mathfrak{Tr}_n$ is a canonical isomorphism.

Relationship to continuous characters of $\mathbb{Z}_p^\times$. Since p is odd, the exponential map defines an isomorphism of algebraic $\mathbb{F}_p$-groups

$$\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{W}^*_{n+1,\mathbb{F}_p} \tag{4}$$

where $\mathbb{W}^*_{n+1,\mathbb{F}_p}$ refers to the group scheme of units in the Witt ring scheme $\mathbb{W}_{n+1,\mathbb{F}_p}$ (see [Greenberg 1962]) and therefore an isomorphism

$$\mathbb{G}_m(\mathbb{F}_p)\times\mathbb{W}_n(\mathbb{F}_p)=\mathbb{Z}/(p-1)\times\mathbb{Z}/p^n\xrightarrow{\simeq}\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p). \tag{5}$$

Accordingly, we can think of the p-adic trace as a character of $\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p)$. Composing with the quotient $\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p)$, we see that the p-adic trace can be interpreted as a continuous ℓ-adic character of $\mathbb{Z}_p^\times$.

Conversely, for every continuous character $\chi:\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_\ell^\times$, there is a nonnegative integer n such that $\chi(\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p))=\{1\}$. The smallest such n is known as the depth of χ. We propose to think of $\mathcal{K}_\chi:=\mathfrak{Tr}_n^{-1}(\chi)$ as *the geometrization of* χ, when $\chi:\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_\ell^\times$ is a continuous character of depth n. We do not discuss how to vary n in the present text.

We note that choosing an isomorphism of the form (5) is unappetizing. We hope, in time, to give a construction which does not depend on this choice.

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