
ARTHUR PACKETS FOR p -ADIC GROUPS BY WAY OF MICROLOCAL VANISHING CYCLES OF PERVERSE SHEAVES, WITH EXAMPLES

by

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Abstract. — In this article we propose a geometric description of Arthur packets for p -adic groups using vanishing cycles of perverse sheaves. Our approach is inspired by the 1992 book by Adams, Barbasch and Vogan on the Langlands classification of admissible representations of real groups and follows the direction indicated by Vogan in his 1993 paper on the Langlands correspondence. We introduce and study a functor built from vanishing cycles from the category of equivariant perverse sheaves on the moduli space of certain Langlands parameters to local systems on the regular part of the conormal bundle for this variety. By establishing the main properties of this functor, we show that it plays the role of the microlocalisation functor in the work of Adams, Barbasch and Vogan. Using this, we define ABV-packets for pure rational forms of p -adic groups and we also give a geometric description of the transfer coefficients that appear in Arthur's work. This article includes conjectures modelled on Vogan's work, including the prediction that Arthur packets are ABV-packets for p -adic groups. We verify these conjectures in several examples and while doing so, we show how to calculate the transfer coefficients that appear in Arthur's main local result in the endoscopic classification of representations, using purely geometric tools.

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Résumé. — Dans cet article nous proposons une description géométrique des paquets d'Arthur pour les groupes p -adiques en utilisant les cycles évanescents des faisceaux pervers. Notre approche s'inspire du livre d'Adams, Barbasch et Vogan sur la classification de Langlands des représentations admissibles des groupes réels et suit la direction suggérée par Vogan dans son article sur la correspondance de Langlands. Nous définissons et étudions un foncteur, construit à partir des cycles évanescents, de la catégorie des faisceaux pervers équivariant sur l'espace de modules de certains paramètres de Langlands vers la catégorie des systèmes locaux sur la partie régulière du fibré conormal de cette variété. En établissant les principales propriétés de ce foncteur, nous montrons qu'il joue le rôle du foncteur de microlocalisation dans les travaux d'Adams, Barbasch et Vogan. En utilisant cela, nous définissons les ABV-paquets des formes rationnelles pures des groupes p -adiques et nous donnons également une description géométrique des coefficients de transfert qui apparaissent dans les travaux d'Arthur. Cet article contient des conjectures inspirées des travaux de Vogan et en particulier la prédiction que les paquets d'Arthur sont les ABV-paquets pour les groupes p -adiques.

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1. Introduction

1.1. Motivation. — Let F be a local field of characteristic zero and G be a connected reductive linear algebraic group over F . According to the local Langlands

conjecture, the set $\Pi(G(F))$ of isomorphism classes of irreducible admissible representations of $G(F)$ can be naturally partitioned into finite subsets, called L-packets. Moreover, the local Langlands conjecture predicts that if an L-packet contains one tempered representation, then all the representations in that L-packet are tempered, so tempered L-packets provide a partition of tempered irreducible admissible representations. Tempered L-packets enjoy some other very nice properties. For instance, every tempered L-packet determines a stable distribution on $G(F)$ by a non-trivial linear combination of the distribution characters of the representations in the packet. Tempered L-packets also have an endoscopy theory, which leads to a parametrization of the distribution characters of the representations in the packet.

These properties fail for non-tempered L-packets. To remedy this, in 1989 Arthur introduced what are now known as Arthur packets, which enlarge the non-tempered L-packets in such a way that these last two properties do extend to the non-tempered case. Arthur's motivation was global, arising from the classification of automorphic representations, so the local meaning of Arthur packets was unclear when they first appeared.

In 1992, shortly after Arthur packets were introduced, Adams, Barbasch and Vogan suggested a purely local description of Arthur packets for real groups using microlocal analysis of certain stratified complex varieties built from Langlands parameters. Then, in 1993 Vogan used similar tools to make a prediction for a local description of Arthur packets for p -adic groups. Since these constructions are purely local, and since the initial description of Arthur packets was global in nature, it was not easy to compare ABV-packets with Arthur packets. The conjecture that Arthur packets are ABV-packets has remained open since the latter were introduced.

When Arthur finished his monumental work on the classification of automorphic representations of symplectic and special orthogonal groups in 2013, the situation changed dramatically. Not only did he prove his own conjectures on Arthur packets given in [Art89], but he also gave a local characterisation of them, using twisted endoscopy. This opened the door to comparing Arthur packets with ABV-packets and motivated us to compare Arthur's work with Vogan's constructions in the p -adic case. This article is the first in a series making that comparison.

1.2. Background. — To begin, let us briefly review Arthur's main local result in the endoscopic classification of representations. Let G be a quasi-split connected reductive algebraic group over a p -adic field F . An Arthur parameter for G is a homomorphism, $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$, where L_F is the local Langlands group, to the Langlands group ${}^L G = \widehat{G} \rtimes W_F$, satisfying a number of conditions. One important condition is that the image of $\psi(W_F)$ under the projection onto \widehat{G} must have compact closure. When G is symplectic or special orthogonal, Arthur [Art13, Theorem 1.5.1] assigns to any ψ a multiset $\Pi_\psi(G(F))$ over $\Pi(G(F))$, known as the Arthur packet of G associated with ψ . It is a deep result of Mœglin [Mœg11] that $\Pi_\psi(G(F))$ is actually a subset of $\Pi(G(F))$. Endoscopy theory [Art13, Theorem 2.2.1] in this case

gives rise to a canonical map

$$(1) \quad \begin{aligned} \Pi_\psi(G(F)) &\rightarrow \widehat{\mathcal{S}}_\psi \\ \pi &\mapsto \langle \cdot, \pi \rangle_\psi \end{aligned}$$

to $\widehat{\mathcal{S}}_\psi$, the set of irreducible characters of $\mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0 Z(\widehat{G})^{\Gamma_F}$. If the Arthur parameter $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ is trivial on the $\mathrm{SL}(2, \mathbb{C})$ factor then $\Pi_\psi(G(F))$ is a tempered L-packet and the map (1) is a bijection. In general, $\Pi_\psi(G(F))$ contains the L-packet $\Pi_{\phi_\psi}(G(F))$, where ϕ_ψ is the Langlands parameter given by $\phi_\psi(u) := \psi(u, d_u)$, where for $u \in L_F$ we set $d_u = \mathrm{diag}(|u|^{1/2}, |u|^{-1/2})$ with $||$ the pullback of the norm map on W_F . The map (1) determines a stable distribution on $G(F)$ by

$$(2) \quad \Theta_\psi^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle z_\psi, \pi \rangle_\psi \Theta_\pi,$$

where z_ψ is the image of $\psi(1, -1)$ in \mathcal{S}_ψ with $(1, -1) \in L_F \times \mathrm{SL}(2, \mathbb{C})$ where -1 is the non-trivial central element of $\mathrm{SL}(2, \mathbb{C})$.

In this article we express Arthur's conjectural generalisation of (1) for inner twists of G using pure rational forms of G as articulated by Vogan. A pure rational form (also known as a *pure inner form*) of G is a cocycle $\delta \in Z^1(F, G)$. An *inner rational form* is a cocycle $\sigma \in Z^1(F, \mathrm{Inn}(G))$. Using the maps

$$Z^1(F, G) \rightarrow Z^1(F, G_{\mathrm{ad}}) = Z^1(F, \mathrm{Inn}(G)) \rightarrow Z^1(F, \mathrm{Aut}(G)),$$

every pure rational form of G determines an inner rational form of G and every inner rational form of G determines a rational form of G . Following [Vog93], a representation of a pure rational form of G is defined to be a pair (π, δ) , where δ is a pure rational form of G and π is an equivalence class of admissible representations of $G_\delta(F)$. The action of $G(\bar{F})$ by conjugation defines an equivalence relation on such pairs, which is compatible with the equivalence relation on pure rational forms $Z^1(F, G)$ producing $H^1(F, G)$. Again following [Vog93], we write $\Pi_{\mathrm{pure}}(G/F)$ for the equivalence classes of such pairs. Then, after choosing a representative for each class in $H^1(F, G)$, we may write

$$\Pi_{\mathrm{pure}}(G/F) = \bigsqcup_{[\delta] \in H^1(F, G)} \Pi(G_\delta(F), \delta),$$

where $\Pi(G_\delta(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi(G_\delta(F))\}$.

An *inner twist* of G is a pair (G_1, φ) where G_1 is a rational form of G together with an isomorphism of algebraic groups φ from $G_1 \otimes_F \bar{F}$ to $G \otimes_F \bar{F}$ such that $\gamma \mapsto \varphi \circ \gamma(\varphi)^{-1}$ is a 1-cocycle in $Z^1(\Gamma_F, \mathrm{Inn}(G))$ [Art13, Section 9.1]. Every inner rational form σ of G determines an *inner twist* $(G_\sigma, \varphi_\sigma)$ such that the action of $\gamma \in \Gamma_F$ on $G_\sigma(\bar{F})$ is given through the σ -twisted action on $G(\bar{F})$. We use the notation $(G_\delta, \varphi_\delta)$ for the inner twist of G determined by the pure rational form δ . An Arthur parameter ψ for G is *relevant* to G_δ if any Levi subgroup of ${}^L G$ that ψ factors through is the dual group of a Levi subgroup of G_δ . In [Art13, Conjecture 9.4.2], Arthur assigns to any relevant ψ a multiset $\Pi_\psi(G_\delta(F))$ over $\Pi(G_\delta(F))$, which is called the Arthur

packet for G_δ associated to ψ . Moeglin's work shows that, since G_δ comes from a pure rational form, $\Pi_\psi(G_\delta(F))$ is again a subset of $\Pi(G_\delta(F))$. To extend (1) to this case, Arthur replaces the group \mathcal{S}_ψ with a generally non-abelian group $\mathcal{S}_{\psi, \text{sc}}$ [Art13, Section 9.2], which is a central extension of \mathcal{S}_ψ by $\widehat{Z}_{\psi, \text{sc}}$; compare with (24). Let $\tilde{\zeta}_{G_\delta}$ be a character of $\widehat{Z}_{\psi, \text{sc}}$ and let $\text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_{G_\delta})$ be the set of isomorphism classes of $\tilde{\zeta}_{G_\delta}$ -equivariant representations of $\mathcal{S}_{\psi, \text{sc}}$ and $\langle \cdot, \pi \rangle_{\psi, \text{sc}}$ is the character of the associated representation of $\mathcal{S}_{\psi, \text{sc}}$.

Endoscopy theory [Art13, Conjecture 9.4.2] gives a map

$$(3) \quad \Pi_\psi(G_\delta(F)) \rightarrow \text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_{G_\delta});$$

the character of the representation attached to an irreducible representation π of the inner twist $(G_\delta, \varphi_\delta)$ is denoted by $\langle \cdot, \pi \rangle_{\psi, \text{sc}}$. The map (3) depends only on (1) and the pure rational form δ . For any Arthur parameter ψ for G and any pure rational form δ of G we define

$$\Pi_\psi(G_\delta(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi_\psi(G_\delta(F))\}$$

where, if ψ is not relevant to G_δ , then $\Pi_\psi(G_\delta^*(F))$ and thus $\Pi_\psi(G_\delta(F), \delta)$ is empty. Now we introduce

$$(4) \quad \Pi_\psi(G/F) := \{[\pi, \delta] \in \Pi_{\text{pure}}(G/F) \mid (\pi, \delta) \in \Pi_\psi(G_\delta(F), \delta)\}.$$

After choosing a representative pure rational form δ for every class in $H^1(F, G)$, we have

$$\Pi_{\text{pure}, \psi}(G/F) = \bigsqcup_{[\delta] \in H^1(F, G)} \Pi_\psi(G_\delta(F), \delta).$$

Now, set

$$A_\psi := \pi_0(Z_{\widehat{G}}(\psi)) = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0$$

and let $\chi_\delta : \pi_0(Z(\widehat{G})^{\Gamma_F}) \rightarrow \mathbb{C}^\times$ be the character matching $[\delta] \in H^1(F, G)$ under the Kottwitz isomorphism $H^1(F, G) \cong \text{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1)$. Let $\text{Rep}(A_\psi, \chi_\delta)$ denote the set of equivalence classes of representations of A_ψ such that the pullback of the representations along

$$\pi_0(Z(\widehat{G})^{\Gamma_F}) \rightarrow \pi_0(Z_{\widehat{G}}(\psi))$$

is χ_δ . In Proposition 2.10.3 we show that (3) defines a canonical map

$$(5) \quad \Pi_{\text{pure}, \psi}(G/F) \rightarrow \text{Rep}(A_\psi)$$

and we write $\langle \cdot, [\pi, \delta] \rangle_\psi$ for the representation attached to $[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)$. built from canonical maps

$$(6) \quad \Pi_\psi(G_\delta(F), \delta) \rightarrow \text{Rep}(A_\psi, \chi_\delta).$$

The maps (6) depend only on δ and (1), as discussed in Section 2.10. When $\delta = 1$, (6) recovers (1) and if ψ is tempered then (6) gives a canonical bijection

$$(7) \quad \Pi_{\phi_\psi}(G_\delta(F), \delta) \rightarrow \Pi(A_\psi, \chi_\delta),$$

where $\Pi(A_\psi, \chi_\delta)$ denotes the set of χ_δ -equivariant characters of A_ψ .

1.3. Main results. — In this article we give a geometric and categorical approach to calculating a generalisation of (5), and therefore of (6) also, which applies to all quasi-split connected reductive algebraic groups G over p -adic fields, by assuming the local Langlands correspondence for its pure rational forms, as articulated by Vogan in [Vog93]. The local Langlands correspondence is known for split symplectic and orthogonal groups by the work of Arthur and others. In [Art13, Chapter 9] Arthur sets the foundation for adapting his work to inner forms of these groups, which can be seen as a step toward the version proposed by Vogan in [Vog93]. Building on Arthur’s work, Vogan’s version of the local Langlands correspondence is known for unitary groups by work of [Mok15] and [KMSW14]; it is expected that similar arguments should yield the result for symplectic and orthogonal groups, but that has not been done yet.

Our approach is based on ideas developed for real groups in [ABV92] and on results from [Vog93] for p -adic groups. We conjecture that this geometric approach produces a map that coincides with (6) from Arthur, after specializing to the case of quasi-split symplectic and special orthogonal p -adic groups. The generalisation of (6) that we propose leads quickly to what should be a generalisation of Arthur packets.

We now sketch our generalisation of (6). Let F be a p -adic field and let G be any quasi-split connected reductive algebraic group over F . Every Langlands parameter ϕ for G determines an “infinitesimal parameter” $\lambda_\phi : W_F \rightarrow {}^L G$ by $\lambda_\phi(w) := \phi(w, d_w)$ where $d_w = \text{diag}(|w|^{1/2}, |w|^{-1/2})$. The map $\phi \mapsto \lambda_\phi$ is not injective, but the preimage of any infinitesimal parameter falls into finitely many equivalence classes of Langlands parameters under \hat{G} -conjugation. Set $\lambda_\psi := \lambda_{\phi_\psi}$. Let $\Pi_{\text{pure}, \lambda_\psi}(G/F)$ be the set of $[\pi, \delta] \in \Pi_{\text{pure}}(G/F)$ such that the Langlands parameter ϕ , whose associated L-packet contains π , satisfies $\lambda_\phi = \lambda_\psi$. The generalisation of (6) that we define takes the form of a map

$$(8) \quad \Pi_{\text{pure}, \lambda_\psi}(G/F) \rightarrow \text{Rep}(A_\psi).$$

The genesis of the map (8) is the interesting part, as it represents a geometrisation and categorification of (6).

To order to define (8), in Section 3 we review the definition of a variety V_λ , following [Vog93], that parametrises the set $P_\lambda({}^L G)$ of Langlands parameters ϕ for G such that $\lambda_\phi = \lambda$. The variety V_λ is equipped with an action of $Z_{\hat{G}}(\lambda)$. Then, again following [Vog93], we consider the category $\text{Per}_{Z_{\hat{G}}(\lambda)}(V_\lambda)$ of equivariant perverse sheaves on V_λ . Together with (7), the version of the Langlands correspondence that applies to G and its pure rational forms determines a *bijection* between $\Pi_{\text{pure}, \lambda}(G/F)$ and isomorphism classes of simple objects in $\text{Per}_{Z_{\hat{G}}(\lambda)}(V_\lambda)$:

$$(9) \quad \begin{aligned} \Pi_{\text{pure}, \lambda}(G/F) &\rightarrow \text{Per}_{Z_{\hat{G}}(\lambda)}(V_\lambda)_{\text{iso}}^{\text{simple}}, \\ [\pi, \delta] &\mapsto \mathcal{P}(\pi, \delta). \end{aligned}$$

Inspired by an analogous result in [ABV92] for real groups, in Proposition 5.6.1 we show that every Arthur parameter ψ determines a particular element in the conormal bundle to V_λ

$$(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V_{\lambda_\psi}),$$

where $C_\psi \subseteq V_{\lambda_\psi}$ is the $Z_{\widehat{G}}(\lambda_\psi)$ -orbit of $x_\psi \in V_\lambda$, such that the $Z_{\widehat{G}}(\lambda_\psi)$ -orbit of (x_ψ, ξ_ψ) is the unique open orbit $T_{C_\psi}^*(V_{\lambda_\psi})_{\text{sreg}}$ in $T_{C_\psi}^*(V_{\lambda_\psi})$. Then we use (x_ψ, ξ_ψ) to show that A_ψ is the equivariant fundamental group of $T_{C_\psi}^*(V_\lambda)_{\text{reg}}$. Thus, (x_ψ, ξ_ψ) determines an equivalence of categories

$$\text{Loc}_{Z_{\widehat{G}}(\lambda)}(T_{C_\psi}^*(V_\lambda)_{\text{sreg}}) \rightarrow \text{Rep}(A_\psi),$$

where $\text{Rep}(A_\psi)$ denotes the category of representations of A_ψ . This means that the spectral transfer factors $\langle \cdot, \pi \rangle_{\psi, \text{sc}}$ for ψ appearing in (3) can be interpreted as equivariant local systems on $T_{C_\psi}^*(V_{\lambda_\psi})_{\text{sreg}}$.

In Section 6.3 we use the vanishing cycles functor to define an exact functor

$$(10) \quad \text{NEv}_\psi : \text{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda) \rightarrow \text{Rep}(A_\psi)$$

which plays the role of the microlocalisation functor as it appears in [ABV92] for real groups. Vanishing cycles of perverse sheaves on V_λ are fundamental tools for understanding the singularities on the boundaries of strata in V_λ and their appearance here is quite natural. Passing to isomorphism classes of objects, this functor defines a map

$$\text{Per}_{Z_{\widehat{G}}(\lambda)}(V_\lambda)^{\text{simple}}_{/\text{iso}} \rightarrow \text{Rep}(A_\psi)_{/\text{iso}}.$$

When composed with (9), this defines (8).

1.4. Conjecture. — We now explain the conjectured relation between (5) and (8). With reference to (10), consider the support of (8), which we call the *ABV-packet* for ψ :

$$(11) \quad \Pi_{\text{pure}, \psi}^{\text{ABV}}(G/F) := \{[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F) \mid \text{NEv}_\psi \mathcal{P}(\pi, \delta) \neq 0\}.$$

We can break the ABV-packet $\Pi_{\text{pure}, \psi}^{\text{ABV}}(G/F)$ apart according to pure rational forms of G :

$$\Pi_{\text{pure}, \psi}^{\text{ABV}}(G/F) = \bigsqcup_{[\delta] \in H^1(F, G)} \Pi_\psi^{\text{ABV}}(G_\delta(F), \delta),$$

where

$$\Pi_\psi^{\text{ABV}}(G_\delta(F), \delta) = \{(\pi, \delta) \in \Pi(G_\delta(F), \delta) \mid \text{NEv}_\psi \mathcal{P}(\pi, \delta) \neq 0\}.$$

We may now state a simplified version of the main conjecture of this article; see Conjecture 1 in Section 7.3 for a stronger form. *Let ψ be an Arthur parameter for a quasi-split symplectic or special orthogonal p -adic group G . Then*

$$\Pi_{\text{pure}, \psi}(G/F) = \Pi_{\text{pure}, \psi}^{\text{ABV}}(G/F).$$

Moreover, for all pure rational forms δ of G and for all $[\pi, \delta] \in \Pi_{\text{pure}, \lambda_\psi}(G/F)$,

$$\langle s, [\pi, \delta] \rangle_\psi = \text{trace}_{a_s} \text{NEv}_\psi \mathcal{P}(\pi, \delta),$$

for all $s \in Z_{\widehat{G}}(\psi)$, where a_s is the image of s under $Z_{\widehat{G}}(\psi) \in A_\psi$. In particular, taking the case when δ is trivial, if $\pi \in \Pi_{\lambda_\psi}(G(F))$ then

$$\langle s, \pi \rangle_\psi = \text{trace}_{a_s} \text{NEv}_\psi \mathcal{P}(\pi),$$

with $s \in Z_{\widehat{G}}(\psi)$ and $a_s \in A_\psi$ as above.

The pithy version of this conjecture is *Arthur packets are ABV-packets for p -adic groups*, but that statement obscures the fact that Arthur packets are defined separately for each inner rational form (more precisely the corresponding inner twist), while ABV-packets treat all pure rational forms in one go. More seriously, this pithy version of the conjecture obscures the fact that the conjecture proposes a completely geometric approach to calculating the characters $\langle \cdot, \pi \rangle_{\psi, \text{sc}}$ appearing in Arthur's endoscopic classification of representations.

To simplify the discussion, in this overview we have only described ABV-packets for Arthur parameters; however, as we see in this article, it is possible to attach an ABV-packet to each Langlands parameter. Consequently, there are more ABV-packets than Arthur packets. So, while the conjecture above asserts that every Arthur packet is an ABV-packet, it is certainly not true that every ABV-packet is an Arthur packet. If validated, the conjecture gives credence to the idea that ABV-packets may be thought of as generalised Arthur packets.

Although we do not prove the conjecture above in this article, we do have in mind a strategy for a proof using twisted spectral endoscopic transfer and its geometric counterpart for perverse sheaves on Vogan varieties; we use this strategy to prove Conjectures 1 and 2 for unipotent representations of odd orthogonal groups in forthcoming work [CFMX].

1.5. Examples. — Our objective in Part II of this article is to show how to use vanishing cycles of perverse sheaves to calculate the local transfer coefficients $\langle s_\psi s, \pi \rangle_\psi$ that appear in Arthur's endoscopic classification [Art13, Theorem 1.5.1]. We do this by independently calculating both sides of Conjecture 1 in examples:

$$(12) \quad \langle s_\psi s, \pi \rangle_\psi = (-1)^{\dim C_\psi - \dim C_\pi} \text{trace}_s \mathbf{NEv}_\psi \mathcal{P}(\pi),$$

for every $s \in Z_{\widehat{G}}(\psi)$. By making these calculations, we wish to demonstrate that the functor \mathbf{NEv} provides a practical tool for calculating Arthur packets, the associated stable distributions and their transfer under endoscopy. We also verify the Kazhdan-Lusztig conjecture for p -adic groups as it applies to our examples.

Specifically, in Part II we consider certain admissible representations of the p -adic groups: $\text{SL}(2)$ and its inner form; $\text{PGL}(4)$; split $\text{SO}(3)$, $\text{SO}(5)$, $\text{SO}(7)$ and their pure rational forms. There are a variety of reasons why we have chosen to present this specific set of examples. The groups $\text{SO}(3)$, $\text{SO}(5)$, and $\text{SO}(7)$ are the first few groups in the family $\text{SO}(2n+1)$, and this is the family we study in [CFMX] for unipotent representations. The group $\text{SO}(7)$ is the first in this family to exhibit some of the more general phenomena that meaningfully illuminate the conjectures from Part I. Moreover, since $\text{SO}(3) \times \text{SO}(3)$ is an elliptic endoscopic group for $\text{SO}(5)$ and $\text{SO}(5) \times \text{SO}(3)$ is an elliptic endoscopic group for $\text{SO}(7)$, we are also able to use these examples to show how to use geometric tools to compute Langlands-Shelstad transfer of invariant distributions for endoscopic groups. Not only was this ultimately a useful feature for doing the geometric calculations, but presenting these examples side by side allows one to see certain relationships that hold more generally for endoscopic groups. We also include two examples – for $\text{SL}(2)$ and $\text{PGL}(4)$ – that show how the

problem of calculating Arthur packets and Arthur's transfer coefficients is reduced to unipotent representations.

1.6. Related work. — Using techniques different from those employed in this article (namely, microlocalisation of regular holonomic D -modules, rather than vanishing cycles of perverse sheaves) one of the authors of this article has calculated many other examples of ABV-packets in his PhD thesis [Mra17]. Specifically, if π is a unipotent representation of $\mathrm{PGL}(n)$, $\mathrm{SL}(n)$, $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$, of any of its pure rational forms, and if the image of Frobenius of the infinitesimal parameter of π is *regular* semisimple in the dual group, then all ABV-packets containing π have been calculated by finding the support of the microlocalisation of the relevant D -modules. This work overlaps with Sections 10 and 12, here. However, we found it difficult to calculate the finer properties of the microlocalisation of these D -modules required to determine the local transfer coefficients appearing in Arthur's work. This is one of the reasons we use vanishing cycles of perverse sheaves in this article.

1.7. Disclaimer. — To acknowledge the debt we owe to [ABV92] and [Vog93], we refer to the packets appearing in this article as ABV-packets for p -adic groups, though it must be pointed out that they appear neither in [ABV92] nor in [Vog93]. For real groups, the definition of "ABV-packets" uses an exact functor $Q_C^{\mathrm{mic}} : \mathrm{Per}_H(V) \rightarrow \mathrm{Loc}_H(T_C^*(V)_{\mathrm{reg}})$ introduced in [ABV92, Theorem 24.8] whose properties are established using stratified Morse theory, which we have not used in this article; and for p -adic groups, [Vog93] uses the microlocal Euler characteristic $\chi_C^{\mathrm{mic}} : \mathrm{Per}_H(V) \rightarrow \mathbb{Z}$ derived from the microlocalisation functor, which we also have not used in this article. We have elected to use vanishing cycles, or more precisely the functor Ev , in place of stratified Morse theory or microlocalisation because we found Ev more amenable to the many calculations we performed in Part II and because we found some theoretical advantages to using vanishing cycles.

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PART I. ARTHUR PACKETS AND MICROLOCAL VANISHING CYCLES

Here are the main features of Part I, by section.

In Section 2 we review the main local result from [Art13], adapted to pure rational forms of quasi-split connected reductive groups over p -adic fields.

In Section 3 we describe Vogan's parameter variety for p -adic groups and review Vogan's perspective on the local Langlands conjecture for pure rational forms of quasi-split connected reductive groups over p -adic fields, based on [Vog93].

Theorem 4.1.1 shows that the Vogan variety for an arbitrary infinitesimal parameter coincides with the Vogan variety for an unramified infinitesimal parameter. This theorem also shows that the category of equivariant perverse sheaves is related to the category of equivariant perverse sheaves on a graded Lie algebra, thereby putting tools from [Lus95c] at our disposal.

Proposition 5.1.1 shows that Arthur parameters determine conormal vectors to Vogan's parameter space and further that representations of the component group attached to the Arthur parameter correspond exactly to equivariant local systems on the orbit of that conormal vector, as in the case of real groups [ABV92].

In Section 6 we use vanishing cycles to define two exact functors – denoted by \mathbf{Evs} and \mathbf{NEvs} – from equivariant perverse sheaves on the Vogan variety to equivariant local systems on the strongly regular part of the conormal bundle associated to its stratification. Sections 6.3 through 6.9 establish the main properties of \mathbf{Evs} , including Theorem 6.7.5 which determines the rank of these local systems. Theorem 6.10.1 shows that \mathbf{NEvs} replaces microlocalisation by showing that it enjoys properties parallel to Q^{mic} from [ABV92, Theorem 24.8],

In Section 7 we express Vogan's conjectures from [Vog93] in terms of vanishing cycles; see Conjectures 1 and 2. One of the most interesting features of the vanishing cycles approach to Arthur packets is that it suggests two different parametrizations of Arthur packets, as determined by the two functor \mathbf{Evs} and \mathbf{NEvs} . The conjectures predict that the one determined by the functor \mathbf{NEvs} coincides with Arthur's work.

2. Arthur packets and pure rational forms

The goal of this section is primarily to set some notation and recall the characters of A_ψ and $A_{\psi, \text{sc}}$ appearing in Arthur's work as they pertain to pure rational forms.

2.1. Local Langlands group. — Let F be a p -adic field; let $q = q_F$ be the cardinality of the residue field for F . Let \bar{F} be an algebraic closure of F and set $\Gamma_F := \text{Gal}(\bar{F}/F)$. There is an exact sequence

$$1 \longrightarrow I_F \longrightarrow \Gamma_F \longrightarrow \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1,$$

where I_F is the inertia subgroup of Γ_F and $\bar{\mathbb{F}}_q$ is an algebraic closure of \mathbb{F}_q . Since $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$, it contains a dense subgroup $W_{k_F} \cong \mathbb{Z}$, in which 1 corresponds to

the automorphism $x \mapsto x^{q_F}$ in $\bar{\mathbb{F}}_q$. We fix a lift Fr in Γ_F of $x \mapsto x^{q_F}$ in W_{k_F} . The Weil group W_F of F is the preimage of W_{k_F} in Γ_F ,

$$1 \longrightarrow I_F \longrightarrow W_F \overset{\curvearrowright}{\longrightarrow} W_{k_F} \longrightarrow 1,$$

topologized so that the compact subgroup I_F is open in W_F . Let

$$|\cdot|_F : W_F \longrightarrow \mathbb{R}^\times$$

be the norm homomorphism, trivial on I_F and sending Fr to q_F . Then $|\cdot|_F$ is continuous with respect to this topology for W_F .

The *local Langlands group* of F is the trivial extension of W_F by $\text{SL}(2, \mathbb{C})$:

$$1 \longrightarrow \text{SL}(2, \mathbb{C}) \longrightarrow L_F \overset{\curvearrowright}{\longrightarrow} W_F \longrightarrow 1.$$

2.2. L-groups. — Let G be a connected reductive linear algebraic group over F . Let

$$\Psi_0(G) = (X^*, \Delta, X_*, \Delta^\vee)$$

be the based root datum of G . The dual based root datum is

$$\Psi_0^\vee(G) := (X_*, \Delta^\vee, X^*, \Delta).$$

A *dual group* of G is a complex connected reductive algebraic group \hat{G} together with a bijection

$$\eta_{\hat{G}} : \Psi_0^\vee(G) \cong \Psi_0(\hat{G}).$$

The Galois group Γ_F acts on $\Psi_0(G)$ and $\Psi_0^\vee(G)$; see [Bor79, Section 1.3]. This action induces a homomorphism

$$\mu : \Gamma_F \longrightarrow \text{Aut}(\Psi_0(G)) \cong \text{Aut}(\Psi_0^\vee(G)).$$

Let \hat{G} be a dual group of G . Then we can compose $\eta_{\hat{G}}$ with μ and get a homomorphism

$$\mu_{\hat{G}} : \Gamma_F \longrightarrow \text{Aut}(\Psi_0(\hat{G})).$$

An *L-group data* for G is a triple $(\hat{G}, \rho, \text{Spl}_{\hat{G}})$, where \hat{G} is a dual group of G , $\rho : \Gamma_F \longrightarrow \text{Aut}(\hat{G})$ is a continuous homomorphism and $\text{Spl}_{\hat{G}} := (B, T, \{X_\alpha\})$ is a splitting of \hat{G} such that ρ preserves $\text{Spl}_{\hat{G}}$ and induces $\mu_{\hat{G}}$ on $\Psi_0(\hat{G})$ (see [Bor79, Sections 1, 2] for details.)

The *L-group* of G determined by the L-group data $(\hat{G}, \rho, \text{Spl}_{\hat{G}})$ is

$${}^L G := \hat{G} \rtimes W_F,$$

where the action of W_F on \hat{G} factors through ρ . Since ρ induces $\mu_{\hat{G}}$ on $\Psi_0(\hat{G})$ and since $\text{Aut}(\Psi_0(\hat{G}))$ is finite, the action of W_F on \hat{G} factors through a finite quotient of W_F . We remark that the L-group, ${}^L G$, only depends on \hat{G} and ρ and is unique up to conjugation by elements in \hat{G} fixed by Γ_F .

Henceforth we fix an L-group, ${}^L G$, of G and make ${}^L G$ a topological group by giving \hat{G} the complex topology and W_F the profinite topology.

2.3. Langlands parameters. — If $\phi : L_F \rightarrow {}^L G$ is a group homomorphism that commutes with the projections $L_F \rightarrow W_F$ and ${}^L G \rightarrow W_F$, then we may define $\phi^\circ : L_F \rightarrow \widehat{G}$ by $\phi(w, x) = \phi^\circ(w, x) \rtimes w$. Then we have the following map of split short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{SL}(2, \mathbb{C}) & \longrightarrow & L_F & \overset{\curvearrowright}{\longrightarrow} & W_F \longrightarrow 1 \\ & & \downarrow & \nearrow \phi^\circ & \downarrow \phi & & \parallel \\ 1 & \longrightarrow & \widehat{G} & \longrightarrow & {}^L G & \overset{\curvearrowright}{\longrightarrow} & W_F \longrightarrow 1. \end{array}$$

A *Langlands parameter* for G is a homomorphism $\phi : L_F \rightarrow {}^L G$ such that

- (P.i) ϕ is continuous;
- (P.ii) ϕ commutes with the projections $L_F \rightarrow W_F$ and ${}^L G \rightarrow W_F$;
- (P.iii) $\phi^\circ|_{\mathrm{SL}(2, \mathbb{C})} : \mathrm{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$ is induced from a morphism of algebraic groups;
- (P.iv) the image of $\phi|_{W_F}$ consists of semisimple elements in ${}^L G$.

See [Bor79, Section 8.2(i)] and Section 4.2 for the meaning of semisimple elements in L-groups.

Let $P({}^L G)$ be the set of Langlands parameters for G . For $\phi \in P({}^L G)$, we refer to

$$A_\phi := \pi_0(Z_{\widehat{G}}(\phi)) = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^0$$

as the *component group* for ϕ .

Langlands parameters are *equivalent* if they are conjugate under \widehat{G} . The set of equivalence classes of Langlands parameters of G is denoted by $\Phi(G/F)$; it is independent of the choice of L-group ${}^L G$ made above.

2.4. Arthur parameters. — If $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ is a group homomorphism that commutes with the projections $L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow L_F \rightarrow W_F$ and ${}^L G \rightarrow W_F$, then we define $\psi^\circ : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$ by $\psi(w, x, y) = \phi^\circ(w, x, y) \rtimes w$, where $(w, x) \in L_F$ and $y \in \mathrm{SL}(2, \mathbb{C})$.

An *Arthur parameter* for G is a homomorphism $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ such that

- (Q.i) $\psi|_{L_F}$ is a Langlands parameter for G ;
- (Q.ii) $\psi^\circ|_{\mathrm{SL}(2, \mathbb{C})} : \mathrm{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$ is induced from a morphism of algebraic groups;
- (Q.iii) the image $\psi^\circ|_{W_F} : W_F \rightarrow \widehat{G}$ is bounded (its closure is compact).

Following [Vog93, Definition 4.2], the set of Arthur parameters for G will be denoted by $Q({}^L G)$. The set of \widehat{G} -conjugacy classes of Arthur parameters will be denoted by $\Psi(G/F)$.

For $\psi \in Q({}^L G)$, we refer to

$$A_\psi := \pi_0(Z_{\widehat{G}}(\psi)) = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0$$

as the *component group* for ψ .

2.5. Langlands parameters of Arthur type. — Define $d : W_F \rightarrow \mathrm{SL}(2, \mathbb{C})$ by

$$(13) \quad d_w := \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

Note that $w \mapsto (w, d_w)$ is a section of $L_F \rightarrow W_F$. For $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$, define $\phi_\psi : L_F \rightarrow {}^L G$ by

$$\phi_\psi(w, x) = \psi(w, x, d_w).$$

This defines a map

$$(14) \quad \begin{array}{ccc} Q({}^L G) & \rightarrow & P({}^L G) \\ \psi & \mapsto & \phi_\psi. \end{array}$$

We will refer to ϕ_ψ as the *Langlands parameter for ψ* . The function $\psi \mapsto \phi_\psi$ is neither injective nor surjective. Langlands parameters in the image of the map $Q({}^L G) \rightarrow P({}^L G)$ are called *Langlands parameters of Arthur type*. The function

$$\Psi(G/F) \rightarrow \Phi(G/F),$$

induced from $Q({}^L G) \rightarrow P({}^L G)$, is injective.

2.6. Pure rational forms. — We suppose now that the connected reductive algebraic group G over F is quasi-split.

An *inner rational form* σ of G is a 1-cocycle of Γ_F in G_{ad} , where G_{ad} is the adjoint group of G . It determines an *inner twist* $(G_\sigma, \varphi_\sigma^*)$ of G as follows. Let $G_\sigma(\bar{F}) := G^*(\bar{F})$ and φ_σ be the identity map. The action of $\gamma \in \Gamma_F$ on $G_\sigma(\bar{F})$ is given through the twisted Galois action on $G(\bar{F})$, i.e., $\gamma : g \mapsto \mathrm{Ad}(\sigma(\gamma))(\gamma \cdot g)$ for $g \in G(\bar{F})$, where $\gamma \cdot g$ refers to the action of Γ_F on $G(\bar{F})$ defining G over F . We will represent the inner twist by G_σ , and identify $G_\sigma(F)$ as a subgroup of $G(\bar{F})$ through φ_σ . Two inner rational forms σ_1, σ_2 of G are equivalent if they give the same cohomology class in $H^1(F, G_{\mathrm{ad}})$, or equivalently $G_{\sigma_1}(F)$ and $G_{\sigma_2}(F)$ are conjugate under $G(\bar{F})$. There is a canonical isomorphism

$$H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}, \mathbb{C}^1)$$

where $\widehat{G}_{\mathrm{sc}}$ is the simply connected cover of the derived group of \widehat{G} . The character of $Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_F}$ determined by $[\sigma] \in H^1(F, G_{\mathrm{ad}})$ will be denoted ζ_σ .

A *pure rational form* δ of G is a 1-cocycle of Γ_F in G . It determines an inner rational form $\sigma := \delta(\sigma)$ by the canonical map

$$(15) \quad Z^1(F, G) \rightarrow Z^1(F, G_{\mathrm{ad}}).$$

We will denote the inner twist G_σ by G_δ . Two pure rational forms of G are equivalent if they give the same cohomology class in $H^1(F, G_{\mathrm{ad}})$. There is also a canonical isomorphism

$$H^1(F, G) \cong \mathrm{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1).$$

The character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ corresponding to the equivalence class of δ will be denoted by χ_δ . By [Kot84, Proposition 6.4], the homomorphism $G \rightarrow G_{\mathrm{ad}}$ induces a

commuting diagram:

$$\begin{array}{ccc} H^1(F, G) & \longrightarrow & H^1(F, G_{\text{ad}}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}(\pi_0(Z(\widehat{G})^{\Gamma_F}), \mathbb{C}^1) & \longrightarrow & \text{Hom}(Z(\widehat{G}_{\text{sc}})^{\Gamma_F}, \mathbb{C}^1). \end{array}$$

So ζ_σ is the image of χ_δ and we will also denote it by ζ_δ .

2.7. Langlands packets for pure rational forms. — An isomorphism class of representations of a pure rational form of G is a pair (π, δ) , where π is an isomorphism class of admissible representations of $G_\delta(F)$. Then $G(\bar{F})$ -conjugation defines an equivalence relation on such pairs, which is compatible with the equivalence relation on pure rational forms $Z^1(F, G)$. We denote the equivalence class of (π, δ) by $[\pi, \delta]$, and following [Vog93], write $\Pi_{\text{pure}}(G/F)$ for the set of these equivalence classes. The local Langlands correspondence for pure rational forms of G can be stated as in the following conjecture. *There is a natural bijection between $\Pi_{\text{pure}}(G/F)$ and \widehat{G} -conjugacy classes of pairs (ϕ, ρ) with $\phi \in P(^L G)$ and $\rho \in \text{Irrep}(A_\phi)$.* We will call the pair (ϕ, ρ) in this conjecture a complete Langlands parameter. For $\phi \in P(^L G)$, we define the corresponding *pure Langlands packet*

$$\Pi_{\text{pure}, \phi}(G/F)$$

to be consisting of $[\pi, \delta]$ in $\Pi_{\text{pure}}(G/F)$, such that they are associated with \widehat{G} -conjugacy classes of (ϕ, ρ) for any $\rho \in \text{Irrep}(A_\phi)$ under the local Langlands correspondence for pure rational forms. This is also known as the Langlands-Vogan packet.

2.8. Arthur packets for quasi-split symplectic or special orthogonal groups.

— From now on until the end of Section 2, we will assume G is a quasi-split symplectic or special orthogonal group over F . In [Art13, Theorem 1.5.1], Arthur assigns to $\psi \in Q(^L G)$ a multiset $\Pi_\psi(G(F))$ over $\Pi(G(F))$, which is usually referred to as the Arthur packet of G associated with ψ . It is a deep result of Mœglin [Mœg11] that $\Pi_\psi(G(F))$ is actually a subset of $\Pi(G(F))$. Arthur [Art13, Theorem 2.2.1] also associates $\Pi_\psi(G(F))$ with a canonical map

$$(16) \quad \begin{aligned} \Pi_\psi(G(F)) &\rightarrow \widehat{\mathcal{S}}_\psi \\ \pi &\mapsto \langle \cdot, \pi \rangle_\psi \end{aligned}$$

where

$$(17) \quad \mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0 Z(\widehat{G})^{\Gamma_F},$$

and $\widehat{\mathcal{S}}_\psi$ denotes the set of irreducible characters of \mathcal{S}_ψ . We use (16) to define a stable virtual representation of $G(F)$ by

$$(18) \quad \eta_\psi^G := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \pi,$$

where $s_\psi \in \mathcal{S}_\psi$ is the image of $\psi(1, 1, -1)$ under the mapping $Z_{\widehat{G}}(\psi) \rightarrow \mathcal{S}_\psi$ and where $(1, 1, -1) \in L_F$ with -1 denoting the non-trivial central element in $\mathrm{SL}(2, \mathbb{C})$. Every semisimple $s \in Z_{\widehat{G}}(\psi)$ determines an element x of \mathcal{S}_ψ and thus a new virtual representation

$$(19) \quad \eta_{\psi, s}^G := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi x, \pi \rangle_\psi \pi.$$

Turning to the stable distributions on $G(F)$, we set

$$(20) \quad \Theta_\psi^G := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \Theta_\pi,$$

and

$$(21) \quad \Theta_{\psi, s}^G := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi x, \pi \rangle_\psi \Theta_\pi.$$

The pair (ψ, s) also determines an endoscopic datum $(G', {}^L G', s, \xi)$ for G and an Arthur parameter ψ' for G' so that $\psi = \xi \circ \psi'$. In fact, G' is a product group, whose factors consist of symplectic, special orthogonal and general linear groups. So one can extend the above discussions about G to G' without difficulty, as done in [Art13].

Arthur's main local result shows that, for locally constant compactly supported function f on $G(F)$, we have

$$(22) \quad \Theta_{\psi, s}^G(f) = \Theta_{\psi'}^{G'}(f'),$$

where f' is the Langlands-Shelstad transfer of f from $G(F)$ to $G'(F)$. It is in this sense that the maps (16) are compatible with spectral endoscopic transfer to $G(F)$.

On the other hand, there is an involution θ of $G := \mathrm{GL}(N)$ over F such that $(G, {}^L G, s, \xi_N)$ is a twisted endoscopic datum for $G^+(F) := \mathrm{GL}(N, F) \rtimes \langle \theta \rangle$ in the sense of [KS99, Section 2.1], for suitable semisimple $s \in \widehat{G}^\theta$, the component of $\widehat{\theta}$ in $\widehat{G}^+ := \widehat{G} \rtimes \langle \widehat{\theta} \rangle$, where $\widehat{\theta}$ is the dual involution. Arthur's main local result also shows that, for locally constant compactly supported function f^θ on $G^\theta(F) := G(F) \rtimes \theta$,

$$(23) \quad \Theta_\psi^G(f) = \Theta_{\psi_N, s}^{G^+}(f^\theta),$$

where f is the Langlands-Kottwitz-Shelstad transfer of f^θ from $G^\theta(F)$ to $G^*(F)$ and $\Theta_{\psi_N, s}^{G^+}$ is the twisted character of a particular extension of the Speh representation of $\mathrm{GL}(N, F)$ associated with Arthur parameter $\psi_N := \xi_N \circ \psi$ to the disconnected group $G^+(F)$. It is in this sense that the maps (16) are compatible with twisted spectral endoscopic transfer from $G(F)$.

Arthur shows that the map (16) is uniquely determined by: the stability of Θ_ψ^G ; property (22) for all endoscopic data G' ; and property (23) for twisted endoscopy of $\mathrm{GL}(N)$. In particular, the endoscopic character identities that are used to pin down $\langle \cdot, \pi \rangle_\psi$ involve values at *all* elements of \mathcal{S}_ψ .

When ψ is trivial on the second $\mathrm{SL}(2, \mathbb{C})$, it becomes a tempered Langlands parameter. In this case, Arthur shows (16) is a bijection. By the Langlands classification of $\Pi(G(F))$, which is in terms of tempered representations, this bijection extends to all Langlands parameters of G . Moreover, it follows from Arthur's results that there

is a bijection between $\Pi(G(F))$ and \widehat{G} -conjugacy classes of pairs (ϕ, ϵ) for $\phi \in P({}^L G)$ and $\epsilon \in \widehat{\mathcal{S}}_\phi$.

2.9. Arthur packets for inner rational forms. — A conjectural description of Arthur packets for *inner twists* of G is presented in [Art13, Chapter 9], though the story is far from complete. Let σ be an inner rational form of G . An Arthur parameter ψ of G_σ is said to be *relevant* if any Levi subgroup of ${}^L G_\sigma$ that ψ factors through is the L-group of a Levi subgroup of G_σ . We denote the subset of relevant Arthur parameter by $Q_{\text{rel}}(G_\sigma)$. In [Art13, Conjecture 9.4.2], Arthur assigns to $\psi \in Q_{\text{rel}}(G_\sigma)$ a multiset $\Pi_\psi(G_\sigma(F))$ over $\Pi(G_\sigma(F))$, which is called the Arthur packet of G_σ associated with ψ . This time Mœglin's results [Mœg11] only show $\Pi_\psi(G_\sigma(F))$ is a subset of $\Pi(G_\sigma(F))$ in case when σ comes from a pure rational form; see also [Art13, Conjecture 9.4.2, Remark 2]. For the purpose of comparison with the geometric construction of Arthur packets, in this article we define $\Pi_\psi(G_\sigma(F))$ simply as the image of this multiset in $\Pi(G_\sigma(F))$.

To extend (16) to this case, one must replace the group \mathcal{S}_ψ with a larger, finite, generally non-abelian group $\mathcal{S}_{\psi, \text{sc}}$, which is a central extension

$$(24) \quad 1 \longrightarrow \widehat{Z}_{\psi, \text{sc}} \longrightarrow \mathcal{S}_{\psi, \text{sc}} \longrightarrow \mathcal{S}_\psi \longrightarrow 1$$

of \mathcal{S}_ψ by the finite abelian group

$$\widehat{Z}_{\psi, \text{sc}} := Z(\widehat{G}_{\text{sc}}^*) / Z(\widehat{G}_{\text{sc}}^*) \cap S_{\psi, \text{sc}}^0.$$

To explain the group in this exact sequence, we introduce the following notations. Set

$$S_\psi := Z_{\widehat{G}^*}(\psi) \quad \text{and} \quad \bar{S}_\psi := Z_{\widehat{G}^*}(\psi) / Z(\widehat{G})^{\Gamma_F}.$$

So \bar{S}_ψ is the image of S_ψ in $\widehat{G}_{\text{ad}}^*$, whose preimage in \widehat{G}^* is $S_\psi Z(\widehat{G})$. Let $S_{\psi, \text{sc}}$ be the preimage of \bar{S}_ψ under the projection $\widehat{G}_{\text{sc}}^* \rightarrow \widehat{G}_{\text{ad}}^*$, which is the same as the preimage of $S_\psi Z(\widehat{G})$ in $\widehat{G}_{\text{sc}}^*$. Let $S_{\psi, \text{sc}}^\sharp$ be the preimage of S_ψ in $\widehat{G}_{\text{sc}}^*$ and $\widehat{Z}_{\text{sc}}^\sharp$ be the preimage of $Z(\widehat{G})^{\Gamma_F}$ in $\widehat{G}_{\text{sc}}^*$. Let us write $Z(\widehat{G}^*)$ (resp. $Z(\widehat{G}_{\text{sc}}^*)$) for \widehat{Z} (resp. \widehat{Z}_{sc}). It is clear that $\widehat{Z}_{\text{sc}}^{\Gamma_F} \hookrightarrow \widehat{Z}_{\text{sc}}^\sharp$. Then we have the following commutative diagram, which is exact on each row:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \widehat{Z}^{\Gamma_F} & \longrightarrow & S_\psi & \longrightarrow & \bar{S}_\psi \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & \widehat{Z}_{\text{sc}}^\sharp & \longrightarrow & S_{\psi, \text{sc}}^\sharp & \longrightarrow & \bar{S}_\psi \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \widehat{Z}_{\text{sc}} & \longrightarrow & S_{\psi, \text{sc}} & \longrightarrow & \bar{S}_\psi \longrightarrow 1. \end{array}$$

Note $S_{\psi, \text{sc}} = S_{\psi, \text{sc}}^\sharp \widehat{Z}_{\text{sc}}$, and hence $S_{\psi, \text{sc}}^0 = (S_{\psi, \text{sc}}^\sharp)^0$. After passing to the component groups, we have the following commutative diagram, which is again exact on each

row:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \widehat{Z}_\psi^{\Gamma_F} & \longrightarrow & A_\psi & \longrightarrow & \mathcal{S}_\psi \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \parallel \\
 1 & \longrightarrow & \widehat{Z}_{\psi, \text{sc}}^\# & \longrightarrow & \mathcal{S}_{\psi, \text{sc}}^\# & \longrightarrow & \mathcal{S}_\psi \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \widehat{Z}_{\psi, \text{sc}} & \longrightarrow & \mathcal{S}_{\psi, \text{sc}} & \longrightarrow & \mathcal{S}_\psi \longrightarrow 1.
 \end{array}$$

Here $A_\psi, \mathcal{S}_\psi, \mathcal{S}_{\psi, \text{sc}}^\#, \mathcal{S}_{\psi, \text{sc}}$ are the corresponding component groups and

$$\widehat{Z}_\psi^{\Gamma_F} := \widehat{Z}^{\Gamma_F} / \widehat{Z}^{\Gamma_F} \cap S_\psi^0$$

$$\widehat{Z}_{\psi, \text{sc}}^\# := \widehat{Z}_{\text{sc}}^\# / \widehat{Z}_{\text{sc}}^\# \cap S_{\psi, \text{sc}}^0$$

$$\widehat{Z}_{\psi, \text{sc}} := \widehat{Z}_{\text{sc}} / \widehat{Z}_{\text{sc}} \cap S_{\psi, \text{sc}}^0$$

Let ζ_σ be the character of $\widehat{Z}_{\text{sc}}^{\Gamma_F}$ corresponding to the equivalence class of σ . We will also fix an extension of ζ_σ to \widehat{Z}_{sc} and denote that by $\tilde{\zeta}_\sigma$. By [Art99, Lemma 2.1], an Arthur parameter ψ of G_σ is relevant if and only if the restriction of ζ_σ to $\widehat{Z}_{\text{sc}}^{\Gamma_F} \cap S_{\psi, \text{sc}}^0$ is trivial.

Lemma 2.9.1. — $\widehat{Z}_{\text{sc}}^{\Gamma_F} \cap S_{\psi, \text{sc}}^0 = \widehat{Z}_{\text{sc}} \cap S_{\psi, \text{sc}}^0$.

Proof. — It suffices to show $\widehat{Z}_{\text{sc}} \cap S_{\psi, \text{sc}}^0 \subseteq \widehat{Z}_{\text{sc}}^{\Gamma_F}$. Let $L_F \times \text{SL}(2, \mathbb{C})$ act on $\widehat{G}_{\text{sc}}^*$ by conjugation of the preimage of $\psi(L_F \times \text{SL}(2, \mathbb{C}))$ in ${}^L G_{\text{sc}}$. Then we can define the group cohomology $H_\psi^0(L_F \times \text{SL}(2, \mathbb{C}), \widehat{G}_{\text{sc}}^*)$, which is the group of fixed points in \widehat{G}_{sc} under the action of $L_F \times \text{SL}(2, \mathbb{C})$. It is clear that $H_\psi^0(L_F \times \text{SL}(2, \mathbb{C}), \widehat{G}_{\text{sc}}^*) \subseteq S_{\psi, \text{sc}}^\#$. In fact, it is also not hard to show that

$$(H_\psi^0(L_F \times \text{SL}(2, \mathbb{C}), \widehat{G}_{\text{sc}}^*))^0 = (S_{\psi, \text{sc}}^\#)^0.$$

As a result, we have

$$\widehat{Z}_{\text{sc}} \cap S_{\psi, \text{sc}}^0 \subseteq \widehat{Z}_{\text{sc}} \cap (H_\psi^0(L_F \times \text{SL}(2, \mathbb{C}), \widehat{G}_{\text{sc}}^*))^0 \subseteq \widehat{Z}_{\text{sc}} \cap H_\psi^0(L_F \times \text{SL}(2, \mathbb{C}), \widehat{G}_{\text{sc}}^*) = \widehat{Z}_{\text{sc}}^{\Gamma_F}.$$

This finishes the proof. \square

So, if ψ is relevant, it follows from Lemma 2.9.1 that $\tilde{\zeta}_\sigma$ descends to a character of $\widehat{Z}_{\psi, \text{sc}}$. Let $\text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_\sigma)$ be the set of isomorphism classes of $\tilde{\zeta}_\sigma$ -equivariant representations of $\mathcal{S}_{\psi, \text{sc}}$. In [Art13, Conjecture 9.4.2], Arthur conjectures a map

$$(25) \quad \Pi_\psi(G_\sigma(F)) \rightarrow \text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_\sigma)$$

and writes $\langle \cdot, \pi \rangle_{\psi, \text{sc}}$ for the character of the associated representation of $\mathcal{S}_{\psi, \text{sc}}$. Because of our definition of $\Pi_\psi(G_\sigma(F))$ here, one can not replace $\text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_\sigma)$ by the subset $\Pi(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_\sigma)$ of $\tilde{\zeta}_\sigma$ -equivariant irreducible characters of $\mathcal{S}_{\psi, \text{sc}}$ as in Arthur's original formulation. The map (25) is far from being canonical for it depends on (16) and various other choices implicitly.

When ψ is a tempered Langlands parameter, Arthur states all these results as a theorem [Art13, Theorem 9.4.1]. In particular, he claims (25) gives a bijection

$$(26) \quad \Pi_\phi(G_\sigma(F)) \rightarrow \Pi(\mathcal{S}_{\phi, \text{sc}}, \tilde{\zeta}_\sigma).$$

By the Langlands classification of $\Pi(G_\sigma(F))$, which is in terms of tempered representations, this bijection extends to all relevant Langlands parameters of G_σ . Moreover, it follows from [Art13, Theorem 9.4.1] that there is a bijection between $\Pi(G_\sigma(F))$ and \widehat{G} -conjugacy classes of pairs (ϕ, ϵ) for $\phi \in P_{\text{rel}}({}^L G_\sigma^*)$ and $\epsilon \in \Pi(\mathcal{S}_{\phi, \text{sc}}, \tilde{\zeta}_\sigma)$.

2.10. Pure Arthur packets. — Let δ be a pure rational form of G and ψ be an Arthur parameter of G_δ . Let χ_δ be the character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ corresponding to the equivalence class of δ . We will also denote its pull-back to $Z(\widehat{G})^{\Gamma_F}$ by χ_δ . Let $\zeta_\delta := \zeta_{\sigma(\delta)}$ be the character of $Z(\widehat{G}_{\text{sc}})^{\Gamma_F}$, which is also the pull-back of χ_δ along

$$Z(\widehat{G}_{\text{sc}})^{\Gamma_F} \rightarrow \pi_0(Z(\widehat{G})^{\Gamma_F}).$$

Lemma 2.10.1. — χ_δ is trivial on $\widehat{Z}^{\Gamma_F} \cap S_\psi^0$ if and only if ζ_δ is trivial on $\widehat{Z}_{\text{sc}}^{\Gamma_F} \cap S_{\psi, \text{sc}}^0$.

Proof. — One just needs to notice that S_ψ^0 is the product of $(\widehat{Z}^{\Gamma_F})^0$ with the image of $S_{\psi, \text{sc}}^0$ in S_ψ . \square

As a direct consequence, we have the following corollary.

Corollary 2.10.2. — An Arthur parameter ψ of G_δ is relevant if and only if χ_δ is trivial on $\widehat{Z}^{\Gamma_F} \cap S_\psi^0$.

Let us assume ψ is relevant. Then χ_δ descends to a character of $\widehat{Z}_\psi^{\Gamma_F}$. Let $\text{Rep}(A_\psi, \chi_\delta)$ be the set equivalence classes of χ_δ -equivariant representations of A_ψ . Let $\tilde{\zeta}_\delta$ be a character of \widehat{Z}_{sc} extending ζ_δ , so that its restriction to $\widehat{Z}_{\text{sc}}^\#$ is the pull-back of χ_δ . Since ψ is relevant, $\tilde{\zeta}_\delta$ descends to a character of $\widehat{Z}_{\psi, \text{sc}}$. Let $\text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}_\delta)$ be the set of equivalence classes of $\tilde{\zeta}_\delta$ -equivariant representations of $\mathcal{S}_{\psi, \text{sc}}$.

Proposition 2.10.3. — Let χ a character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$. Let $\tilde{\zeta}$ be a character of $Z(\widehat{G}_{\text{sc}})$. Suppose the pull-back of χ along $\widehat{Z}_{\text{sc}}^\# \rightarrow Z(\widehat{G})^{\Gamma_F} \rightarrow \pi_0(Z(\widehat{G})^{\Gamma_F})$ coincides with the restriction of $\tilde{\zeta}$ to $\widehat{Z}_{\text{sc}}^\# \hookrightarrow Z(\widehat{G}_{\text{sc}})$. Then there is a canonical bijection

$$(27) \quad \text{Rep}(A_\psi, \chi) \rightarrow \text{Rep}(\mathcal{S}_{\psi, \text{sc}}, \tilde{\zeta}).$$

Proof. — Since

$$\text{Ker}(\mathcal{S}_{\psi, \text{sc}}^\# \rightarrow A_\psi) = \text{Ker}(\widehat{Z}_{\psi, \text{sc}}^\# \rightarrow \widehat{Z}_\psi^{\Gamma_F}),$$

there is a canonical bijection

$$\text{Rep}(A_\psi, \chi) \rightarrow \text{Rep}(\mathcal{S}_{\psi, \text{sc}}^\#, \zeta^\#),$$

where $\zeta^\#$ is the pull-back of χ_δ to $\widehat{Z}_{\psi, \text{sc}}^\#$. Since

$$\mathcal{S}_{\psi, \text{sc}} = \widehat{Z}_{\psi, \text{sc}} \mathcal{S}_{\psi, \text{sc}}^\# \quad \text{and} \quad \widehat{Z}_{\psi, \text{sc}} \cap \mathcal{S}_{\psi, \text{sc}}^\# = \widehat{Z}_{\psi, \text{sc}}^\#,$$

there is also a canonical bijection

$$\mathrm{Rep}(\mathcal{S}_{\psi, \mathrm{sc}}, \tilde{\zeta}) \rightarrow \mathrm{Rep}(\mathcal{S}_{\psi, \mathrm{sc}}^{\sharp}, \zeta^{\sharp}).$$

Combining the two isomorphisms above, we obtain the canonical bijection promised above. \square

Let us take δ among various other choices to be made in defining (25). To emphasize this choice, we will define

$$\Pi_{\psi}(G_{\delta}(F), \delta) := \{(\pi, \delta) \mid \pi \in \Pi_{\psi}(G_{\delta}(F))\}.$$

Then by composing (25) with (27) modulo isomorphisms, we can have a canonical map

$$(28) \quad \begin{aligned} \Pi_{\psi}(G_{\delta}(F), \delta) &\rightarrow \mathrm{Rep}(A_{\psi}, \chi_{\delta}) \\ (\pi, \delta) &\mapsto \langle \cdot, (\pi, \delta) \rangle_{\psi} \end{aligned}$$

which only depends on δ and (16). In particular, it becomes (16) when $\delta = 1$. For equivalent pure rational forms δ_1 and δ_2 of G , it follows from the construction of (25) that the following diagram commutes.

$$\begin{array}{ccc} \Pi_{\psi}(G_{\delta_1}(F), \delta_1) & \xrightarrow{\cong} & \Pi_{\psi}(G_{\delta_2}(F), \delta_2) \\ \downarrow & & \downarrow \\ \mathrm{Rep}(A_{\psi}, \chi_{\delta_1})_{/\mathrm{iso}} & \xlongequal{\quad} & \mathrm{Rep}(A_{\psi}, \chi_{\delta_2})_{/\mathrm{iso}}. \end{array}$$

As a result, (28) is also well-defined for the equivalence class $[\pi, \delta]$.

Let ψ be an Arthur parameter of G . For pure rational form δ such that ψ is not relevant, we will define $\Pi_{\psi}(G_{\delta}(F), \delta)$ to be empty. Then we can define the *pure Arthur packet* associated with ψ to be

$$(29) \quad \Pi_{\mathrm{pure}, \psi}(G/F) = \bigsqcup_{[\delta] \in H^1(F, G)} \Pi_{\psi}(G_{\delta}(F), \delta)$$

as a subset of $\Pi_{\mathrm{pure}}(G/F)$. It is equipped with a canonical map

$$(30) \quad \begin{aligned} \Pi_{\mathrm{pure}, \psi}(G/F) &\rightarrow \mathrm{Rep}(A_{\psi}) \\ [\pi, \delta] &\mapsto \langle \cdot, [\pi, \delta] \rangle_{\psi} \end{aligned}$$

When ψ is a tempered Langlands parameter, this induces a bijection

$$\begin{aligned} \Pi_{\mathrm{pure}, \phi}(G/F) &\rightarrow \Pi(A_{\phi}) \\ [\pi, \delta] &\mapsto \langle \cdot, [\pi, \delta] \rangle_{\phi}. \end{aligned}$$

This bijection also extends to all Langlands parameters ϕ of G , according to the discussion in the end of Section 2.9. Combined with the local Langlands correspondence for each pure rational form of G , we can conclude the local Langlands correspondence for pure rational forms of G appearing in Section 2.7.

2.11. Virtual representations of pure rational forms. — Let $\mathbb{K}\Pi_{\mathrm{pure}}(G/F)$ be the free abelian group generated by the set $\Pi_{\mathrm{pure}}(G/F)$. Define $\eta_{\psi} \in \mathbb{K}\Pi_{\mathrm{pure}}(G/F)$

by

$$(31) \quad \eta_\psi := \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)} e(\delta) \langle a_\psi, [\pi, \delta] \rangle_\psi [\pi, \delta],$$

where $e(\delta) = e(G_\delta)$ is the Kottwitz sign [Kot83] of the group G_δ , and a_ψ is the image of $\psi(1, 1, -1)$ in A_ψ . Using (29) we have

$$\eta_\psi = \sum_{[\delta] \in H^1(F, G)} e(\delta) \eta_\psi^\delta$$

where, for each pure rational form δ of G ,

$$\eta_\psi^\delta := \sum_{(\pi, \delta) \in \Pi_\psi(G_\delta(F), \delta)} \langle a_\psi, (\pi, \delta) \rangle_\psi [\pi, \delta].$$

For semisimple $s \in Z_{\widehat{G}}(\psi)$, we define $\eta_{\psi, s} \in \mathbb{K}\Pi_{\text{pure}}(G/F)$ by

$$(32) \quad \eta_{\psi, s} = \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)} e(\delta) \langle a_\psi a_s, (\pi, \delta) \rangle_\psi [\pi, \delta],$$

where a_s is the image of s in A_ψ . As above, we can break this into summands indexed by pure rational form by writing

$$\eta_{\psi, s} = \sum_{[\delta] \in H^1(F, G)} e(\delta) \eta_{\psi, s}^\delta$$

where, for each pure rational form δ of G ,

$$\eta_{\psi, s}^\delta := \sum_{(\pi, \delta) \in \Pi_\psi(G_\delta(F), \delta)} \langle a_\psi a_s, (\pi, \delta) \rangle_\psi [\pi, \delta].$$

Then $\eta_{\psi, 1}^\delta = \eta_\psi^\delta$ and $\eta_{\psi, 1} = \eta_\psi$. We note that, with reference to (18) and (19),

$$\eta_\psi^1 = \eta_\psi^G \quad \text{and} \quad \eta_{\psi, s}^1 = \eta_{\psi, s}^G.$$

Turning from virtual representations to distributions, we see that each η_ψ^δ and $\eta_{\psi, s}^\delta$ determines a distribution on $G_\delta(F)$ by

$$\Theta_{\psi, s}^\delta := \sum_{(\pi, \delta) \in \Pi_\psi(G_\delta(F), \delta)} \langle a_\psi a_s, (\pi, \delta) \rangle_\psi \Theta_\pi.$$

This extends (20) and (21) from $G(F)$ to $G_\delta(F)$ arising from pure rational forms δ of G :

$$\Theta_\psi^1 = \Theta_\psi^G \quad \text{and} \quad \Theta_{\psi, s}^1 = \Theta_{\psi, s}^G.$$

3. Equivariant perverse sheaves on parameter varieties

In this section we drop the quasi-split hypothesis and let G be an arbitrary connected reductive algebraic group over a p -adic field F .

3.1. Infinitesimal parameters. — An *infinitesimal parameter* for G is a homomorphism $\lambda : W_F \rightarrow {}^L G$ such that

(R.i) λ is continuous;

- (R.ii) λ is a section of ${}^L G \rightarrow W_F$;
- (R.iii) the image of λ consists of semisimple elements in ${}^L G$.

Let $R({}^L G)$ be the set of infinitesimal parameters for G . The *component group* for λ is

$$(33) \quad A_\lambda := \pi_0(Z_{\widehat{G}}(\lambda)) = Z_{\widehat{G}}(\lambda)/Z_{\widehat{G}}(\lambda)^0.$$

The set of \widehat{G} -conjugacy classes of infinitesimal parameters is denoted by $\Lambda(G/F)$.

For any Langlands parameter $\phi : L_F \rightarrow {}^L G$, define the *infinitesimal parameter* of ϕ by

$$\begin{aligned} \lambda_\phi : W_F &\rightarrow {}^L G \\ w &\mapsto (w, d_w), \end{aligned}$$

where $d : W_F \rightarrow \mathrm{SL}(2, \mathbb{C})$ was defined in Section 2.1. This defines

$$(34) \quad \begin{aligned} P({}^L G) &\rightarrow R({}^L G) \\ \phi &\mapsto \lambda_\phi. \end{aligned}$$

The function $\phi \mapsto \lambda_\phi$ is surjective but not, in general, injective. For any fixed $\lambda \in R({}^L G)$, set

$$P_\lambda({}^L G) := \{\phi \in P({}^L G) \mid \lambda_\phi = \lambda\}.$$

We write $\Phi_\lambda(G/F)$ for the set of $Z_{\widehat{G}}(\lambda)$ -conjugacy classes of Langlands parameters with infinitesimal parameter λ .

With reference to Section 2.7, for any quasi-split G over F , we set

$$\Pi_{\mathrm{pure}, \lambda}(G/F) := \bigcup_{\phi \in P_\lambda({}^L G)} \Pi_{\mathrm{pure}, \phi}(G/F),$$

with the union taken in $\Pi_{\mathrm{pure}}(G/F)$. Then, after choosing a representative for each class in $\Phi_\lambda({}^L G)$, we have

$$\Pi_{\mathrm{pure}, \lambda}(G/F) = \bigsqcup_{[\phi] \in \Phi_\lambda({}^L G)} \Pi_{\mathrm{pure}, \phi}(G/F).$$

Now the local Langlands correspondence for pure rational forms of G (cf. Section 2.7) provides a bijection

$$(35) \quad \Pi_{\mathrm{pure}, \lambda}(G/F) \leftrightarrow \{(\phi, \rho) \mid \phi \in P_\lambda({}^L G), \rho \in \mathrm{Irrep}(A_\phi)\} / \sim,$$

where the equivalence on pairs (ϕ, ρ) is defined by $Z_{\widehat{G}}(\lambda)$ -conjugation.

3.2. Vogan varieties. — Fix $\lambda \in R({}^L G)$. Define

$$(36) \quad H_\lambda := Z_{\widehat{G}}(\lambda) := \{g \in \widehat{G} \mid (g \rtimes 1)\lambda(w)(g \rtimes 1)^{-1} = \lambda(w), \forall w \in W_F\}$$

and

$$(37) \quad K_\lambda := Z_{\widehat{G}}(\lambda(I_F)) := \{g \in \widehat{G} \mid (g \rtimes 1)\lambda(w)(g \rtimes 1)^{-1} = \lambda(w), \forall w \in I_F\}.$$

The centraliser K_λ of $\lambda(I_F)$ in \widehat{G} consists of fixed points in \widehat{G} under a finite group of semisimple automorphisms of \widehat{G} , so K_λ is a reductive algebraic group. Since H_λ can be viewed as the group of fixed points in K_λ under the semisimple automorphism

$\text{Ad}(\lambda(\text{Fr}))$, then K_λ is also a reductive algebraic group. Neither H_λ nor K_λ is connected, in general.

Following [Vog93, (4.4)(e)], define

$$(38) \quad V_\lambda := V_\lambda({}^L G) := \{x \in \text{Lie } K_\lambda \mid \text{Ad}(\lambda(\text{Fr}))x = q_F x\},$$

called the *Vogan variety* for λ . Then H_λ acts on V_λ by conjugation.

Lemma 3.2.1. — *V_λ is a conical subvariety in the nilpotent cone of $\text{Lie } K_\lambda$.*

Proof. — Set $\mathfrak{k}_\lambda = \text{Lie } K_\lambda$. Decompose \mathfrak{k}_λ according to the eigenvalues of $\text{Ad}(\lambda(\text{Fr}))$:

$$(39) \quad \mathfrak{k}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{k}_\lambda(\nu).$$

Then, using the Lie bracket in \mathfrak{k}_λ , we have

$$(40) \quad [\cdot, \cdot] : \mathfrak{k}_\lambda(\nu_1) \times \mathfrak{k}_\lambda(\nu_2) \rightarrow \mathfrak{k}_\lambda(\nu_1 \nu_2).$$

It follows that all elements in V_λ are ad-nilpotent in $\hat{\mathfrak{g}}$. So it is enough to show that V_λ does not intersect the centre $\hat{\mathfrak{z}}$ of $\hat{\mathfrak{g}}$. Since the adjoint action of $\lambda(W_F)$ on $\hat{\mathfrak{z}}$ factors through a finite quotient of Γ_F , the $\text{Ad}(\lambda(\text{Fr}))$ -eigenvalues on $\hat{\mathfrak{z}}$ are all roots of unity. In particular, they can not be q_F , so V_λ does not intersect $\hat{\mathfrak{z}}$. This shows that all elements in V_λ are nilpotent in $\hat{\mathfrak{g}}$. It is clear from (38) that $V_\lambda({}^L G)$ is closed under scalar multiplication by \mathbb{C}^\times in $\mathfrak{k}_\lambda^{\text{nilp}}$. \square

With reference to decomposition of $\mathfrak{k}_\lambda = \text{Lie } K_\lambda$ in the proof of Lemma 3.2.1, observe that

$$\mathfrak{k}_\lambda(q_F) = V_\lambda \quad \text{and} \quad \mathfrak{k}_\lambda(1) = \text{Lie } H_\lambda.$$

Proposition 3.2.2. — *For each infinitesimal parameter $\lambda \in R({}^L G)$, the H_λ -equivariant function*

$$P_\lambda({}^L G) \longrightarrow V_\lambda({}^L G),$$

$$\phi \mapsto x_\phi := d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $\varphi := \phi^\circ|_{\text{SL}(2, \mathbb{C})} : \text{SL}(2, \mathbb{C}) \rightarrow \hat{G}$, is surjective. The fibre of $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ over any $x \in V_\lambda({}^L G)$ is a principal homogeneous space for the unipotent radical of $Z_{H_\lambda}(x)$. The induced map between the sets of H_λ -orbits

$$\Phi_\lambda({}^L G) \longrightarrow V_\lambda({}^L G)/H_\lambda,$$

$$[\phi] \mapsto C_\phi$$

is a bijection.

Proof. — Fix $x \in V_\lambda = \mathfrak{k}_\lambda(q_F)$. By Lemma 3.2.1, x is nilpotent. There exists an \mathfrak{sl}_2 -triple (x, y, h) in \mathfrak{k}_λ such that

$$(41) \quad x \in V_\lambda = \mathfrak{k}_\lambda(q_F) \quad \text{and} \quad z \in \mathfrak{h}_\lambda = \mathfrak{k}_\lambda(1) \quad \text{and} \quad y \in \mathfrak{k}_\lambda(q_F^{-1});$$

see, for example, [GR10, Lemma 2.1]. Let $\varphi : \mathrm{SL}(2, \mathbb{C}) \rightarrow K_\lambda$ be the homomorphism defined by

$$\mathrm{d}\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x, \quad \mathrm{d}\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h, \quad \mathrm{d}\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = y,$$

and define $\phi : W_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ by

$$\phi(w, g) = \varphi(g) \varphi(d_w^{-1}) \lambda(w).$$

Then $\phi \in P_\lambda({}^L G)$ and

$$\mathrm{d}(\phi^\circ|_{\mathrm{SL}(2, \mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{d}\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x.$$

This shows the map $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ is surjective.

Now, suppose that ϕ_1 is also mapped to x under the map $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ and set $\varphi_1 := \phi_1^\circ|_{\mathrm{SL}(2, \mathbb{C})}$. Then φ_1 determines an \mathfrak{sl}_2 -triple (x, y_1, z_1) in \mathfrak{k}_λ such that

$$z_1 \in \mathfrak{h}_\lambda = \mathfrak{k}_\lambda(1) \quad \text{and} \quad y_1 \in \mathfrak{k}_\lambda(q_F^{-1}).$$

The two \mathfrak{sl}_2 -triples (x, y, z) and (x, y_1, z_1) are conjugate by an element of $Z_{H_\lambda}(x)$; see, for example, the second part of [GR10, Lemma 2.1]. Thus, φ and φ_1 are conjugate under $Z_{H_\lambda}(x)$. We can also write ϕ_1 as

$$\phi_1(w, g) = \varphi_1(g) \varphi_1(d_w^{-1}) \lambda(w).$$

It is then clear that ϕ and ϕ_1 are also conjugate under $Z_{H_\lambda}(x)$. This shows that the map $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ induces a bijection between H_λ -orbits and also that the fibre above any $x \in V_\lambda$ is in bijection with $Z_{H_\lambda}(x)/Z_{H_\lambda}(\phi)$ for $\phi \mapsto x$ and that $Z_{H_\lambda}(x) = Z_{H_\lambda}(\phi)U$ where U is the unipotent radical of $Z_{H_\lambda}(x)$. \square

We remark that Proposition 3.2.2 is analogous to [ABV92, Proposition 6.17] for real groups. However, Proposition 3.2.2 might appear to contradict with [Vog93][Corollary 4.6]. The apparent discrepancy is explained by the two different incarnations of the Weil-Deligne group: we use $L_F = W_F \times \mathrm{SL}(2, \mathbb{C})$ while [Vog93] uses $W'_F = W_F \rtimes \mathbb{G}_{\mathrm{add}}(\mathbb{C})$ and we use pullback along $W_F \rightarrow L_F$ given by $w \mapsto (w, d_w)$ to define the infinitesimal parameter of a Langlands parameter while [Vog93] uses restriction of a parameter $W'_F \rightarrow {}^L G$ to W_F to define its infinitesimal parameter. We find L_F preferable to W'_F here because it stresses the analogy to the real groups case. However, there is a cost. In the optic of [Vog93], V_λ is exactly a moduli space for Langlands parameters $\phi : W'_F \rightarrow {}^L G$ with $\phi|_{W_F} = \lambda$, while in this article the map $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ from Langlands parameters $\phi : L_F \rightarrow {}^L G$ with $\lambda_\phi = \lambda$ to V_λ is not a bijection, as we saw in Proposition 3.2.2.

3.3. Parameter varieties. — Recall from Section 3.1 that elements of $\Lambda(G/F)$ are \widehat{G} -conjugacy class of elements of $R({}^L G)$. We will use the notation $[\lambda] \in \Lambda(G/F)$ for the class of $\lambda \in R({}^L G)$; then $[\lambda]$ is an *infinitesimal character* in the language of [Vog93]. Consider the variety

$$(42) \quad X_\lambda := X_\lambda({}^L G) := \widehat{G} \times_{H_\lambda} V_\lambda({}^L G).$$

Then $[\lambda] = [\lambda']$ implies $X_\lambda({}^L G) \cong X_{\lambda'}({}^L G)$. Set

$$P_{[\lambda]}({}^L G) := \{\phi \in P({}^L G) \mid \lambda_\phi = \text{Ad}(g)\lambda, \exists g \in \widehat{G}\}.$$

It follows immediately from Proposition 3.2.2 that the function

$$(43) \quad P_{[\lambda]}({}^L G) \rightarrow X_\lambda({}^L G),$$

induced from $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ is \widehat{G} -equivariant, surjective, and the fibre over any $x \in X_\lambda({}^L G)$ is a principal homogeneous space for the unipotent radical of $Z_{\widehat{G}}(x)$.

Let $\text{Hom}_{W_F}(W_F, {}^L G)$ be the set of homomorphisms that satisfy conditions (R.i) and (R.ii). Observe that

$$R({}^L G) = \{\lambda \in \text{Hom}_{W_F}(W_F, {}^L G) \mid \lambda(\text{Fr}) \in {}^L G_{\text{ss}}\}$$

where ${}^L G_{\text{ss}} \subseteq {}^L G$ denotes the set of semisimple elements in ${}^L G$. Now let $\text{Hom}_{W_F}(I_F, {}^L G)$ be the set of continuous homomorphisms that commute with the natural maps $I_F \rightarrow W_F$ and ${}^L G \rightarrow W_F$. As explained in [Pra, Section 10], the set $\text{Hom}_{W_F}(W_F, {}^L G)$ naturally carries the structure of (locally finite-type) variety over \mathbb{C} and its components are indexed by \widehat{G} -conjugacy classes of those $\phi_0 \in \text{Hom}_{W_F}(I_F, {}^L G)$ that lie in the image of $\text{Hom}_{W_F}(W_F, {}^L G) \rightarrow \text{Hom}_{W_F}(I_F, {}^L G)$ given by restriction. We remark that \widehat{G} -orbits in $\text{Hom}_{W_F}(W_F, {}^L G)$ are closed subvarieties.

Now consider the (locally finite-type) variety

$$X({}^L G) := \{(\lambda, x) \in \text{Hom}_{W_F}(W_F, {}^L G) \times \text{Lie } \widehat{G} \mid x \in V_\lambda({}^L G)\}.$$

This (locally finite-type) variety comes equipped with morphisms

$$\begin{array}{ccccc} X({}^L G) & \rightarrow & \text{Hom}_{W_F}(W_F, {}^L G) & \rightarrow & \text{Hom}_{W_F}(I_F, {}^L G) \\ (\lambda, x) & \mapsto & \lambda & \mapsto & \lambda|_{I_F}. \end{array}$$

The components of $X({}^L G)$ are again indexed by the \widehat{G} -conjugacy classes of those $\phi_0 \in \text{Hom}_{W_F}(I_F, {}^L G)$ that lie in the image of $\text{Hom}_{W_F}(W_F, {}^L G) \rightarrow \text{Hom}_{W_F}(I_F, {}^L G)$. The fibre of $X({}^L G) \rightarrow \text{Hom}_{W_F}(W_F, {}^L G)$ above $\lambda \in R({}^L G) \subseteq X({}^L G)$ is precisely the affine variety $X_\lambda({}^L G)$ defined in (42).

Now, with reference to the definition of λ_ϕ from (34) and the definition of x_ϕ in Proposition 3.2.2, consider the map

$$(44) \quad \begin{array}{ccc} P({}^L G) & \rightarrow & X({}^L G) \\ \phi & \mapsto & (\lambda_\phi, x_\phi). \end{array}$$

It follows from Proposition 3.2.2 that the image is $\{(\lambda, x) \in X({}^L G) \mid \lambda \in R({}^L G)\}$ and the fibre of $P({}^L G) \rightarrow X({}^L G)$ above any (λ, x) in its image is a principal homogeneous space for the unipotent radical of $Z_{\widehat{G}}(x)$, and moreover that $P({}^L G) \rightarrow X({}^L G)$ induces a bijection

$$\begin{aligned} \Phi({}^L G) &\longrightarrow X({}^L G)/\widehat{G}, \\ [\phi] &\mapsto S_\phi. \end{aligned}$$

Though the map (44) is neither injective nor surjective, in general, and though $X({}^L G)$ is not of finite type over \mathbb{C} , in general, we refer to $X({}^L G)$ as the *parameter variety* for G .

We note that $X(^L G)$ is stratified into \widehat{G} -orbit varieties, locally closed in $X(^L G)$; this stratification is not finite, in general, but it is closure-finite. For each \widehat{G} -orbit $S \subseteq X(^L G)$, there is some $\lambda \in \text{Hom}_{W_F}(W_F, ^L G)$ such that $S \subseteq X_\lambda(^L G)$. Then \bar{S} , the closure of S in $X(^L G)$, is also contained in $X_\lambda(^L G)$. It is essentially for this reason that this article is concerned with the affine varieties $X_\lambda(^L G)$, for $[\lambda] \in \Lambda(G/F)$, rather than the full parameter variety $X(^L G)$.

3.4. Equivariant perverse sheaves. — The definitive reference for perverse sheaves is [BBD82], and we will use notation from that paper here, but equivariant perverse sheaves do not appear in [BBD82], so we now briefly describe that category and some properties that will be important to us. Our treatment is consistent with [BL94, Section 5].

Let $m : H \times V \rightarrow V$ be a group action in the category of algebraic varieties. So, in particular, H is an algebraic group, but need not be connected. Consider the morphisms

$$H \times H \times V \begin{array}{c} \xrightarrow{m_1, m_2, m_3} \\ \xrightarrow{\quad \quad \quad} \end{array} H \times V \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{m} \\ \xleftarrow{m_0} \end{array} V$$

where $m_0 : H \times V \rightarrow V$ is projection, $s : V \rightarrow H \times V$ is defined by $s(x) = (1, x)$ and $m_1, m_2, m_3 : H \times H \times V \rightarrow H \times V$ are defined by

$$\begin{aligned} m_1(h_1, h_2, x) &= (h_1 h_2, x) \\ m_2(h_1, h_2, x) &= (h_1, m(h_2, x)) \\ m_3(h_1, h_2, x) &= (h_2, x). \end{aligned}$$

These are all smooth morphisms. An object in $\text{Per}_H(V)$ is a pair (\mathcal{A}, α) where $\mathcal{A} \in \text{Per}(V)$ and

$$(45) \quad \alpha : m^*[\dim H]\mathcal{A} \rightarrow m_0^*[\dim H]\mathcal{A}$$

is an isomorphism in $\text{Per}(H \times V)$ such that

$$(46) \quad s^*(\alpha) = \text{id}_{\mathcal{A}}[\dim H]$$

and such that the following diagram in $\text{Per}(H \times H \times V)$, which makes implicit use of [BBD82, 1.3.17] commutes:

$$(47) \quad \begin{array}{ccc} m_2^*[\dim H]m^*[\dim H]\mathcal{A} & \xrightarrow{m_2^*[\dim H](\alpha)} & m_2^*[\dim H]m_0^*[\dim H]\mathcal{A} \\ \downarrow m \circ m_1 = m \circ m_2 & & m_0 \circ m_2 = m \circ m_3 \downarrow \\ m_1^*[\dim H]m^*[\dim H]\mathcal{A} & & m_3^*[\dim H]m^*[\dim H]\mathcal{A} \\ \downarrow m_1^*[\dim H](\alpha) & & m_3^*[\dim H](\alpha) \downarrow \\ m_1^*[\dim H]m_0^*[\dim H]\mathcal{A} & \xleftarrow{m_0 \circ m_3 = m_0 \circ m_1} & m_3^*[\dim H]m_0^*[\dim H]\mathcal{A}. \end{array}$$

We remark that ${}^p H^{\dim H} m^* = m^*[\dim H]$ on $\text{Per}(V)$ and ${}^p H^{\dim H} m_i^* = m_i^*[\dim H]$ on $\text{Per}(H \times V)$ for $i = 1, 2, 3$; see [BBD82, 4.2.4]. This does *not* require connected H .

Morphisms of H -equivariant perverse sheaves $(\mathcal{A}, \alpha) \rightarrow (\mathcal{B}, \beta)$ are morphisms of perverse sheaves $\phi : A \rightarrow B$ for which the diagram

$$(48) \quad \begin{array}{ccc} m^*[\dim H]A & \xrightarrow{m^*[\dim H](\phi)} & m^*[\dim H]B \\ \alpha \downarrow & & \downarrow \beta \\ m_0^*[\dim H]A & \xrightarrow{m_0^*[\dim H](\phi)} & m_0^*[\dim H]B \end{array}$$

commutes. This defines $\text{Per}_H(V)$, the category of H -equivariant perverse sheaves on V .

The category $\text{Per}_H(V)$ comes equipped with the forgetful functor

$$\text{Per}_H(V) \rightarrow \text{Per}(V)$$

trivial on morphisms and given on objects by $(\mathcal{A}, \alpha) \rightarrow \mathcal{A}$. This is a special case of a more general construction called equivariant pullback. Let $m : H \times V \rightarrow V$ and $m' : H' \times V' \rightarrow V'$ be actions. Let $u : H' \rightarrow H$ be a morphism in the category of algebraic groups and suppose H' acts on V' and H acts on V . A morphism $f : V' \rightarrow V$ is *equivariant* (with respect to u) if

$$\begin{array}{ccc} H' \times V' & \xrightarrow{m'} & V' \\ u \times f \downarrow & & \downarrow f \\ H \times V & \xrightarrow{m} & V \end{array}$$

commutes. Then for every $i \in \mathbb{Z}$ there is a functor ${}^p\text{H}_u^i f^* : \text{Per}_H(V) \rightarrow \text{Per}_{H'}(V')$ making

$$\begin{array}{ccc} \text{Per}_{H'}(V') & \xleftarrow{{}^p\text{H}_u^i f^*} & \text{Per}_H(V) \\ \text{forget} \downarrow & & \downarrow \text{forget} \\ \text{Per}(V') & \xleftarrow{{}^p\text{H}^i f^*} & \text{Per}(V) \end{array}$$

commute; we call this equivariant pullback. The forgetful functor above is just ${}^p\text{H}_1^0 \text{id}_V^*$, where $u : 1 \rightarrow H$.

The category $\text{Per}_H(V)$ also comes equipped with the forgetful functor

$$\text{Per}_H(V) \rightarrow \text{Per}_{H^0}(V)$$

where H^0 is the identity component of H . The category $\text{Per}_{H^0}(V)$ is easier to study than $\text{Per}_H(V)$, since the functor $\text{Per}_{H^0}(V) \rightarrow \text{Per}(V)$ is faithful, which is generally not the case for $\text{Per}_H(V) \rightarrow \text{Per}(V)$ when H is not connected. The following lemma shows how $\text{Per}_H(V)$ is related to $\text{Per}_{H^0}(V)$.

Lemma 3.4.1. — *Let $m : H \times V \rightarrow V$ be a group action in the category of algebraic varieties. Suppose V is smooth and connected. We have a sequence of functors*

$$\text{Rep}(\pi_0(H)) \xrightarrow{E \mapsto E_V[\dim V]} \text{Per}_H(V) \xleftarrow[\pi_*]{\text{forget: } \mathcal{P} \mapsto \mathcal{P}_0} \text{Per}_{H^0}(V)$$

such that:

(a) for every $E \in \text{Rep}(\pi_0(H))$, $(E_V[\dim V])_0 \cong \mathbb{1}_V^{\dim E}[\dim V]$;

- (b) the functor $\mathrm{Rep}(\pi_0(H)) \rightarrow \mathrm{Per}_H(V)$ is fully faithful and its essential image is the category of perverse local systems $\mathcal{L}[\dim V] \in \mathrm{Per}_H(V)$ such that $(\mathcal{L}[\dim V])_0 \cong \mathbb{1}_V^{\dim \mathcal{L}}[\dim V]$;
- (c) the forgetful functor $\mathrm{Per}_H(V) \rightarrow \mathrm{Per}_{H^0}(V)$ is exact and admits an adjoint $\pi_* : \mathrm{Per}_{H^0}(V) \rightarrow \mathrm{Per}_H(V)$, both left and right;
- (d) every $\mathcal{P} \in \mathrm{Per}_H(V)$ is a summand of $\pi_* \mathcal{P}_0$.

Proof. — The identity $\mathrm{id}_V : V \rightarrow V$ is equivariant with respect to the inclusion $u : H^0 \rightarrow H$ of the identity component of H . Consider the functor

$${}^p\mathrm{H}_u^0 \mathrm{id}_V^* : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_{H^0}(V).$$

The trivial map $0 : V \rightarrow 0$ is equivariant with respect to the quotient $\pi_0 : H \rightarrow \pi_0(H)$.

$$\begin{array}{ccc} H \times V & \longrightarrow & V \\ \downarrow & & \downarrow \\ \pi_0(H) \times 0 & \longrightarrow & 0 \end{array}$$

Consider the functor

$${}^p\mathrm{H}_{\pi_0}^{\dim H} 0^* : \mathrm{Per}_{\pi_0(H)}(0) \rightarrow \mathrm{Per}_H(V).$$

Then

$$({}^p\mathrm{H}_{\pi_0}^{\dim H} 0^*) ({}^p\mathrm{H}_u^0 \mathrm{id}_V^*) \cong {}^p\mathrm{H}_0^{\dim H} 0^*$$

and we have a sequence of functors

$$\mathrm{Per}_{\pi_0(H)}(0) \longrightarrow \mathrm{Per}_H(V) \xrightarrow{\text{forget}} \mathrm{Per}_{H^0}(V)$$

The tensor category $\mathrm{Per}_{\pi_0(H)}(0)$ is equivalent to $\mathrm{Rep}(\pi_0(H))$, the category of representations of the finite group $\pi_0(H)$. Property (a) now follows from the canonical isomorphism of functors above.

Since V is smooth, the functor $\mathrm{Rep}(\pi_0(H)) \rightarrow \mathrm{Per}_H(V)$ is given explicitly by $E \mapsto E_V[\dim V]$; this functor is full and faithful [BBD82, Corollaire 4.2.6.2] from which we also find the adjoint functors $\mathrm{Per}_H(V) \rightarrow \mathrm{Rep}(\pi_0(H))$ and Property (b). Connectedness of V plays a role here.

To see Property (c), set $\tilde{V} = H \times_{H^0} V$ and consider the closed embedding $i : V \rightarrow \tilde{V}$ given by $i(x) = [1, x]_{H^0}$. By descent, equivariant pullback

$${}^p\mathrm{H}_u^0 i^* : \mathrm{Per}_H(\tilde{V}) \rightarrow \mathrm{Per}_{H^0}(V)$$

is an equivalence. Now consider the morphism

$$\begin{aligned} c : \tilde{V} &\rightarrow V \\ [h, x]_{H^0} &\mapsto h \cdot x. \end{aligned}$$

Then $c : \tilde{V} \rightarrow V$ is an H -equivariant finite étale cover with group $\pi_0(H) = H/H^0$. In fact, $\tilde{V} \cong V \times H/H^0$ and c is simply the composition of this isomorphism with projection $V \times H/H^0 \rightarrow V$. Since c is proper and semismall, the adjoint to pullback

$${}^p\mathrm{H}^0 c^* : \mathrm{Per}(V) \rightarrow \mathrm{Per}(\tilde{V})$$

takes perverse sheaves to perverse sheaves,

$${}^p\mathrm{H}^0 c_* : \mathrm{Per}(\tilde{V}) \rightarrow \mathrm{Per}(V)$$

and coincides with ${}^p\mathrm{H}^0 c_!$; see also [BBD82, Corollaire 2.2.6]. To see that the adjoint extends to a functor of equivariant perverse sheaves, define

$${}^p\mathrm{H}_H^0 c_* : \mathrm{Per}_H(\tilde{V}) \rightarrow \mathrm{Per}_H(V)$$

as follows. On objects, ${}^p\mathrm{H}_H^0 c_*(\tilde{\mathcal{A}}, \tilde{\alpha}) = (\mathcal{A}, \alpha)$ with $\mathcal{A} = {}^p\mathrm{H}^0 c_* \tilde{\mathcal{A}}$ while the isomorphism $\alpha : {}^p\mathrm{H}^{\dim H} m^* \mathcal{A} \rightarrow {}^p\mathrm{H}^{\dim H} m_0^* \mathcal{A}$ in $\mathrm{Per}(H \times V)$ is defined by the following diagram of isomorphisms.

$$\begin{array}{ccc} {}^p\mathrm{H}^{\dim H} m^* \mathcal{A} & \xrightarrow{\alpha} & {}^p\mathrm{H}^{\dim H} m_0^* \mathcal{A} \\ \parallel & & \parallel \\ {}^p\mathrm{H}^{\dim H} m^* ({}^p\mathrm{H}^0 c_* \tilde{\mathcal{A}}) & & {}^p\mathrm{H}^{\dim H} m_0^* ({}^p\mathrm{H}^0 c_* \tilde{\mathcal{A}}) \\ \downarrow \text{smooth base change} & & \downarrow \text{smooth base change} \\ {}^p\mathrm{H}^0(\mathrm{id}_H \times c)_* {}^p\mathrm{H}^{\dim H}(\tilde{m})^* A & \xrightarrow{{}^p\mathrm{H}^0(\mathrm{id}_H \times c)_*(\tilde{\alpha})} & {}^p\mathrm{H}^0(\mathrm{id}_H \times c)_* {}^p\mathrm{H}^{\dim H}(\tilde{m}_0)^* A \end{array}$$

It is straightforward to verify that α satisfies (46) and (47) as they apply here and also that if $\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}$ is a map in $\mathrm{Per}_H(\tilde{V})$ then ${}^p\mathrm{H}^i c_!(\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}})$ satisfies condition (48), so is a map in $\mathrm{Per}_H(V)$. By this definition of ${}^p\mathrm{H}_H^i c_* : \mathrm{Per}_H(\tilde{V}) \rightarrow \mathrm{Per}_H(V)$, it follows immediately that the diagram

$$\begin{array}{ccc} \mathrm{Per}_H(\tilde{V}) & \xrightarrow{{}^p\mathrm{H}_H^0 c_*} & \mathrm{Per}_H(V) \\ \text{forget} \downarrow & & \downarrow \text{forget} \\ \mathrm{Per}(\tilde{V}) & \xrightarrow{c_* = {}^p\mathrm{H}^0 c_*} & \mathrm{Per}(V) \end{array}$$

commutes. Now, we define the adjoint $\pi_* : \mathrm{Per}_{H^0}(V) \rightarrow \mathrm{Per}_H(V)$ by the following diagram

$$\begin{array}{ccc} \mathrm{Per}_{H^0}(V) & \xrightarrow{\pi_*} & \mathrm{Per}_H(V) \\ \swarrow \text{equiv.} & & \searrow {}^p\mathrm{H}_H^0 c_* \\ & \mathrm{Per}_H(\tilde{V}) & \end{array}$$

This shows Property (c).

Property (d) follows from the Decomposition Theorem applied to $c : \tilde{V} \rightarrow V$. \square

3.5. Equivariant perverse sheaves on parameter varieties. — Our fundamental object of study is the category $\mathrm{Per}_{\hat{G}}(X_\lambda)$ of \hat{G} -equivariant perverse sheaves on $X({}^L G)$, for fixed $[\lambda] \in \Lambda(G/F)$. Consider the closed embedding

$$\begin{array}{ccc} V_\lambda & \rightarrow & X_\lambda \\ x & \mapsto & [1, x]_{H_\lambda}. \end{array}$$

By a simple application of descent, the functor obtained by equivariant pullback along $V_\lambda \rightarrow X_\lambda$,

$$\mathrm{Per}_{H_\lambda}(V_\lambda) \leftarrow \mathrm{Per}_{\widehat{G}}(X_\lambda),$$

is an equivalence. Consequently, it may equally be said that our fundamental object of study is the category $\mathrm{Per}_{H_\lambda}(V_\lambda)$ of H_λ -equivariant perverse sheaves on V_λ .

Now define

$$(49) \quad \tilde{X}_\lambda := \widehat{G} \times_{H_\lambda^0} V_\lambda.$$

Then

$$\begin{aligned} V_\lambda &\rightarrow \tilde{X}_\lambda \\ x &\mapsto [1, x]_{H_\lambda^0} \end{aligned}$$

induces an equivalence

$$\mathrm{Per}_{\widehat{G}}(\tilde{X}_\lambda) \rightarrow \mathrm{Per}_{H_\lambda^0}(V_\lambda).$$

Define

$$\begin{aligned} c_\lambda : \tilde{X}_\lambda &\rightarrow X_\lambda \\ [h, x]_{H_\lambda^0} &\mapsto [h, x]_{H_\lambda}. \end{aligned}$$

Arguing as in Section 3.4, it follows that there is a sequence of exact functors

$$\begin{array}{ccccc} \mathrm{Rep}(A_\lambda) & \xrightarrow{E \mapsto E_{X_\lambda}[\dim X_\lambda]} & \mathrm{Per}_{\widehat{G}}(X_\lambda) & \begin{array}{c} \xleftarrow{(c_\lambda)^*} \\ \xrightarrow{(c_\lambda)_*} \end{array} & \mathrm{Per}_{\widehat{G}}(\tilde{X}_\lambda) \\ & & \downarrow \text{equiv} & & \downarrow \text{equiv} \\ & & \mathrm{Per}_{H_\lambda}(V_\lambda) & & \mathrm{Per}_{H_\lambda^0}(V_\lambda) \end{array}$$

enjoying the properties of Lemma 3.4.1.

3.6. Langlands component groups are equivariant fundamental groups. —

Every simple object in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ takes the form $\mathcal{IC}(C, \mathcal{L})$, where C is an H_λ -orbit in V_λ and \mathcal{L} is a simple equivariant local system on C . Thus, simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ are parametrized by pairs (C, ρ) where C is an H_λ -orbit in V_λ and ρ is an isomorphism class of irreducible representations of the equivariant fundamental group A_C of C . To calculate that group, we may pick a base point $x \in C$ so

$$(50) \quad A_C \cong \pi_1(C, x)_{H_\lambda^0}.$$

We are left with a canonical bijection:

$$(51) \quad \mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}} \leftrightarrow \{(C, \gamma) \mid H_\lambda\text{-orbit } C \subseteq V_\lambda, \rho \in \mathrm{Irrep}(A_C)\}.$$

Lemma 3.6.1. — *For any Langlands parameter $\phi : L_F \rightarrow {}^L G$,*

$$A_{C_\phi} = A_\phi,$$

where C_ϕ is the H_{λ_ϕ} -orbit of x_ϕ in V_{λ_ϕ} ; see Proposition 3.2.2.

Proof. — Recall from Section 2.3 that the component group for a Langlands parameter ϕ is given by $A_\phi = \pi_0(Z_{\widehat{G}}(\phi)) = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^0$. Since $\lambda_\phi(W_F) \subseteq \phi(L_F)$, $A_\phi = \pi_0(Z_{H_{\lambda_\phi}}(\phi))$. On the other hand, the equivariant fundamental group of C_ϕ is $\pi_1(C_\phi, x_\phi)_{H_{\lambda_\phi}} = \pi_0(Z_{H_{\lambda_\phi}}(x_\phi))$. From the proof of Proposition 3.2.2 we see that

$Z_{H_{\lambda_\phi}}(x_\phi) = Z_{H_{\lambda_\phi}}(\phi)U$, where U is a connected unipotent group. It follows that

$$\pi_0(Z_{H_{\lambda_\phi}}(x_\phi)) = \pi_0(Z_{H_{\lambda_\phi}}(\phi)U) = \pi_0(Z_{H_{\lambda_\phi}}(\phi)),$$

which concludes the proof. \square

The following proposition is one of the fundamental ideas in [Vog93]. Because our set up is slightly different, however, we include a proof here.

Proposition 3.6.2. — *Suppose G is quasi-split. The local Langlands correspondence for pure rational forms determines a bijection between the set of isomorphism classes of simple objects in $\text{Per}_{H_\lambda}(V_\lambda)$ and those of $\Pi_{\text{pure},\lambda}(G/F)$ as defined in Section 3.1:*

$$\text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}} \leftrightarrow \Pi_{\text{pure},\lambda}(G/F).$$

Proof. — We have already seen (35) that the local Langlands correspondence for pure rational forms gives a bijection between $\Pi_{\text{pure},\lambda}(G/F)$ and

$$\{([\phi], \epsilon) \mid [\phi] \in \Phi_\lambda({}^L G), \epsilon \in \text{Irrep}(A_\phi)\}$$

Proposition 3.2.2 gives a canonical bijection between $\Phi_\lambda({}^L G)$ and the set of H_λ -orbits in V_λ . When $C \leftrightarrow [\phi]$ under this bijection, Lemma 3.6.1, gives a bijection between $\text{Irrep}(A_C)$ and $\text{Irrep}(A_\phi)$. \square

We introduce some convenient notation for use below. For $[\pi, \delta] \in \Pi_\lambda(G/F)$, let $\mathcal{P}(\pi, \delta) = \mathcal{IC}(C_{\pi,\delta}, \mathcal{L}_{\pi,\delta})$ be a simple perverse sheaf in the isomorphism class determined by $[\pi, \delta]$ using Proposition 3.6.2:

$$\begin{aligned} \Pi_{\text{pure},\lambda}(G/F) &\rightarrow \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}} \\ [\pi, \delta] &\mapsto \mathcal{P}(\pi, \delta) = \mathcal{IC}(C_{\pi,\delta}, \mathcal{L}_{\pi,\delta}) \end{aligned}$$

Conversely, for a simple perverse sheaf $\mathcal{P} = \mathcal{IC}(C, \mathcal{L})$ in $\text{Per}_{H_\lambda}(V_\lambda)$, let $\chi_{\mathcal{P}}$ be the character of $\pi_0(Z(\widehat{G})^{\Gamma_F})$ obtained by pullback along

$$(52) \quad \pi_0(Z(\widehat{G})^{\Gamma_F}) \rightarrow \pi_0(Z_{\widehat{G}}(x))$$

from the representation of $\pi_0(Z_{\widehat{G}}(x))$ determined by the choice of a base point $x \in C$ and the equivariant local system \mathcal{L} on C . Let $\delta_{\mathcal{P}} \in Z^1(F, G)$ be a pure rational form of G representing the class determined by $\chi_{\mathcal{P}}$ under the Kottwitz isomorphism. Let $\pi_{\mathcal{P}}$ be an admissible representation of $G_{\delta_{\mathcal{P}}}(F)$ such that $[\pi_{\mathcal{P}}, \delta_{\mathcal{P}}]$ matches \mathcal{P} under Proposition 3.6.2:

$$\begin{aligned} \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}} &\rightarrow \Pi_{\text{pure},\lambda}(G/F) \\ \mathcal{P} &\mapsto [\pi_{\mathcal{P}}, \delta_{\mathcal{P}}]. \end{aligned}$$

4. Reduction to unramified parameters

Let G be an arbitrary connected reductive algebraic group over a p -adic field F .

4.1. Unramification. — In this section we show that the study of $\text{Per}_{\widehat{G}}(X_\lambda)$ may be reduced to the study of $\text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{\text{nr}}})$ for a split connected reductive group G_λ and an unramified infinitesimal parameter $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$. Moreover, we show how

the tools developed in [Lus95c] may be brought to bear on $\text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}})$. The group G_λ that appears in Theorem 4.1.1 is sometimes an endoscopic group for G , but not in general; nonetheless, the principle of functoriality applies here through a map of L -groups $r_\lambda : {}^L G_\lambda \rightarrow {}^L G$.

Theorem 4.1.1. — *Let $\lambda : W_F \rightarrow {}^L G$ be an infinitesimal parameter.*

- (a) *There is a connected reductive group G_λ , split over F , and an infinitesimal parameter $\lambda_{nr} : W_F \rightarrow {}^L G_\lambda$ for G_λ , trivial on I_F , and an inclusion of L -groups $r_\lambda : {}^L G_\lambda \rightarrow {}^L G$ such that the following diagram commutes*

$$\begin{array}{ccc} W_F & \xrightarrow{\lambda} & {}^L G \\ \uparrow & & \uparrow r_\lambda \\ W_F & \xrightarrow{\lambda_{nr}} & {}^L G_\lambda, \end{array}$$

where $W_F \rightarrow W_F$ is trivial on I_F and $\text{Fr} \mapsto \text{Fr}$ (chosen in Section 2.1).

- (b) *By equivariant pullback, the inclusion of L -groups $r_\lambda : {}^L G_\lambda \rightarrow {}^L G$ defines an equivalence*

$$\text{Per}_{\widehat{G}}(\tilde{X}_\lambda) \rightarrow \text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}})$$

where \tilde{X}_λ is defined in Section 3.5, (49).

- (c) *There is a sequence of exact functors*

$$\text{Rep}(A_\lambda) \xrightarrow{E \mapsto E_{X_\lambda}[\dim X_\lambda]} \text{Per}_{\widehat{G}}(X_\lambda) \xrightleftharpoons[(c_\lambda)_*]{(c_\lambda)^*} \text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}})$$

enjoying the properties of Lemma 3.4.1, where A_λ is defined by (33).

- (d) *There is a connected complex reductive algebraic group M_λ , a co-character $\iota : \mathbb{G}_m \rightarrow M_\lambda$ and an integer n such that*

$$\text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{nr}}) \equiv \text{Per}_{M_\lambda^\iota}(\mathfrak{m}_{\lambda,n}),$$

where $\mathfrak{m}_{\lambda,n}$ is the weight- n space of $\text{Ad}(\iota)$ acting on $\mathfrak{m}_\lambda = \text{Lie } M_\lambda$.

The proof of Theorem 4.1.1 will be given in Section 4.5.

4.2. Elliptic and hyperbolic semisimple elements in L -groups. — Recall that a semisimple element x of a complex reductive group H is called *hyperbolic* (resp. *elliptic*) if for every torus $D \subset H$ containing x and every rational character $\chi : D \rightarrow \mathbb{G}_m(\mathbb{C})$ of D , $\chi(x)$ is a positive real number (resp. $\chi(x)$ has complex norm 1). An arbitrary semisimple element can be uniquely decomposed as a commuting product of hyperbolic and elliptic semisimple elements. An element commutes with x if and only if it commutes with its hyperbolic and elliptic parts separately.

Recall that an element $g \in {}^L G$ is semisimple if $\text{Int}(g)$ is a semisimple automorphism of \widehat{G} . Then $g = f \rtimes w \in {}^L G$ is semisimple if and only if $f' \in \widehat{G}$ is semisimple where $(f \rtimes w)^N = f' \rtimes w^N$ and w^N acts trivially on \widehat{G} .

The hyperbolic and elliptic parts of a semisimple $g = f \rtimes \text{Fr} \in {}^L G$ are defined as follows. Let N be as above, so $(f \rtimes \text{Fr})^N = f' \rtimes \text{Fr}^N$ and Fr^N acts trivially on \widehat{G} . Then $f' \in \widehat{G}$ is semisimple. Let $s' \in \widehat{G}$ be the hyperbolic part of f' and let $t' \in \widehat{G}$ be

the elliptic part of f' . Let s be the unique hyperbolic element of \widehat{G} such that $s^N = s'$. It is clear that s is independent of N . Set $t = s^{-1}f$. We call $s \rtimes 1$ the *hyperbolic part* of $f \rtimes \text{Fr}$ and $t \rtimes \text{Fr}$ the *elliptic part* of $f \rtimes w$. Then $\text{Ad}(s) \in \text{Aut}(\widehat{\mathfrak{g}})$ is the hyperbolic part of the semisimple automorphism $\text{Ad}(f \rtimes \text{Fr}) \in \text{Aut}(\widehat{\mathfrak{g}})$ and $\text{Ad}(t \rtimes \text{Fr}) \in \text{Aut}(\widehat{\mathfrak{g}})$ is the elliptic part of the semisimple automorphism $\text{Ad}(f \rtimes \text{Fr}) \in \text{Aut}(\widehat{\mathfrak{g}})$. Moreover, ${}^{\text{Fr}}s = t^{-1}st$, so

$$(s \rtimes 1)(t \rtimes \text{Fr}) = (t \rtimes \text{Fr})(s \rtimes 1).$$

Lemma 4.2.1. — *Write $\lambda(\text{Fr}) = f_\lambda \rtimes \text{Fr}$; let $s_\lambda \rtimes 1$ be the hyperbolic part of $\lambda(\text{Fr})$ and let $t_\lambda \rtimes \text{Fr}$ be the elliptic part of $\lambda(\text{Fr})$. Then $s_\lambda \in H_\lambda^0$ and K_λ is normalized by $f_\lambda \rtimes \text{Fr}$ and by $t_\lambda \rtimes \text{Fr}$.*

Proof. — let I'_F be the kernel of $\rho : \Gamma_F \rightarrow \text{Aut}(\widehat{G})$ restricted to I_F . Then I'_F is an open subgroup of I_F and I'_F is normalized by Fr^N in W_F , with N as above. Set $I_F^0 = \lambda^{-1}(1 \rtimes I'_F) \subseteq I'_F$. By continuity of λ , I_F^0 is an open subgroup of I_F . Then $\lambda(\text{Fr}^N)$ normalises $\lambda(I_F^0)$. Since $\lambda(\text{Fr}^N)$ also normalises $\lambda(I_F)$, we see $\lambda(\text{Fr}^N)$ acts on the finite group $\lambda(I_F)/\lambda(I_F^0)$. In particular, replacing N by a larger integer if necessary, it follows that $\lambda(\text{Fr}^N)$ acts on $\lambda(I_F)/\lambda(I_F^0)$ trivially.

Recall the notation $\lambda(\text{Fr}) = f_\lambda \rtimes \text{Fr}$ and $\lambda(\text{Fr}^N) = f' \rtimes \text{Fr}^N$. We now show $f' \in Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda$. For any $h \rtimes w \in \lambda(I_F)$,

$$\lambda(\text{Fr}^N)(h \rtimes w)(\lambda(\text{Fr}^N))^{-1} = h \rtimes ww'$$

for some $w' \in I_F^0$. Since $\lambda(\text{Fr}^N) = f' \rtimes \text{Fr}^N = (1 \rtimes \text{Fr}^N)(f' \rtimes 1)$, we get

$$\text{Fr}^N f'(h \rtimes w) f'^{-1} \text{Fr}^{-N} = h \rtimes ww'.$$

This implies

$$f'hw(f'^{-1}) \rtimes w = \text{Fr}^{-N}(h \rtimes ww')\text{Fr}^N = h \rtimes \text{Fr}^{-N}ww'\text{Fr}^N.$$

Therefore, $f'hw(f'^{-1}) = h$ and $w = \text{Fr}^{-N}ww'\text{Fr}^N$. From the first equality, we can conclude $f'(h \rtimes w)f'^{-1} = h \rtimes w$. Hence $f' \in Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda$.

Since some power of f' will lie in $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$, replacing N by a larger integer if necessary, we may conclude that f' actually belongs to $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$. In particular, we can take both s' and t' in K_λ^0 .

Since $\lambda(\text{Fr}^N) = \lambda(\text{Fr})^{-1}\lambda(\text{Fr}^N)\lambda(\text{Fr})$, we have

$$f' \rtimes \text{Fr}^N = (f_\lambda \rtimes \text{Fr})^{-1}(f' \rtimes \text{Fr}^N)(f_\lambda \rtimes \text{Fr}) = ((f_\lambda \rtimes \text{Fr})^{-1}f'(f_\lambda \rtimes \text{Fr})) \rtimes \text{Fr}^N.$$

Thus, $f' = \lambda(\text{Fr})^{-1}f'\lambda(\text{Fr})$. Since $\lambda(\text{Fr})$ normalises $Z_{\widehat{G}}(\lambda(I_F))^0 = K_\lambda^0$, we have

$$f' = \lambda(\text{Fr})^{-1}f'\lambda(\text{Fr}) = (\lambda(\text{Fr})^{-1}s'\lambda(\text{Fr}))(\lambda(\text{Fr})^{-1}t'\lambda(\text{Fr})),$$

where, as above, s' is the hyperbolic part of f' and t' is the elliptic part of f' . Since the decomposition of a semisimple element of \widehat{G} into hyperbolic and elliptic parts is unique, we have

$$s' = \lambda(\text{Fr})^{-1}s'\lambda(\text{Fr}) \quad \text{and} \quad t' = \lambda(\text{Fr})^{-1}t'\lambda(\text{Fr}).$$

In particular, it now follows that $s' \in Z_{\widehat{G}}(\lambda)^0 = H_\lambda^0$. Since $s_\lambda^N = s'$, it follows that $s_\lambda \in Z_{\widehat{G}}(\lambda)^0 = H_\lambda^0$, also.

The Frobenius element Fr normalises I_F , so $\lambda(\text{Fr}) = f_\lambda \rtimes \text{Fr}$ normalises $\lambda(I_F)$ and hence normalises K_λ as well. Since $s_\lambda \in H_\lambda^0 = Z_{\widehat{G}}(\lambda)^0 \subseteq Z_{\widehat{G}}(\lambda(I_F)) = K_\lambda^0$, it follows now that s_λ normalises K_λ ; likewise, $t_\lambda \rtimes \text{Fr}$ normalises K_λ . \square

4.3. Construction of the unramified parameter. — Define

$$(53) \quad J_\lambda := Z_{\widehat{G}}(\lambda(I_F)) \cap Z_{\widehat{G}}(t_\lambda \rtimes \text{Fr}) = Z_{K_\lambda}(t_\lambda \rtimes \text{Fr}).$$

Lemma 4.2.1 shows that J_λ is a complex reductive algebraic group. Recall the definition of s_λ and t_λ from Lemma 4.2.1. It follows from Section 4.2 that $s_\lambda \in J_\lambda^0$ and t_λ normalises J_λ^0 .

We now have the following complex reductive groups attached to $\lambda \in R({}^L G)$:

$$H_\lambda \subseteq J_\lambda \subseteq K_\lambda \subseteq \widehat{G}.$$

Let G_λ be the split connected reductive algebraic group over F so that

$$(54) \quad {}^L G_\lambda = J_\lambda^0 \times W_F.$$

Define

$$(55) \quad r_\lambda : {}^L G_\lambda \rightarrow {}^L G \quad \text{by} \quad h \times 1 \mapsto h \rtimes 1 \quad \text{and} \quad 1 \times \text{Fr} \mapsto t_\lambda \rtimes \text{Fr}.$$

Then $r_\lambda : {}^L G_\lambda \rightarrow {}^L G$ is a homomorphism of L-groups. Using Lemma 4.2.1, we define an unramified (*i.e.*, trivial on I_F) homomorphism

$$(56) \quad \begin{aligned} \lambda_{\text{nr}} : W_F &\longrightarrow {}^L G_\lambda \\ \text{Fr} &\mapsto s_\lambda \times \text{Fr}. \end{aligned}$$

Lemma 4.3.1. — *Let $\lambda : W_F \rightarrow {}^L G$ be an infinitesimal parameter. Define the parameter $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$ as above. Then*

$$V_{\lambda_{\text{nr}}} = V_\lambda \quad \text{and} \quad H_{\lambda_{\text{nr}}} = H_\lambda^0.$$

Consequently,

$$\text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}}) = \text{Per}_{H_\lambda^0}(V_\lambda).$$

Proof. — Applying (36) to $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$ gives

$$H_{\lambda_{\text{nr}}} = Z_{J_\lambda^0}(\lambda_{\text{nr}}) = Z_{J_\lambda^0}(s_\lambda) = H_\lambda^0.$$

Applying (37) to $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$ gives

$$K_{\lambda_{\text{nr}}} = Z_{J_\lambda^0}(\lambda_{\text{nr}}|_{I_F}) = J_\lambda^0.$$

Applying (38) to $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$ gives

$$V_{\lambda_{\text{nr}}} = V_{\lambda_{\text{nr}}}({}^L G_\lambda) = \{x \in \text{Lie } Z_{\widehat{G}_\lambda}(\lambda_{\text{nr}}|_{I_F}) \mid \text{Ad}(\lambda_{\text{nr}}(\text{Fr}))x = q_F x\}.$$

Since $\widehat{G}_\lambda = J_\lambda^0$ and $\lambda_{\text{nr}}|_{I_F} = 1$, and since Fr acts trivially on J_λ^0 in ${}^L G_\lambda$, we have

$$(57) \quad V_{\lambda_{\text{nr}}} = \{x \in \mathfrak{j}_\lambda \mid \text{Ad}(s_\lambda)x = q_F x\}.$$

Then $V_\lambda = V_{\lambda_{\text{nr}}}$ because $\text{Ad}(f_\lambda \rtimes \text{Fr})x = qx$ if and only if $\text{Ad}(t_\lambda \rtimes \text{Fr})x = x$ and $\text{Ad}(s_\lambda)x = qx$. \square

Lemma 4.3.1 tells us that the category $\text{Per}_{H_\lambda^0}(V_\lambda)$ determined by $\lambda : W_F \rightarrow {}^L G$ can always be apprehended as the category for an *unramified* infinitesimal parameter $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_\lambda$. Note, however, that it is $\text{Per}_{H_\lambda}(V_\lambda)$, not $\text{Per}_{H_\lambda^0}(V_\lambda)$ which is needed to study Arthur packets of admissible representations of pure rational forms of $G(F)$; fortunately, Lemma 3.4.1 describes the relation between these two categories.

Remark 4.3.2. — Without defining G_λ^+ itself, let us set ${}^L G_\lambda^+ := J_\lambda \times W_F$ and define $\lambda_{\text{nr}}^+ : W_F \rightarrow {}^L G_\lambda^+$ by the composition of λ_{nr} and ${}^L G_\lambda \hookrightarrow {}^L G_\lambda^+$. Then (54) may also be used to define $r_\lambda^+ : {}^L G_\lambda^+ \hookrightarrow {}^L G$ and extends r_λ . Arguing as in the proof of Lemma 4.3.1, it follows that

$$V_{\lambda_{\text{nr}}^+} = V_\lambda \quad \text{and} \quad H_{\lambda_{\text{nr}}^+} = H_\lambda,$$

so

$$\text{Per}_{H_{\lambda_{\text{nr}}^+}}(V_{\lambda_{\text{nr}}^+}) = \text{Per}_{H_\lambda}(V_\lambda).$$

We pursue this perspective elsewhere.

4.4. Construction of the cocharacter. — From Section 4.3, recall the definition of $s_\lambda \in \widehat{G}$ and the fact that s_λ lies in the identity component of the subgroup $J_\lambda \subseteq \widehat{G}$. Decompose the Lie algebra \mathfrak{j}_λ of J_λ according to $\text{Ad}(s_\lambda)$ -eigenvalues:

$$\mathfrak{j}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{j}_\lambda(\nu), \quad \mathfrak{j}_\lambda(\nu) := \{x \in \mathfrak{j}_\lambda \mid \text{Ad}(s_\lambda)(x) = \nu x\}.$$

Following [Lus95c], define

$$\mathfrak{j}_\lambda^\dagger := \bigoplus_{r \in \mathbb{Z}} \mathfrak{j}_\lambda(q_F^r).$$

Lemma 4.4.1. — *There is a connected reductive algebraic subgroup M_λ of J_λ^0 and a cocharacter $\iota : \mathbb{G}_m \rightarrow M_\lambda$ such that*

$$M_\lambda^\iota = H_{\lambda_{\text{nr}}} \quad \text{and} \quad \mathfrak{m}_\lambda = \mathfrak{j}_\lambda^\dagger,$$

where $\mathfrak{m}_\lambda := \text{Lie } M_\lambda$ and an integer n so that, for every $r \in \mathbb{Z}$,

$$\mathfrak{m}_{\lambda, rn} = \mathfrak{j}_\lambda(q_F^r),$$

where $\mathfrak{m}_{\lambda, rn} := \{x \in \mathfrak{m} \mid \text{Ad}(\iota(t))x = t^{rn}x, \forall t \in \mathbb{G}_m\}$. In particular,

$$V_\lambda = \mathfrak{j}_\lambda(q_F) = \mathfrak{m}_{\lambda, n}.$$

Proof. — Decompose the Lie algebra \mathfrak{j}_λ of J_λ according to $\text{Ad}(s_\lambda)$ -eigenvalues:

$$\mathfrak{j}_\lambda = \bigoplus_{\nu \in \mathbb{C}^*} \mathfrak{j}_\lambda(\nu).$$

Fix a maximal torus S of J_λ^0 such that $s_\lambda \in S$ and denote the set of roots determined by this choice by $R(S, J_\lambda^0)$. For $\alpha \in R(S, J_\lambda^0)$, denote the root space in \mathfrak{j}_λ by \mathfrak{u}_α . Then

$$(58) \quad \mathfrak{j}_\lambda(\nu) = \bigoplus_{\substack{\alpha \in R(S, J_\lambda^0) \\ \alpha(s_\lambda) = \nu}} \mathfrak{u}_\alpha.$$

Let $\langle \cdot, \cdot \rangle$ be the natural pairing between $X^*(S)$ and $X_*(S)$. First, let us consider all $\alpha \in R(S, J_\lambda^0)$ such that $\alpha(s_\lambda)$ are integral powers of q . For these roots we can choose $\chi \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ so that $\langle \alpha, \chi \rangle = r$ if $\alpha(s_\lambda) = q^r$ for some integer r . Let n be an integer such that $n\chi \in X_*(S)$, and we set $t = (n\chi)(\zeta q^{1/n}) \in S$, where ζ is a primitive n -th root of unity. Now for $\alpha \in R(S, J_\lambda^0)$ such that $\alpha(s_\lambda) = q^r$, we have

$$\alpha(t) = \alpha((n\chi)(\zeta q^{1/n})) = \alpha(\chi(\zeta q^{1/n}))^n = (\zeta q^{1/n})^{rn} = q^r = \alpha(s_\lambda).$$

Next, consider those $\alpha \in R(S, J_\lambda^0)$ such that $\alpha(s_\lambda)$ are not integral powers of q . We have two cases: if $\langle \alpha, \chi \rangle \in \mathbb{Z}$, then $\alpha(t)$ is an integral power of q ; if $\langle \alpha, \chi \rangle \notin \mathbb{Z}$, then $\alpha(t) \in \zeta^l \mathbb{R}_{>0}$ for some $0 < l < n$. Since s_λ is hyperbolic, $\alpha(s_\lambda) \in \mathbb{R}_{>0}$ for all $\alpha \in R(S, J_\lambda^0)$, so $\alpha(s_\lambda) \neq \alpha(t)$ in either case. Therefore, we can define $M_\lambda = Z_{J_\lambda}(s_\lambda t^{-1})^0$ and take $\iota = n\chi$. \square

4.5. Proof of Theorem 4.1.1. — The essential facts about the groups K_λ , H_λ , J_λ and M_λ are summarized in the following diagram.

$$\begin{array}{ccccccc} & & & & \widehat{G} & & \\ & & & & \uparrow & & \\ & & & & K_\lambda := Z_{\widehat{G}}(\lambda(I_F)) & & \\ & & & & \uparrow & & \\ M_\lambda^0 = M_\lambda & \twoheadrightarrow & J_\lambda^0 & \twoheadrightarrow & J_\lambda := Z_{K_\lambda}(t_\lambda \rtimes \text{Fr}) & \twoheadrightarrow & \pi_0(J_\lambda) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ M_\lambda^\iota & \xlongequal{\quad} & H_\lambda^0 & \twoheadrightarrow & H_\lambda = Z_{J_\lambda}(s_\lambda) & \twoheadrightarrow & \pi_0(H_\lambda) \end{array}$$

From the definitions of G_λ (54), λ_{nr} (56) and $r_\lambda : {}^L G_\lambda \rightarrow {}^L G$ (55), we have

$$(59) \quad r_\lambda(\lambda_{\text{nr}}(\text{Fr})) = r_\lambda(s_\lambda \times \text{Fr}) = (s_\lambda \times 1)(t_\lambda \rtimes \text{Fr}) = f_\lambda \rtimes \text{Fr} = \lambda(\text{Fr}).$$

Now, Theorem 4.1.1 follows from a direct application of Lemmas 3.4.1 and 4.4.1, as in the diagram below.

$$\begin{array}{ccccc} \text{Rep}(A_\lambda) & & \text{Per}_{\widehat{G}}(X_\lambda) & \xrightleftharpoons[(c_\lambda)_*]{(c_\lambda)^*} & \text{Per}_{\widehat{G}_\lambda}(X_{\lambda_{\text{nr}}}) \\ \parallel & & \text{equiv} \downarrow & & \downarrow \text{equiv} \\ \text{Rep}(\pi_0(H_\lambda)) & \longrightarrow & \text{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{\text{forget}} & \text{Per}_{H_\lambda^0}(V_\lambda) \\ & & & & \parallel \\ & & & & \text{Per}_{M_\lambda^\iota}(\mathfrak{m}_{\lambda,n}) \end{array}$$

4.6. Further properties of Vogan varieties. — From (58) in the proof of Theorem 4.1.1 we get a very concrete description of V_λ as a variety, for any $\lambda \in R({}^L G)$:

$$V_\lambda \cong \mathbb{A}^d, \quad \text{for } d = |\{\alpha \in R(S, J_\lambda^0) \mid \alpha(s_\lambda) = q_F\}|.$$

Proposition 4.6.1. — *The space V_λ is stratified into H_λ -orbits, of which there are finitely many, with a unique open orbit.*

Proof. — With Proposition 4.4.1 in hand, this follows immediately from [Lus95c, Proposition 3.5] and [Lus95c, Section 3.6]. \square

A different proof is given in [Vog93, Proposition 4.5].

Proposition 4.6.2. — *Every H_λ -orbit in V_λ is a conical variety.*

Proof. — By Proposition 4.4.1, it suffices to prove that every M_λ^ι -orbit C in $\mathfrak{m}_{\lambda,n}$ is a conical variety. Arguing as in the proof of [GR10, Lemma 2.1], for $x \in C$, we can find a homomorphism $\varphi : \mathrm{SL}(2, \mathbb{C}) \rightarrow M_\lambda$ such that for $t \in \mathbb{C}^*$

$$\varphi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in M_\lambda^\iota \quad \text{and} \quad d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x.$$

Then

$$\mathrm{Ad} \left(\varphi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) (x) = d\varphi \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix} = t^2 x,$$

so $t^2 x \in C$. \square

5. Arthur parameters and the conormal bundle

The goal of Section 5 is to show that every Arthur parameter $\psi \in Q_\lambda(^L G)$ with infinitesimal parameter λ may be apprehended as a regular conormal vector to the associated stratum at an associated point, that is $\xi_\psi \in T_{C_\psi, x_\psi}^*(V_\lambda)_{\mathrm{reg}}$.

In the rest of this section, G is an arbitrary connected reductive linear algebraic group over the p -adic field F unless noted otherwise.

5.1. Regular conormal vectors. — For $\lambda \in R(^L G)$ and every H_λ -orbit $C \subseteq V_\lambda$, let $T_C^*(V_\lambda)_{\mathrm{reg}} \subset T_C^*(V_\lambda)$ be the subvariety defined by

$$(60) \quad T_C^*(V_\lambda)_{\mathrm{reg}} := T_C^*(V_\lambda) \setminus \bigcup_{C \subsetneq \overline{C_1}} \overline{T_{C_1}^*(V_\lambda)}.$$

Also define

$$T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}} := \bigcup_C T_C^*(V_\lambda)_{\mathrm{reg}},$$

the union taken over all H_λ -orbits C in V_λ . Then $T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}}$ is open subvariety of $T_{H_\lambda}^*(V_\lambda)$ and each $T_C^*(V_\lambda)_{\mathrm{reg}}$ is a component in $T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}}$.

We may compose (14) and (34):

$$(61) \quad \begin{array}{ccccc} Q(^L G) & \rightarrow & P(^L G) & \rightarrow & R(^L G) \\ \psi & \mapsto & \phi_\psi & \mapsto & \lambda_{\phi_\psi}. \end{array}$$

To simplify notation, we set $\lambda_\psi := \lambda_{\phi_\psi}$. We will refer to λ_ψ as the *infinitesimal parameter* of ψ . Using Proposition 3.2.2, define

$$x_\psi := x_{\phi_\psi} \in V_{\lambda_\psi}$$

and let $C_\psi \subseteq V_{\lambda_\psi}$ be the H_{λ_ψ} -orbit of $x_\psi \in V_{\lambda_\psi}$.

Proposition 5.1.1. — *Let $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ be an Arthur parameter. Let $\lambda_\psi : W_F \rightarrow {}^L G$ be its infinitesimal parameter. Then ψ determines a regular conormal vector*

$$\xi_\psi \in T_{C_\psi, x_\psi}^*(V_\lambda)_{\mathrm{reg}},$$

with the property that the H_λ -orbit of (x_ψ, ξ_ψ) in $T_{C_\psi}^(V_\lambda)$ is open and dense in $T_{C_\psi}^*(V_\lambda)_{\mathrm{reg}}$. The equivariant fundamental group of this orbit is A_ψ .*

The proof of Proposition 5.1.1 will be given in Section 5.8.

5.2. Cotangent space to the Vogan variety. — Consider

$$(62) \quad {}^t V_\lambda := \{x \in \mathfrak{k}_\lambda \mid \mathrm{Ad}(\lambda(\mathrm{Fr}))(x) = q_F^{-1}x\},$$

which clearly comes equipped with an action of H_λ just as V_λ comes equipped with an action of H_λ . Compare ${}^t V_\lambda$ with V_λ defined in (38). In fact, the variety ${}^t V_\lambda$ has already appeared: see the proof of Proposition 3.2.2. We note

$${}^t V_\lambda = \mathfrak{k}_\lambda(q_F^{-1}) = \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda, -n},$$

where \mathfrak{k} and \mathfrak{m}_n are defined in Sections 3.2 and 4.4, respectively.

For $\phi : L_F \rightarrow {}^L G$, we can define

$$(63) \quad \begin{aligned} P_\lambda({}^L G) &\longrightarrow {}^t V_\lambda, \\ \phi &\mapsto x_\phi := d\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where $\varphi := \phi^\circ|_{\mathrm{SL}(2, \mathbb{C})} : \mathrm{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$. This map satisfies all the properties of the map $P_\lambda({}^L G) \rightarrow V_\lambda({}^L G)$ in Proposition 3.2.2, from which it follows that there is a canonical bijection between H_λ -orbits in V_λ and H_λ -orbits in ${}^t V_\lambda$, so that the following diagram commutes.

$$\begin{array}{ccc} P_\lambda({}^L G)/H_\lambda & \xlongequal{\quad} & P_\lambda({}^L G)/H_\lambda \\ \downarrow & & \downarrow \\ V_\lambda/H_\lambda & \xrightarrow{\cong} & {}^t V_\lambda/H_\lambda \end{array}$$

Proposition 5.2.1. — *There is an H_λ -equivariant isomorphism*

$$T^*(V_\lambda) \simeq V_\lambda \times {}^t V_\lambda,$$

and consequently,

$$T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda, n} \oplus \mathfrak{m}_{\lambda, -n}.$$

Proof. — As V_λ is an affine H_λ -space there is a standard H_λ -equivariant isomorphism $T^*(V_\lambda) \simeq V_\lambda \times V_\lambda^*$, so it suffices to exhibit an H_λ -equivariant isomorphism

$$V_\lambda^* \cong {}^t V_\lambda.$$

To do this, let J_λ be the reductive group defined in (53) and write \mathfrak{j}_λ for $\mathrm{Lie} J_\lambda$, as in Section 4.3. From Proposition 4.4.1, we have

$$V_\lambda = \mathfrak{j}_\lambda(q_F) \quad \text{and} \quad \mathfrak{h}_\lambda = \mathfrak{j}_\lambda(1) \quad \text{and} \quad {}^t V_\lambda = \mathfrak{j}_\lambda(q_F^{-1}).$$

As J_λ is reductive, its Lie algebra decomposes into a direct sum of its centre and a semisimple Lie algebra, $\mathfrak{j}_\lambda \simeq Z(\mathfrak{j}_\lambda) \oplus [\mathfrak{j}_\lambda, \mathfrak{j}_\lambda]$. We choose any non-degenerate symmetric bilinear form on $Z(\mathfrak{j}_\lambda)$ and extend to a bilinear form on \mathfrak{j}_λ using the Cartan-Killing form, while insisting that the direct sum decomposition above is orthogonal, that is, the components in the direct sum are pairwise perpendicular. The result is a non-degenerate, symmetric, J_λ -invariant bilinear pairing

$$(\cdot | \cdot) : \mathfrak{j}_\lambda \times \mathfrak{j}_\lambda \rightarrow \mathbb{A}^1.$$

Now, if $\mathfrak{j}_\lambda(\nu)$ and $\mathfrak{j}_\lambda(\nu')$ are two $\text{Ad}(s_\lambda)$ -weight spaces, then the invariance of the pairing implies that $(\mathfrak{j}_\lambda(\nu) | \mathfrak{j}_\lambda(\nu')) \neq 0$ if and only if $\nu' = \nu^{-1}$. Since the pairing is non-degenerate this gives an $Z_{J_\lambda}(s_\lambda) = H_\lambda$ -equivariant isomorphism

$$V_\lambda^* = \mathfrak{j}_\lambda(q_F)^* \cong \mathfrak{j}_\lambda(q_F^{-1}) = {}^t V_\lambda.$$

A similar argument using the cocharacter $\iota : \mathbb{G}_m \rightarrow M_\lambda$ and the graded Lie algebra

$$\begin{aligned} \mathfrak{m}_\lambda &= \cdots \oplus \mathfrak{m}_{\lambda,2n} \oplus (\mathfrak{m}_{\lambda,n} \oplus \mathfrak{m}_{\lambda,0} \oplus \mathfrak{m}_{\lambda,-n}) \oplus \mathfrak{m}_{\lambda,-2n} \oplus \cdots \\ &= \cdots \oplus \mathfrak{m}_{\lambda,2n} \oplus (V_\lambda \oplus \mathfrak{h}_\lambda \oplus {}^t V_\lambda) \oplus \mathfrak{m}_{\lambda,-2n} \oplus \cdots \end{aligned}$$

produces an $M_\lambda^\iota = H_\lambda^0$ -equivariant isomorphism

$$(64) \quad V_\lambda^* = \mathfrak{m}_{\lambda,n}^* \cong \mathfrak{m}_{\lambda,-n} = {}^t V_\lambda.$$

This allows us to view $T^*(V_\lambda)$ as a subspace of \mathfrak{m}_λ , even with H_λ -action, and gives H_λ -equivariant isomorphisms

$$T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1}) = \mathfrak{m}_{\lambda,n} \oplus \mathfrak{m}_{\lambda,-n},$$

as desired. \square

In the remainder of the paper we identify ${}^t V_\lambda$ with V_λ^* , using Proposition 5.2.1.

5.3. Conormal bundle to the Vogan variety. —

Proposition 5.3.1. — *Let $C \subseteq V_\lambda$ be an H_λ -orbit in V_λ ; then*

$$T_C^*(V_\lambda) = \{(x, \xi) \in T^*(V_\lambda) \mid x \in C, [x, \xi] = 0\},$$

where $[\cdot, \cdot]$ denotes the Lie bracket on \mathfrak{j}_λ and where we use Proposition 5.2.1 to identify $T^*(V_\lambda) \cong \mathfrak{j}_\lambda(q_F) \oplus \mathfrak{j}_\lambda(q_F^{-1})$. Consequently,

$$T_{H_\lambda}^*(V_\lambda) = \{(x, \xi) \in T^*(V_\lambda) \mid [x, \xi] = 0\}.$$

Proof. — The map $\mathfrak{h}_\lambda \rightarrow T_x(C)$ given by $X \mapsto [x, X]$ is a surjection. So for any $\xi \in \mathfrak{j}_\lambda(q_F^{-1})$, we have $\xi \in T_{C,x}^*(V_\lambda)$ if and only if $0 = (\xi | [x, X]) = ([\xi, x] | X)$ for all $X \in \mathfrak{h}_\lambda$. As we saw in the proof of Proposition 5.2.1, the pairing restricts non-degenerately to \mathfrak{h}_λ , so this is also equivalent to require $[x, \xi] = 0$. \square

Corollary 5.3.2. — $T_{H_\lambda}^*(V_\lambda) \hookrightarrow (\cdot | \cdot)^{-1}(0)$.

Proof. — If $(x, \xi) \in V_\lambda \times V_\lambda^*$ lies in $T_{H_\lambda}^*(V_\lambda)$ then $[x, \xi] = 0$. Choose an \mathfrak{sl}_2 -triple (x, y, z) such that $y \in V_\lambda^*$, and $z \in \mathfrak{h}_\lambda$. Then,

$$(x | \xi) = \frac{1}{2}([z, x] | \xi) = \frac{1}{2}(z | [x, \xi]) = 0. \quad \square$$

5.4. Orbit duality. — Consider the H_λ -equivariant isomorphism

$$(65) \quad \begin{aligned} T^*(V_\lambda) &\rightarrow T^*(V_\lambda^*) \\ (x, \xi) &\mapsto (\xi, x), \end{aligned}$$

where we use the form $(\cdot | \cdot)$ to identify the dual to V_λ^* with V_λ . Just as every H_λ -orbit $C \subset V_\lambda$ determines the conormal bundle

$$T_C^*(V_\lambda) = \{(x, \xi) \in V_\lambda \times V_\lambda^* \mid x \in C, [x, \xi] = 0\},$$

every H_λ -orbit $B \subset V_\lambda^*$ determines a conormal bundle in $T^*(V_\lambda^*)$:

$$T_B^*(V_\lambda^*) = \{(\xi, x) \in V_\lambda^* \times V_\lambda \mid \xi \in B, [\xi, x] = 0\}.$$

Lemma 5.4.1. — *For every H_λ -orbit C in V_λ there is a unique H_λ -orbit C^* in V_λ^* so that (65) restricts to an isomorphism*

$$\overline{T_C^*(V_\lambda)} \cong \overline{T_{C^*}^*(V_\lambda^*)}.$$

The rule $C \mapsto C^$ is a bijection from H_λ -orbits in V_λ to H_λ -orbits in V_λ^* .*

Proof. — This is a well-known result. See [Pja75, Corollary 2] for the case when H_λ is connected. The result extends easily to the case when H_λ is not connected. \square

The orbit C^* is called the *dual orbit* of $C \subseteq V_\lambda$; likewise, the dual orbit of $B \subseteq V_\lambda^*$ is denoted by B^* .

Lemma 5.4.2. — *If $(x, \xi) \in T_C^*(V_\lambda)_{\text{reg}}$ then $\xi \in C^*$, so*

$$T_C^*(V_\lambda)_{\text{reg}} \subseteq \{(x, \xi) \in C \times C^* \mid [x, \xi] = 0\}.$$

Proof. — Since $(x, \xi) \in T_C^*(V_\lambda)_{\text{reg}}$, then (x, ξ) is not contained in any other closures of conormal bundles except for that of C . On the other hand, $(\xi, x) \in T_{B_\xi}^*(V_\lambda^*)$ where B_ξ is the H_λ -orbit of ξ in V_λ^* , so $\overline{T_C^*(V_\lambda)} \cong \overline{T_{B_\xi}^*(V_\lambda^*)}$. Hence $B_\xi = C^*$, i.e., $\xi \in C^*$. \square

Proposition 5.4.3. — *If $(x, \xi) \in T_C^*(V_\lambda)_{\text{reg}}$ then $(x, \xi) \in C \times C^*$ and $[x, \xi] = 0$ and $(x | \xi) = 0$.*

Proof. — Combine Lemma 5.3.2 with Lemma 5.4.2. \square

We remark that $(x, \xi) \in C \times C^*$ implies neither $[x, \xi] = 0$ nor $(x | \xi) = 0$ in general.

Although transposition in j_λ is not H_λ -equivariant, it does induce another canonical bijection

$$C \mapsto {}^t C \quad \text{and} \quad B \mapsto {}^t B$$

between H_λ -orbits in V_λ and H_λ -orbits in V_λ^* , and vice versa. Unlike the bijection of Lemma 5.4.1, this bijection preserves equivariant fundamental groups (50):

$$A_C \cong A_{{}^t C} \quad \text{and} \quad A_B \cong A_{{}^t B}.$$

For $C \subseteq V_\lambda$ (resp. $B \subseteq V_\lambda^*$) we refer to ${}^t C$ (resp. ${}^t B$) as the *transposed orbit* of C (resp. B). Composing orbit transposition with orbit duality defines an involution

$$(66) \quad C \mapsto \widehat{C} := {}^t C^*$$

on the set of H_λ -orbits in V_λ .

5.5. Strongly regular conormal vectors. — We say that $(x, \xi) \in T_C^*(V_\lambda)$ is *strongly regular* if its H_λ -orbit is open and dense in $T_C^*(V_\lambda)$. We write $T_C^*(V_\lambda)_{\text{sreg}}$ for the strongly regular part of $T_C^*(V_\lambda)_{\text{reg}}$. We set

$$(67) \quad T_{H_\lambda}^*(V_\lambda)_{\text{sreg}} := \bigcup_C T_C^*(V_\lambda)_{\text{sreg}}.$$

Proposition 5.5.1. —

$$T_{H_\lambda}^*(V_\lambda)_{\text{sreg}} \subseteq T_{H_\lambda}^*(V_\lambda)_{\text{reg}}$$

and if $(x, \xi) \in T_C^*(V_\lambda)$ is strongly regular then its H_λ -orbit is $T_C^*(V_\lambda)_{\text{sreg}}$.

Proof. — First we show $T_C^*(V_\lambda)_{\text{sreg}} \subseteq T_C^*(V_\lambda)_{\text{reg}}$. From the definition of $T_C^*(V_\lambda)_{\text{reg}}$ (60) it is clear that it is open and dense in $T_C^*(V_\lambda)$. Fix $(x, \xi) \in T_C^*(V_\lambda)$ and let $\mathcal{O}_{H_\lambda}(x, \xi)$ denote the H_λ -orbit of (x, ξ) . If (x, ξ) is not regular, then $(x, \xi) \in \overline{T_{C_1}^*(V_\lambda)}$ for some $C_1 \neq C$ with $C \subset \bar{C}_1$, so all of $\mathcal{O}_{H_\lambda}(x, \xi)$ and its closure also does not intersect $T_C^*(V_\lambda)_{\text{reg}}$. Suppose, for a contradiction, that (x, ξ) is strongly regular also. Then the closure of $\mathcal{O}_{H_\lambda}(x, \xi)$ is $T_C^*(V_\lambda)$, which certainly does intersect $T_C^*(V_\lambda)_{\text{reg}}$. So, if (x, ξ) is not regular, then it is not strongly regular.

Now suppose $T_{C,x}^*(V_\lambda)_{\text{sreg}}$ is not empty, then it is enough to show $T_{C,x}^*(V_\lambda)_{\text{sreg}}$ forms a single $Z_{H_\lambda}(x)$ -orbit. Note

$$T_{C,x}^*(V_\lambda)_{\text{sreg}} = \{\xi \in T_{C,x}^*(V_\lambda) \mid [\text{Lie}(Z_{H_\lambda}(x)), \xi] = T_{C,x}^*(V_\lambda)\}$$

which is open, dense and connected in $T_{C,x}^*(V_\lambda)$. Moreover, $Z_{H_\lambda}(x)$ -orbits in $T_{C,x}^*(V_\lambda)_{\text{sreg}}$ are open, and hence they are also closed in $T_{C,x}^*(V_\lambda)_{\text{sreg}}$. By the connectedness of $T_{C,x}^*(V_\lambda)_{\text{sreg}}$, we can conclude it is a single $Z_{H_\lambda}(x)$ -orbit. \square

The equivariant fundamental group of $T_C^*(V_\lambda)_{\text{sreg}}$ will be denoted by $A_{T_C^*(V_\lambda)_{\text{sreg}}}$. Since H_λ acts transitively on $T_C^*(V_\lambda)_{\text{sreg}}$,

$$(68) \quad A_{T_C^*(V_\lambda)_{\text{sreg}}} \cong \pi_0(Z_{H_\lambda}(x, \xi)) = Z_{H_\lambda}(x, \xi)/Z_{H_\lambda}(x, \xi)^0,$$

for every $(x, \xi) \in T_C^*(V_\lambda)_{\text{sreg}}$. Consequently, each $(x, \xi) \in T_C^*(V_\lambda)_{\text{sreg}}$ determines an equivalence

$$\text{Loc}_{H_\lambda}(T_C^*(V_\lambda)_{\text{sreg}}) \rightarrow \text{Rep}(A_{T_C^*(V_\lambda)_{\text{sreg}}}).$$

5.6. From Arthur parameters to strongly regular conormal vectors. — For $\psi \in Q(^L G)$, define

$$\psi_0 := \psi^0|_{\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})} : \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$$

and

$$\psi_1 := \psi_0|_{\text{SL}(2, \mathbb{C}) \times 1} : \text{SL}(2, \mathbb{C}) \rightarrow \widehat{G} \quad \text{and} \quad \psi_2 := \psi_0|_{1 \times \text{SL}(2, \mathbb{C})} : \text{SL}(2, \mathbb{C}) \rightarrow \widehat{G}.$$

Set

$$(69) \quad x_\psi := d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \quad y_\psi := d\psi_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \quad \text{and} \quad \xi_\psi := d\psi_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}}.$$

It follows easily from these definitions that

$$x_\psi, y_\psi \in V_{\lambda_\psi} \quad \text{and} \quad \xi_\psi \in V_{\lambda_\psi}^*$$

and

$$(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V_\lambda).$$

Proposition 5.6.1. — *For any $\psi \in Q(^L G)$,*

$$(x_\psi, \xi_\psi) \in T_{H_{\lambda_\psi}}^*(V_{\lambda_\psi})_{\text{reg}}.$$

Proof. — Set $\lambda = \lambda_\psi$. Define $f_\lambda, s_\lambda, t_\lambda \in \widehat{G}$ as in Section 4.3. Then

$$s_\lambda \rtimes 1 = \psi(1, d_{\text{Fr}}, d_{\text{Fr}}) \quad \text{and} \quad t_\lambda \times \text{Fr} = \psi(\text{Fr}, 1, 1).$$

Recall $\lambda_{\text{nr}} : W_F \rightarrow J_\lambda^0$ from Section 4.3. By Proposition 4.4.1,

$$V_\lambda = V_{\lambda_{\text{nr}}} = \mathfrak{j}_{\lambda, 2}.$$

Since the image of $\psi_0 : \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$ lies in J_λ^0 , we may define

$$\psi_{\text{nr}} : W_F \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow J_\lambda^0$$

such that its restriction to W_F is trivial and its restriction to $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ is ψ_0 . Let

$$\iota_\psi : \mathbb{G}_m \longrightarrow J_\lambda^0$$

be the cocharacter obtained by composing

$$\mathbb{G}_m \rightarrow W_F \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}), \quad z \mapsto 1 \times \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \times \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

with of $\psi_{\text{nr}} : L_F \times \text{SL}(2, \mathbb{C}) \rightarrow J_\lambda^0$. Then

$$\iota_\psi(q_F^{1/2}) = \lambda_{\text{nr}}(\text{Fr}).$$

Recall $H_\lambda \subseteq J_\lambda \subseteq K_\lambda \subseteq \widehat{G}$ from Sections 3.2 and 4.3. For the rest of the proof we set $J = J_\lambda$. We must show that the orbit $\mathcal{O}_{Z_{H_\lambda}(x_\psi)}(\xi_\psi)$ is open and dense in $T_{C_\psi, x_\psi}^*(V_\lambda)$, where $C_\psi = \mathcal{O}_{H_\lambda}(x_\psi)$. With Lemma 4.6.1 in hand, it is enough to show the tangent space to the orbit $\mathcal{O}_{Z_{H_\lambda}(x_\psi)}(\xi_\psi)$ at ξ_ψ is isomorphic to $T_{C_\psi, x_\psi}^*(V_\lambda)$; in other words, it is enough to show

$$[\text{Lie} Z_{H_\lambda}(x_\psi), \xi_\psi] = \{\xi \in \mathfrak{j}_{-2} \mid [x_\psi, \xi] = 0\}.$$

The adjoint action of $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ on \mathfrak{j} through ψ_{nr} gives two commuting representations of $\text{SL}(2, \mathbb{C})$, which induce the weight decomposition

$$(70) \quad \mathfrak{j}_n = \bigoplus_{r+s=n} \mathfrak{j}_{r,s}$$

where $r, s \in \mathbb{Z}$. Note $\text{Lie}(H_\lambda) = \mathfrak{j}_0$. So it is enough to show

$$(71) \quad \mathfrak{j}_0 \cap \text{Lie}(Z_{\widehat{G}}(x_\psi)), \xi_\psi] = \mathfrak{j}_{-2} \cap \text{Lie}(Z_{\widehat{G}}(x_\psi)).$$

For this we can consider the following diagram in case $r + s = 0$.

$$\begin{array}{ccc} \mathfrak{j}_{r,s} & \xrightarrow{\text{ad}(x_\psi)} & \mathfrak{j}_{r+2,s} \\ \text{ad}(\xi_\psi) \downarrow & & \downarrow \text{ad}(\xi_\psi) \\ \mathfrak{j}_{r,s-2} & \xrightarrow{\text{ad}(x_\psi)} & \mathfrak{j}_{r+2,s-2} \end{array}$$

It is easy to see

$$\begin{aligned} \text{LHS(71)} &= \bigoplus_{r+s=0} \text{ad}(\xi_\psi) \left(\ker(\text{ad}(x_\psi)|_{\mathfrak{j}_{r,s}}) \right) \\ \text{RHS(71)} &= \bigoplus_{r+s=0} \ker(\text{ad}(x_\psi)|_{\mathfrak{j}_{r,s-2}}) \end{aligned}$$

By \mathfrak{sl}_2 -representation theory, $\text{ad}(x_\psi)$ in the diagram are injective for $r < 0$ and surjective for $r \geq 0$. So we only need to consider $r \geq 0$ and hence $s \leq 0$. In this case, the two instances of $\text{ad}(\xi_\psi)$ in the diagram above are surjective by \mathfrak{sl}_2 -representation theory again.

It is obvious that $\text{LHS(71)} \subseteq \text{RHS(71)}$. For the other direction, let us choose $x \in \mathfrak{j}_{r,s-2}$ such that $[x_\psi, x] = 0$. So x is primitive for the action of the first \mathfrak{sl}_2 , and it generates an irreducible representation V . Let \tilde{x} be a preimage of x in $\mathfrak{g}_{r,s}$ and W be the representation of the first \mathfrak{sl}_2 generated by \tilde{x} . Then $\text{ad}(\xi_\psi)$ induces a morphism of \mathfrak{sl}_2 -representations from W to V . By the semisimplicity of W , this morphism admits a splitting and we can denote the image of x by ξ . It is clear that $\xi \in \mathfrak{j}_{r,s}$ and $[x_\psi, \xi] = 0$. This finishes the proof. \square

Corollary 5.6.2. — *Let $\psi : W_F \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow {}^L G$ be an Arthur parameter with infinitesimal parameter λ . If $C_\psi \subseteq V_\lambda$ is the H_λ -orbit of x_ψ , then*

$$\widehat{C}_\psi = C_{\widehat{\psi}},$$

where $\widehat{C}_\psi = {}^t C_\psi^*$ (66) and where the map $\widehat{\psi} : W_F \times \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow {}^L G$ is defined by $\widehat{\psi}(w, x, y) := \psi(w, y, x)$.

5.7. Arthur component groups are equivariant fundamental groups. — Recall the definition of $T_{C_\psi}^*(V_\lambda)_{\text{sreg}}$ from Section 5.5 as well as the notation $A_{T_{C_\psi}^*(V_\lambda)_{\text{sreg}}}$ for its equivariant fundamental group. Also recall $A_\psi := \pi_0(Z_{\widehat{G}}(\psi))$ from Section 2.4.

Proposition 5.7.1. —

$$A_{T_{C_\psi}^*(V_\lambda)_{\text{sreg}}} = A_\psi.$$

Proof. — We use the notation from the proof of Proposition 5.6.1 and set $t_\psi := t_{\lambda_\psi}$. It is clear that $Z_{\widehat{G}}(\psi) = Z_J(\psi_{\text{nr}}) = Z_J(\psi_1) \cap Z_J(\psi_2)$. By Lemma 4.3.1, we also have

$$Z_{\widehat{G}}(\lambda)_{(x_\psi, \xi_\psi)} = Z_J(\lambda_{\text{nr}}) \cap Z_J(x_\psi) \cap Z_J(\xi_\psi).$$

First we would like to compute the right hand side of the above identity. Note

$$Z_J(\lambda_{\text{nr}}) \cap Z_J(x_\psi) = (Z_J(\psi_1) \cap Z_J(\lambda_{\text{nr}})) \cdot U$$

where U is the unipotent radical of the left hand side. Moreover,

$$Z_J(\psi_1) \cap Z_J(\lambda_{\text{nr}}) = Z_J(\psi_1) \cap Z_J(t_\psi)$$

and

$$\text{Lie}(U) \subseteq \bigoplus_{\substack{r+s=0 \\ r>0}} \mathfrak{j}_{r,s}$$

in the notation of (70). For $u \in U$, we have

$$\text{Ad}(u)(\xi_\psi) \in \xi_\psi + \bigoplus_{\substack{r+s=-2 \\ s<-2}} \mathfrak{j}_{r,s}.$$

Suppose $\text{Ad}(lu)$ stabilises ξ_ψ for $l \in Z_J(\psi_1) \cap Z_J(t_\psi)$ and $u \in U$. Since $\text{Ad}(l)$ preserves $\mathfrak{j}_{r,s}$, we have

$$\xi_\psi = \text{Ad}(lu)(\xi_\psi) \in \text{Ad}(l)(\xi_\psi) + \bigoplus_{\substack{r+s=-2 \\ s<-2}} \mathfrak{j}_{r,s}$$

Note $\xi_\psi \in \mathfrak{j}_{0,-2}$. It follows $\xi_\psi = \text{Ad}(l)(\xi_\psi)$. Hence $\xi_\psi = \text{Ad}(u)(\xi_\psi)$. As a result,

$$Z_J(\lambda_{\text{nr}}) \cap Z_J(x_\psi) \cap Z_J(\xi_\psi) = (Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi)) \cdot (U \cap Z_J(\xi_\psi)).$$

Since $U \cap Z_J(\xi_\psi)$ is connected, we only need to show

$$Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi) = Z_J(\psi_1) \cap Z_J(\psi_2).$$

Take any $g \in Z_J(\psi_1) \cap Z_J(t_\psi) \cap Z_J(\xi_\psi)$, it suffices to show $\text{Ad}(g)$ stabilises y_ψ . Note

$$[y_\psi, \xi_\psi] = d\psi_2(\ln(|\text{Fr}|)),$$

and

$$[\text{Ad}(g)(y_\psi), \xi_\psi] = [\text{Ad}(g)(y_\psi), \text{Ad}(g)\xi_\psi] = \text{Ad}(g)(d\psi_2(\ln(|\text{Fr}|))) = d\psi_2(\ln(|\text{Fr}|))$$

Since $[\cdot, \xi_\psi]$ is injective on $\mathfrak{j}_{0,2}$ and $\text{Ad}(g)(y_\psi) \in \mathfrak{j}_{0,2}$, it follows that $\text{Ad}(g)(y_\psi) = y_\psi$. This finishes the proof. \square

5.8. Proof of Proposition 5.1.1. — Proposition 5.1.1 is now a direct consequence of Propositions 5.5.1, 5.6.1 and 5.7.1.

5.9. Equivariant Local systems. — We close Section 5 with a practical tool for understanding local systems on strata $C \subseteq V_\lambda$, on $T_C^*(V_\lambda)_{\text{sreg}}$, and on $C^* \subseteq V_\lambda^*$. Pick a base point $(x, \xi) \in T_C^*(V_\lambda)_{\text{sreg}}$. Recalling the structure of $T_C^*(V_\lambda)_{\text{reg}}$ from Lemma 5.4.2 and by using that $T_C^*(V_\lambda)_{\text{sreg}} \subseteq T_C^*(V_\lambda)_{\text{reg}}$ by Proposition 5.5.1. The projections

$$C \longleftarrow T_C^*(V_\lambda)_{\text{sreg}} \longrightarrow C^*$$

induce homomorphisms of fundamental groups:

$$\begin{array}{ccccc} A_C & \longleftarrow & A_{T_C^*(V_\lambda)_{\text{sreg}}} & \longrightarrow & A_{C^*} \\ \parallel & & \parallel & & \parallel \\ Z_{H_\lambda}(x)/Z_{H_\lambda}(x)^0 & \longleftarrow & Z_{H_\lambda}(x, \xi)/Z_{H_\lambda}(x, \xi)^0 & \longrightarrow & Z_{H_\lambda}(\xi)/Z_{H_\lambda}(\xi)^0. \end{array}$$

The horizontal homomorphisms are surjective by an application of [ABV92, Lemma 24.6]. This can be used to enumerate all the simple local systems on H_λ -orbits in V_λ and $T_{H_\lambda}^*(V_\lambda)_{\text{sreg}}$ and V_λ^* .

6. Microlocal vanishing cycles of perverse sheaves

The goal of Section 6 is to introduce the functor appearing in (10) and to establish some of its properties. We begin by stating the main application of Theorem 6.10.1, whose proof will occupy the rest of this section.

Corollary 6.0.1. — *Let G be a quasi-split connected reductive algebraic group over a p -adic field F . Let $\psi \in Q_\lambda({}^L G)$ be an Arthur parameter and let $\lambda : W_F \rightarrow {}^L G$ be its infinitesimal parameter. Vanishing cycles define an exact functor*

$$\text{NEv}_\psi : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Rep}(A_\psi)$$

which induces a function

$$\text{Per}_{H_\lambda}(V_\lambda)_{/iso}^{\text{simple}} \rightarrow \text{Rep}(A_\psi)_{/iso}$$

such that the composition

$$\Pi_{\text{pure}, \lambda}(G/F) \xrightarrow{[\pi, \delta] \mapsto \mathcal{P}(\pi, \delta)} \text{Per}_{H_\lambda}(V_\lambda)_{/iso}^{\text{simple}} \xrightarrow{\text{NEv}_\psi} \text{Rep}(A_\psi)_{/iso}$$

enjoys the following properties, for every $[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F)$:

- (a) $\text{NEv}_\psi \mathcal{P}(\pi, \delta) = 0$ unless $C_\psi \leq C_\phi$, where C_ψ is defined in Section 5.1, ϕ is the Langlands parameter for (π, δ) and C_ϕ is given by Proposition 3.6.2.
- (b) The dimension of the representation $\text{NEv}_\psi \mathcal{P}(\pi, \delta)$ of A_ψ is

$$\text{rank}(\text{R}\Phi_{\xi_\psi} \mathcal{P}(\pi, \delta))_{x_\psi},$$

where $(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V_\lambda)_{\text{sreg}}$ is given by Proposition 5.1.1 and $\text{R}\Phi_{\xi_\psi}$ is the vanishing cycles functor determined ξ_ψ .

- (c) If $C_\psi = C_\phi$ (equivalently, if ϕ_ψ is \widehat{G} -conjugate to ϕ) then

$$\text{NEv}_\psi \mathcal{P}(\pi, \delta) = p_\psi^*(\rho_{\pi, \delta})$$

where $\rho_{(\pi, \delta)}$ is the representation of A_ϕ given by Proposition 3.6.2 and where the map $p_\psi : A_\psi \rightarrow A_\phi$ is the canonical group homomorphism of Section 5.9; in particular,

$$\text{rank NEv}_\psi \mathcal{P}(\pi, \delta) = \text{rank } \rho_{\pi, \delta}.$$

To prove Corollary 6.0.1 we make a study of the vanishing cycles of the equivariant perverse sheaves on V_λ with respect to integral models for V_λ determined by regular covectors $(x, \xi) \in T_{H_\lambda}^*(V_\lambda)_{\text{reg}}$, especially those coming from Arthur parameters using Proposition 5.6.1.

In the rest of this section, unless noted otherwise, G is an arbitrary connected reductive algebraic group over a p -adic field F .

6.1. Background on vanishing cycles. — Although we will use [DK73, Exposés XIII, XIV] freely, we begin by recalling a few essential facts and setting some notation. Let $R := \mathbb{C}[[t]]$ and $K := \mathbb{C}((t))$. Set $S = \text{Spec}(R)$ and $\eta = \text{Spec}(K)$ and $s = \text{Spec}(\mathbb{C})$. Observe that S is a trait with generic fibre η and special fibre s .

$$\eta \xrightarrow{j} S \xleftarrow{i} s$$

Because S is an equal characteristic trait the morphism $s \rightarrow S$ admits canonical section corresponding to $\mathbb{C} \rightarrow \mathbb{C}[[t]]$.

Let $\bar{\eta}$ be a geometric point of S localized at η ; thus, $\bar{\eta}$ is simply a morphism $\text{Spec}(\bar{K}) \rightarrow \eta \rightarrow S$, where \bar{K} is a separable closure of K . Then $\text{Gal}(\bar{\eta}/\eta) \cong \hat{\mathbb{Z}}$. Let \bar{R} be the integral closure of R in \bar{K} ; note that \bar{R} has residue field \mathbb{C} . Set $\bar{S} = \text{Spec}(\bar{R})$.

For any morphism $X \rightarrow S$ we have the cartesian diagram

$$(72) \quad \begin{array}{ccccc} & \bar{X}_{\bar{\eta}} & \xrightarrow{j_{\bar{X}}} & \bar{X} & \xleftarrow{i_{\bar{X}}} & \bar{X}_s \\ & \swarrow b_{X_{\eta}} & & \swarrow b_X & & \swarrow b_{X_s} \\ X_{\eta} & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s & \\ \downarrow & & \downarrow & & \downarrow & \\ & \bar{\eta} & \xrightarrow{j_{\bar{S}}} & \bar{S} & \xleftarrow{i_{\bar{S}}} & \bar{s} \\ & \swarrow b_{\eta} & & \swarrow b_S & & \swarrow b_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s & \end{array}$$

where $\bar{X} = X \times_S \bar{S}$, $\bar{X}_{\bar{\eta}} = \bar{X} \times_{\bar{S}} \bar{\eta}$ and $\bar{X}_s = \bar{X} \times_{\bar{S}} \bar{s}$. We remark that $\bar{X}_s \neq X_s$, generally. From [DK73, Exposé XIII] we recall the nearby cycles functor

$$R\Psi_{X_{\eta}} : D(X_{\eta}) \rightarrow D(X_s \times_s \eta);$$

see [DK73, Section 1.2], especially the remark after [DK73, Construction 1.2.4] for the meaning of the topos $X_s \times_s \eta$. In particular, we recall that, for any $\mathcal{F}_{\eta} \in D(X_{\eta})$, the object $R\Psi_{X_{\eta}} \mathcal{F}_{\eta}$ in $D(X_s \times_s \eta)$ is the sheaf

$$R\Psi_X \mathcal{F}_{\eta} := (i_{\bar{X}})^*(j_{\bar{X}})_*(b_{X_{\eta}})^* \mathcal{F}_{\eta}$$

on $X_{\bar{s}}$ equipped with an action of $\text{Gal}(\bar{\eta}/\eta)$ obtained by transport of structure from the canonical action of $\text{Gal}(\bar{\eta}/\eta)$ on $(b_{X_{\eta}})^* \mathcal{F}_{\eta}$.

The vanishing cycles functor

$$R\Phi_X : D(X) \rightarrow D(X_s \times_s S)$$

is the cone of the canonical natural transformation $i_X^* b_X^* \rightarrow R\Psi_{X_{\eta}} j_X^*$ of functors from $D(X)$ to $D(X_s \times_s S)$ and thus appears as the summit in the following distinguished triangle [DK73, Exposé XIII, (2.1.2.4)] in $D(X_s \times_s S)$, for $\mathcal{F} \in D(X)$:

$$(73) \quad \begin{array}{ccc} & R\Phi_X \mathcal{F} & \\ (1) \swarrow & & \nwarrow \\ i_X^* b_X^* \mathcal{F} & \xrightarrow{\quad} & R\Psi_{X_{\eta}} j_{X_{\eta}}^* \mathcal{F}. \end{array}$$

See [DK73, Exposé XIII, Section 1.2], especially [DK73, Exposé XIII, Construction 1.2.4] for the meaning of the topos $X_s \times_s S$.

We will make free use of other properties of $R\Psi_{X_\eta}$ and $R\Phi_X$ established in [DK73, Exposés XIII, XIV], such as smooth base change [DK73, Exposé XIII, (2.1.7.1)] and proper base change [DK73, Exposé XIII, (2.1.7.2)].

6.2. Calculating vanishing cycles. — We denote the ℓ -adic constant sheaf by $\mathbb{1}$. In this section we calculate the vanishing cycles $R\Phi_X \mathbb{1}_X$ of the constant sheaf $\mathbb{1}_X$ for a short list of S -schemes X . While elementary, these calculations will be used in the proof of Theorem 6.7.5 and will also play a role in the examples appearing in Part II.

In all our applications of vanishing cycles we begin with map of varieties $f : U \rightarrow \mathbb{A}^1$ over \mathbb{C} and then let $f_S : X \rightarrow S$ be the base change of f along $S \rightarrow \mathbb{A}^1$; thus, in particular, $X = U \times_{\mathbb{A}^1} S$. Assuming $U = \text{Spec}(A)$ is affine for a moment, then the coordinate ring for X is

$$\mathcal{O}_X(X) = A \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]] \cong A[[t]]/(f - t),$$

where $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$ and where we identify f with its image in A . Note that the special fibre of X is

$$X_s = f^{-1}(0);$$

note also that this may not be reduced. We use the notation

$$R\Phi_f \mathcal{F} := R\Phi_X \mathcal{F}_X$$

where \mathcal{F}_X is the pullback of $\mathcal{F} \in D_c^b(U)$ along $X \rightarrow U$. Note that $R\Phi_f \mathcal{F}$ is a sheaf on the special fibre \bar{X}_s of \bar{X} and that \bar{X}_s may not coincide with $f^{-1}(0)$.

Lemma 6.2.1. — *If $0 : U \rightarrow \mathbb{A}^1$ is the map defined on coordinate rings by $t \mapsto 0$, where $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$, then, for every $\mathcal{F} \in D_c^b(U)$,*

$$R\Phi_0 \mathcal{F} = \mathcal{F}$$

with obvious monodromy.

Proof. — It follows directly from definitions that $R\Psi_0 \mathcal{F} = 0$, so $R\Phi_0 \mathcal{F} = \mathcal{F}$ is a consequence of (73). \square

Lemma 6.2.2. — *Let $x : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the map defined on coordinate rings by $t \mapsto x$. Then*

$$R\Phi_x \mathcal{L} = 0$$

for every local system \mathcal{L} on \mathbb{A}^1 . More generally, if $f : U \rightarrow \mathbb{A}^1$ is smooth and \mathcal{L}_U is a local system on U , then

$$R\Phi_f \mathcal{L} = 0.$$

Proof. — It follows directly from the definition of $R\Psi_{X_\eta}$ that $R\Psi_{X_\eta} \mathcal{L} = \mathcal{L}|_{\bar{X}_s}$ for $X = \text{Spec}(\mathbb{C}[x][[t]]/(x - t))$. Thus, $R\Phi_x \mathcal{L} = 0$, using (73). The second sentence is now a consequence of the first by smooth base change. See also [DK73, Exposé XIII, Reformulation 2.1.5]. \square

Lemma 6.2.3. — *Let $x^2 : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the map defined on coordinate rings by $t \mapsto x^2$. Then*

$$R\Phi_{x^2} \mathbb{1} = \mathbb{1}_s,$$

with quadratic monodromy. More generally, if $f : U \rightarrow \mathbb{A}^1$ is smooth then

$$R\Phi_{f^2} \mathbb{1} = \mathbb{1}_{f=0}.$$

with quadratic monodromy coming from the cover associated to \sqrt{f} .

Proof. — We first point out that the second claim follows immediately from the first by smooth base change.

Let $X \rightarrow S$ be the base change of $x^2 : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ along $S \rightarrow \mathbb{A}^1$. More explicitly we have $X = \text{Spec}(R[x]/(x^2 - t))$ and $X \rightarrow S$ is given on coordinate rings by $R \rightarrow R[x]/(x^2 - t)$, where $R = \mathbb{C}[[t]]$. Then

$$\bar{X} = \bar{X}^+ \sqcup \bar{X}^- \quad \text{and} \quad \bar{X}^+ \cap \bar{X}^- = \bar{X}_s,$$

with $\bar{X}^\pm \cong \bar{S}$; note also that $X_s = f^{-1}(0) = \text{Spec}(\mathbb{C}[x]/(x^2))$ while $\bar{X}_s = X_s^{\text{red}} = s$. Consequently,

$$\begin{aligned} \bar{X}_{\bar{\eta}} &= \text{Spec}(\bar{K}[x]/(x^2 - t)) \\ &= \text{Spec}(\bar{K}[x]/(x - t^{1/2}) \oplus \bar{K}[x]/(x + t^{1/2})) \\ &= \bar{X}_{\bar{\eta}}^+ \sqcup \bar{X}_{\bar{\eta}}^-, \end{aligned}$$

with $\bar{X}_{\bar{\eta}}^\pm \cong \bar{\eta}$, where the Galois group $\text{Gal}(\bar{\eta}/\eta)$ acts by interchanging these two components.

We will use (73) to compute $R\Phi_{x^2} \mathbb{1} := R\Phi_X \mathbb{1}$. First, note that

$$i_{\bar{X}}^* b_X^* \mathbb{1}_X = i_{\bar{X}}^* \mathbb{1}_{\bar{X}} = \mathbb{1}_{X_s}.$$

The action of $\text{Gal}(\bar{\eta}/\eta)$ on $\mathbb{1}_{\bar{X}_s}$ is trivial. Next, we find $R\Psi_X \mathbb{1}_X$.

$$\begin{aligned} R\Psi_X \mathbb{1}_X &= i_{\bar{X}}^* (j_{\bar{X}})_* b_{X_{\bar{\eta}}}^* j_X^* \mathbb{1}_X \\ &= i_{\bar{X}}^* (j_{\bar{X}})_* \mathbb{1}_{\bar{X}_{\bar{\eta}}} \\ &= i_{\bar{X}}^* (j_{\bar{X}}^+)_* \mathbb{1}_{\bar{X}_{\bar{\eta}}^+} \oplus i_{\bar{X}}^* (j_{\bar{X}}^-)_* \mathbb{1}_{\bar{X}_{\bar{\eta}}^-}, \end{aligned}$$

where $j_{\bar{X}}^\pm : \bar{X}_{\bar{\eta}}^\pm \rightarrow \bar{X}$ is the composition of the component $\bar{X}_{\bar{\eta}}^\pm \rightarrow \bar{X}_{\bar{\eta}}$ and the generic fibre map $j_{\bar{X}} : \bar{X}_{\bar{\eta}} \rightarrow \bar{X}$. Since $j_{\bar{X}}^\pm : \bar{X}_{\bar{\eta}}^\pm \rightarrow \bar{X}$ is an open immersion and X_s is on the boundary of $\bar{X}_{\bar{\eta}}^\pm$ in \bar{X} , we have

$$i_{\bar{X}}^* (j_{\bar{X}}^\pm)_* \mathbb{1}_{\bar{X}_{\bar{\eta}}^\pm} = \mathbb{1}_{X_s}.$$

Therefore,

$$R\Psi_X \mathbb{1}_X = \mathbb{1}_{\bar{X}_s} \oplus \mathbb{1}_{\bar{X}_s}.$$

Note that the monodromy action on $R\Psi_X \mathbb{1}_X$ switches these two summands. Let

$$R\Psi_X \mathbb{1}_X \cong \mathbb{1}_{\bar{X}_s}^+ \oplus \mathbb{1}_{\bar{X}_s}^-$$

be the eigenspace decomposition of $R\Psi_X \mathbb{1}_X$ according to this action, so $\text{Gal}(\bar{\eta}/\eta)$ acts trivially on $\mathbb{1}_{\bar{X}_s}^+$ while $\text{Gal}(\bar{\eta}/\eta)$ acts on $\mathbb{1}_{\bar{X}_s}^-$ through the quadratic character $\text{Gal}(\sqrt{\eta}/\eta) \rightarrow \{\pm 1\}$. The canonical natural transformation $i_{\bar{X}}^* b_X^* \rightarrow R\Psi_{X_{\bar{\eta}}} j_X^*$, which

induces the map at the base of (73), is compatible with monodromy, so

$$i_X^* b_X^* \mathbb{1}_X \rightarrow R\Psi_X \mathbb{1}_X$$

is the isomorphism of $\mathbb{1}_{\bar{X}_s}$ onto $\mathbb{1}_{\bar{X}_s}^+$.

$$\mathbb{1}_{\bar{X}_s} \rightarrow \mathbb{1}_{\bar{X}_s}^+ \oplus \mathbb{1}_{\bar{X}_s}^-.$$

Since $R\Phi_X \mathbb{1}_X$ is the cone of this arrow, we have

$$R\Phi_X \mathbb{1}_X = \mathbb{1}_{\bar{X}_s}^-,$$

as claimed. \square

Lemma 6.2.4. — Set $\mathbb{A}_u^2 = \text{Spec}(\mathbb{C}[x, u]_u)$ and $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$. Let $x^2 u : \mathbb{A}_u^2 \rightarrow \mathbb{A}^1$ be the map defined by $t \mapsto x^2 u$ on coordinate rings. Then

$$R\Phi_{x^2 u} \mathbb{1} = \mathcal{L}_{\mathbb{A}_u^1}$$

with quadratic monodromy, where $\mathbb{A}_u^1 = \text{Spec}(\mathbb{C}[u]_u)$ and where $\mathcal{L}_{\mathbb{A}_u^1}$ is the local system for the quadratic character of the fundamental group of $\mathbb{A}_u^1 = \text{Spec}(\mathbb{C}[u]_u)$. More generally, if $(f, g) : U \rightarrow \mathbb{A}_u^2$ is smooth then

$$R\Phi_{f^2 g} \mathbb{1} = \mathcal{L}_{f=0}$$

where $\mathcal{L}_{f=0}$ is the local system for the quadratic character associated to the cover coming from \sqrt{g} and quadratic monodromy coming from the cover associated to adjoining \sqrt{f} .

Proof. — As before, the second claim follows from the first by smooth base change.

Consider the map $a : \mathbb{A}_u^2 \rightarrow \mathbb{A}_u^2$ defined on coordinate rings by $x \mapsto x^2$ and $u \mapsto u$. Define $b : \mathbb{A}_u^2 \rightarrow \mathbb{A}_u^2$ on coordinate rings by $x \mapsto x$ and $u \mapsto u^2$. Define $c : \mathbb{A}_u^2 \rightarrow \mathbb{A}_u^2$ by $x \mapsto x^2 u^{-1}$ and $u \mapsto u$. The following diagram commutes,

$$\begin{array}{ccccc} & & \mathbb{A}_u^2 & & \\ & \swarrow b & \downarrow d & \searrow a & \\ \mathbb{A}_u^2 & & \mathbb{A}_u^2 & & \mathbb{A}_u^2 \\ & \searrow a & \downarrow c & \swarrow b & \\ & & \mathbb{A}_u^2 & & \end{array}$$

where $d : \mathbb{A}_u^2 \rightarrow \mathbb{A}_u^2$ is defined by $x \mapsto xu$ and $u \mapsto u^2$.

Define $f : \mathbb{A}_u^2 \rightarrow \mathbb{A}^1$ by $t \mapsto xu$ where, $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$. Then $f \circ a = x^2 u$, $f \circ b = xu^2$ and $f \circ c = x^2$; also, $f \circ b \circ a = x^2 u^2$. By base change along $S \rightarrow \mathbb{A}^1$, we

get the following commuting diagram of S -schemes

$$\begin{array}{ccccc}
 & & X_d & & \\
 & \swarrow b'_S & \downarrow d_S & \searrow a'_S & \\
 X_a & & X_c & & X_b \\
 & \searrow a_S & \downarrow c_S & \swarrow b_S & \\
 & & X & &
 \end{array}
 .$$

Over the generic fibre, these maps are all Galois quadratic. However, after base change along $\bar{S} \rightarrow S$ and restriction to special fibres, the maps induced by a and c become isomorphisms, while b and d remain quadratic. Here we use the sequence of equalities $\bar{X}_{a,s} = \bar{X}_{c,s} = \bar{X}_s = \text{Spec}(\mathbb{C}[u]_u)$. Observe that $X_a \rightarrow S$ is the pullback of $x^2u : \mathbb{A}_u^2 \rightarrow \mathbb{A}^1$ along $S \rightarrow \mathbb{A}^1$. Then

$$(74) \quad R\Phi_{x^2u} \mathbb{1} = R\Phi_{X_a} \mathbb{1}.$$

By proper base change,

$$\bar{a}_{s,*} R\Phi_{X_a} \mathbb{1} = R\Phi_X a_* \mathbb{1}.$$

Since \bar{a}_s is an isomorphism, this gives

$$R\Phi_{X_a} \mathbb{1} \cong R\Phi_X a_* \mathbb{1}.$$

Let \mathcal{E} be the local system on $\mathbb{A}_x^1 := \text{Spec}(\mathbb{C}[x]_x)$ defined by the non-trivial character of the covering $\mathbb{A}_x^1 \rightarrow \mathbb{A}_x^1$ given on coordinate rings by $x \mapsto x^2$. Then

$$a_* \mathbb{1} = a_*(\mathbb{1} \boxtimes \mathbb{1}) = (\mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathcal{E}^\natural \boxtimes \mathbb{1}),$$

where \mathcal{E}^\natural is the extension by zero of \mathcal{E} from \mathbb{A}_x^1 to \mathbb{A}^1 . Here, and below, we write $\mathbb{1}$ for the constant sheaf on \mathbb{A}^1 and also on \mathbb{A}_u^1 . By the exactness of $R\Phi_X$,

$$R\Phi_X a_* \mathbb{1} = R\Phi_X(\mathbb{1} \boxtimes \mathbb{1}) \oplus R\Phi_X(\mathcal{E}^\natural \boxtimes \mathbb{1}).$$

By Lemma 6.2.2, $R\Phi_X(\mathbb{1} \boxtimes \mathbb{1}) = 0$. Thus,

$$R\Phi_X a_* \mathbb{1} = R\Phi_X(\mathcal{E}^\natural \boxtimes \mathbb{1}).$$

Our goal, therefore, is to calculate $R\Phi_X(\mathcal{E}^\natural \boxtimes \mathbb{1})$.

To determine $R\Phi_X(\mathcal{E}^\natural \boxtimes \mathbb{1})$, first we find $\bar{b}_{s,*} \bar{a}_{s,*} R\Phi_{X_d} \mathbb{1}$ in two ways. On the one hand,

$$\begin{aligned}
 \bar{b}_{s,*} \bar{a}_{s,*} R\Phi_{X_d} \mathbb{1} &= \bar{b}_{s,*} \bar{a}_{s,*} \mathbb{1} && \text{by Lemma 6.2.3} \\
 &= \bar{b}_{s,*} \mathbb{1} && \text{since } \bar{a}_s \text{ is an isomorphism} \\
 &= (\mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \mathcal{L}) && \text{by the decomposition theorem.}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \bar{b}_{s,*} \bar{a}_{s,*} \mathbf{R}\Phi_{X_d} \mathbb{1} \\
&= \bar{b}_{s,*} \mathbf{R}\Phi_{X_b} a_* \mathbb{1} && \text{by proper base change} \\
&= \bar{b}_{s,*} \mathbf{R}\Phi_{X_b} ((\mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathcal{E}^\natural \boxtimes \mathbb{1})) && \text{by the decomposition theorem} \\
&= \bar{b}_{s,*} (\mathbf{R}\Phi_{X_b} (\mathbb{1} \boxtimes \mathbb{1}) \oplus \mathbf{R}\Phi_{X_c} (\mathcal{E}^\natural \boxtimes \mathbb{1})) && \text{by exactness of } \mathbf{R}\Phi_{X_c} \\
&= \bar{b}_{s,*} \mathbf{R}\Phi_{X_b} (\mathcal{E}^\natural \boxtimes \mathbb{1}) && \text{by Lemma 6.2.2} \\
&= \mathbf{R}\Phi_X b_* (\mathcal{E}^\natural \boxtimes \mathbb{1}) && \text{by proper base change} \\
&= \mathbf{R}\Phi_X ((\mathcal{E}^\natural \boxtimes \mathbb{1}) \oplus (\mathcal{E}^\natural \boxtimes \mathcal{L})) && \text{by the decomposition theorem} \\
&= \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathbb{1}) \oplus \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathcal{L}) && \text{by exactness of } \mathbf{R}\Phi_X.
\end{aligned}$$

So,

$$(75) \quad \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathbb{1}) \oplus \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathcal{L}) = (\mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \mathcal{L}).$$

We now find $\mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathcal{L})$ by computing $\bar{c}_{s,*} \mathbf{R}\Phi_{X_c} \mathbb{1}$ in two ways. On the one hand,

$$\begin{aligned}
\bar{c}_{s,*} \mathbf{R}\Phi_{X_c} \mathbb{1} &= \bar{c}_{s,*} \mathbb{1} && \text{by Lemma 6.2.3} \\
&= \mathbb{1} && \text{since } \bar{c}_s \text{ is an isomorphism.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar{c}_{s,*} \mathbf{R}\Phi_{X_c} \mathbb{1} &= \mathbf{R}\Phi_X c_* \mathbb{1} && \text{by proper base change} \\
&= \mathbf{R}\Phi_X ((\mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathcal{E}^\natural \boxtimes \mathcal{L})) && \text{by the decomposition theorem} \\
&= \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathcal{L}) && \text{by Lemma 6.2.2.}
\end{aligned}$$

So,

$$(76) \quad \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathcal{L}) = \mathbb{1}.$$

Combining (74), (75) and (76) it now follows that

$$\mathbf{R}\Phi_{x^2u} \mathbb{1} = \mathbf{R}\Phi_{X_a} \mathbb{1} = \mathbf{R}\Phi_X (\mathcal{E}^\natural \boxtimes \mathbb{1}) = \mathbb{1} \boxtimes \mathcal{L}.$$

This completes the proof of Lemma 6.2.4 □

Set $\mathbb{A}_{u_1 \dots u_e}^{2e} = \text{Spec}(\mathbb{C}[x_1, \dots, x_e, u_1, \dots, u_e]_{u_1 \dots u_e})$ and let $\mathbb{A}_{u_1 \dots u_e}^e$ be the subvariety cut out by the equations $x_1 = \dots = x_e = 0$.

Proposition 6.2.5. — *Consider the function*

$$f : \mathbb{A}_{u_1 \dots u_e}^{2e} \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[t]) \quad \text{given by} \quad \sum_{i=1}^e u_i x_i^2 \leftarrow t,$$

Then

$$\mathbf{R}\Phi_f \mathbb{1} = z_! \mathcal{L}[1 - e],$$

where $z : \mathbb{A}_{u_1 \dots u_e}^e \rightarrow \mathbb{A}_{u_1 \dots u_e}^{2e}$ is the closed immersion and where \mathcal{L} is the local system on $\mathbb{A}_{u_1 \dots u_e}^e$ for the character of the fundamental group of $\mathbb{A}_{u_1 \dots u_e}^e$ given by the product of the quadratic characters of each factor $\mathbb{A}_{u_i}^1 = \text{Spec}(\mathbb{C}[u_i]_{u_i})$.

Proof. — By definition,

$$\mathbf{R}\Phi_f \mathbb{1} = \mathbf{R}\Phi_X \mathbb{1}_X,$$

for

$$X = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_e, u_1, \dots, u_e]_{u_1 \dots u_e}[[t]] / (\sum_{i=1}^e u_i x_i^2 - t))$$

with the obvious structure map $X \rightarrow S$. Set $X_i = \operatorname{Spec}(\mathbb{C}[x_i, u_i]_{u_i}[[t]])$. By Lemma 6.2.4,

$$\mathbf{R}\Phi_{X_i} \mathbb{1}_{X_i} = \mathcal{L}_i,$$

where \mathcal{L}_i is the local system for the quadratic character of the fundamental group of $\bar{X}_{i,s} = \operatorname{Spec}(\mathbb{C}[u_i]_{u_i})$, for each $i = 1, \dots, e$. It follows from the Sebastiani-Thom isomorphism (see [Ill17] and [Mas01]) that

$$\mathbf{R}\Phi_X[-1] \mathbb{1}_X \cong z_! (\mathbf{R}\Phi_{X_1}[-1] \mathbb{1}_{X_1} \boxtimes \dots \boxtimes \mathbf{R}\Phi_{X_e}[-1] \mathbb{1}_{X_e}),$$

where $z : Z \hookrightarrow \bar{X}_s$ is the closed subvariety $\operatorname{Spec}(\mathbb{C}[u_1, \dots, u_e]_{u_1 \dots u_e})$. In other words,

$$\mathbf{R}\Phi_X \mathbb{1}_X \cong z_! (\mathcal{L}_{\bar{X}_{1,s}} \boxtimes \dots \boxtimes \mathcal{L}_{\bar{X}_{e,s}}) [1 - e].$$

This concludes the proof of Proposition 6.2.5 \square

Corollary 6.2.6. — *Let $xy : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ be the map defined on coordinate rings by $t \mapsto xy$, where $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x, y])$. Then*

$$\mathbf{R}\Phi_{xy} \mathbb{1} = \mathbb{1}_s$$

6.3. Brylinski's functor \mathbf{Ev} . — Let $f : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ be the s -morphism obtained by restriction from the non-degenerate, symmetric J_λ -invariant bilinear form (\mid) of Section 5.2. Let $X_\lambda = T^*(V_\lambda) \times_{\mathbb{A}^1} S$ and let $f_S : X_\lambda \rightarrow S$ be base change of f along $S \rightarrow \mathbb{A}^1$.

For any $\xi_0 \in V_\lambda^*$, define $f_{\xi_0} : V_\lambda \rightarrow \mathbb{A}^1$ by $f_{\xi_0}(x) := f(x, \xi_0)$. Let $f_{\xi_0, S} : X_{\xi_0} \rightarrow S$ be the base change of $f_{\xi_0} : V_\lambda \rightarrow \mathbb{A}^1$ along $S \rightarrow \mathbb{A}^1$, so $X_{\xi_0} := V_\lambda \times_{\mathbb{A}^1} S$. The special fibre of $X_{\xi_0} \rightarrow S$ is denoted by $X_{\xi_0, s}$ and the generic fibre by $X_{\xi_0, \eta}$; note that

$$X_{\xi_0, s} = \{x \in V_\lambda \mid (x \mid \xi_0) = 0\}.$$

We define

$$\mathbf{R}\Phi_{f_{\xi_0}} : \mathbf{D}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{D}_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times_s S)$$

by the following diagram,

$$\begin{array}{ccc} \mathbf{D}_{H_\lambda}(V_\lambda) & \xrightarrow{\mathbf{R}\Phi_{f_{\xi_0}}} & \mathbf{D}_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times_s S) \\ \downarrow \text{forget} & & \uparrow \mathbf{R}\Phi_{X_{\xi_0}} \\ \mathbf{D}_{Z_{H_\lambda}(\xi_0)}(V_\lambda) & \xrightarrow{\text{base change}} & \mathbf{D}_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}) \end{array}$$

where base change refers to pull-back along $X_B \rightarrow V_\lambda$.

In [Bry86, Notation 1.14], Brylinski remarks without proof that there is a functor $\mathbf{Ev} : \mathbf{D}(V) \rightarrow \mathbf{D}(T_H^*(V)_{\text{reg}})$ with the property

$$(77) \quad (\mathbf{Ev} \mathcal{F})_{(x, \xi)} = (\mathbf{R}\Phi_{f_\xi} \mathcal{F})_x,$$

for $(x, \xi) \in T^*(V)_{\text{reg}}$. Some properties \mathbf{Ev} are described in [Bry86, Remarque 1.13], [Bry86, Théorème 1.9] and [Bry86, Proposition 1.15], using results attributed to

[Kas83, Théorème 3.2.5]. Sadly, [Kas83, Théorème 3.2.5] does not exist in the published version of the original notes, and we have not been able to procure the original notes, so we have been obliged to build Ev ourselves and establish its main properties here. In this section we describe Ev and put it in a form which will be useful for calculations. Establishing its main properties will occupy the rest of Section 6.

For any H_λ -orbit $B \subseteq V_\lambda^*$, consider the locally closed subvariety $V_\lambda \times B \subseteq T^*(V_\lambda)$ and let $f_B : V_\lambda \times B \rightarrow \mathbb{A}^1$ be the restriction of $f : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ to $V_\lambda \times B$. Let $f_{B,S} : X_B \rightarrow S$ be the base change of f_B along $S \rightarrow \mathbb{A}^1$. Then the special fibre of X_B is the s -scheme

$$X_{B,s} = \{(x, \xi) \in V_\lambda \times B \mid (x \mid \xi) = 0\}.$$

We write

$$(78) \quad \mathbf{R}\Phi_{f_B} : \mathbf{D}(V_\lambda \times B) \rightarrow \mathbf{D}(f_B^{-1}(0) \times_s S),$$

for the composition of the functor $\mathbf{D}(V_\lambda \times B) \rightarrow \mathbf{D}(X_B)$ induced by pullback along $X_B \rightarrow V_\lambda \times B$ and the vanishing cycles functor

$$(79) \quad \mathbf{R}\Phi_{X_B} : \mathbf{D}(X_B) \rightarrow \mathbf{D}(f_B^{-1}(0) \times_s S).$$

Now, as an s -scheme, $V_\lambda \times B$ comes equipped with an H_λ -action. Applying base change along $S \rightarrow s$ gives an action of $H_\lambda \times_s S$ on $(V_\lambda \times B)_S$. Because f_B is H_λ -invariant, this defines an action of $H_\lambda \times_s S$ on $\{(x, \xi, t) \in (V_\lambda \times B)_S \mid f(x, \xi) = t\}$. But this is precisely X_B so $H_\lambda \times_s S$ acts on X_B in the category of S -schemes and we have the exact functor

$$(80) \quad \mathbf{D}_{H_\lambda}(V_\lambda \times B) \rightarrow \mathbf{D}_{H_\lambda \times_s S}(X_B).$$

See [BL94, Section 2] for the equivariant derived category $\mathbf{D}_H(X)$. Combining this with the vanishing cycles functors above defines an exact functor

$$(81) \quad \mathbf{R}\Phi_{f_B} : \mathbf{D}_{H_\lambda \times_s S}(V_\lambda \times B) \rightarrow \mathbf{D}_{H_\lambda}(f_B^{-1}(0) \times_s S).$$

We may now revisit Brylinski's observation [Bry86, Notation 1.14] and give the main definition for Section 6.

Definition 1. — For any H_λ -orbit $C \subseteq V_\lambda$, let

$$(82) \quad \text{Ev}_C : \mathbf{D}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{D}_{H_\lambda}(T_C^*(V_\lambda)_{\text{reg}} \times_s S)$$

be the functor defined by the diagram

$$(83) \quad \begin{array}{ccccc} \mathbf{D}_{H_\lambda}(V_\lambda) & \xrightarrow{\text{Ev}_C} & \mathbf{D}_{H_\lambda}(T_C^*(V_\lambda)_{\text{reg}} \times_s S) \\ \downarrow (\cdot) \boxtimes \mathbb{1}_{C^*} & & \uparrow \text{restriction} \\ \mathbf{D}_{H_\lambda}(V_\lambda \times C^*) & \xrightarrow{\text{base change}} \mathbf{D}_{H_\lambda \times_s S}(X_{C^*}) & \xrightarrow{\mathbf{R}\Phi_{X_{C^*}}} \mathbf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S), \end{array}$$

where:

- (i) $(\cdot) \boxtimes \mathbb{1}_{C^*} : \mathbf{D}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{D}_{H_\lambda}(V_\lambda \times C^*)$ is the pullback along the projection map $V_\lambda \times C^* \rightarrow V_\lambda$;
- (ii) $\mathbf{D}_{H_\lambda}(V_\lambda \times C^*) \rightarrow \mathbf{D}_{H_\lambda \times_s S}(X_{C^*})$ is (80) in the case $B = C^*$;
- (iii) $\mathbf{R}\Phi_{X_{C^*}} : \mathbf{D}_{H_\lambda \times_s S}(X_{C^*}) \rightarrow \mathbf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S)$ is (79) in the case $B = C^*$; and

(iv) $D_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S) \rightarrow D_{H_\lambda}(T_C^*(V_\lambda)_{\text{reg}} \times_s S)$ is obtained by pullback along the inclusion $T_C^*(V_\lambda)_{\text{reg}} \hookrightarrow f_{C^*}^{-1}(0)$, using Proposition 5.4.3.

Since the Lagrangian varieties $T_C^*(V)_{\text{reg}}$ are components of $T_H^*(V)_{\text{reg}}$, as C runs over H -orbits in V , we may assemble the functors Ev_C to define

$$(84) \quad \text{Ev} : D_{H_\lambda}(V_\lambda) \rightarrow D_{H_\lambda}(T_H^*(V_\lambda)_{\text{reg}} \times_s S).$$

We refer to this as the *microlocal vanishing cycles functor*.

Although it plays a role in our calculations in Part II, our notation here will generally hide the action of inertia.

6.4. Stalks. — We begin our study of the properties of Ev by showing that it is indeed the functor that Brylinski promises in [Bry86, Notation 1.14].

Proposition 6.4.1. — *The functor*

$$\text{Ev} : D_{H_\lambda}(V_\lambda) \rightarrow D_{H_\lambda}(T_H^*(V_\lambda)_{\text{reg}} \times_s S)$$

is exact and for every $\mathcal{F} \in D_{H_\lambda}(V_\lambda)$ and every $(x_0, \xi_0) \in T_H^*(V_\lambda)_{\text{reg}}$, there is a canonical isomorphism

$$(\text{Ev} \mathcal{F})_{(x_0, \xi_0)} \cong (\mathbf{R}\Phi_{f_{\xi_0}} \mathcal{F})_{x_0}$$

compatible with the actions of $Z_H(x_0, \xi_0)$ and $\text{Gal}(\bar{\eta}/\eta)$.

Proof. — With reference to diagram (83), we see that Ev_C is exact since it is defined as the composition of four exact functors. Since the Lagrangian varieties $T_C^*(V)_{\text{reg}}$ are components of $T_H^*(V)_{\text{reg}}$, Ev is also exact.

The S -morphism

$$\begin{aligned} i_{\xi_0} : X_{\xi_0} &\rightarrow X_B \\ x &\mapsto (x, \xi_0) \end{aligned}$$

is equivariant for the $Z_{H_\lambda}(\xi_0) \times_s S$ -action on X_{ξ_0} and the $H_\lambda \times_s S$ -action on X_B . By Lemma 6.4.2, below, this induces an equivalence

$$i_{\xi_0}^* : D_{H_\lambda \times_s S}(X_B) \rightarrow D_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}).$$

Consider the following commuting diagram, for $\xi_0 \in C^*$:

$$\begin{array}{ccccc} D_{H_\lambda \times_s \eta}(f_{C^*}^{-1}(\eta)) & \xrightarrow{(j_{C^*, S})^*} & D_{H_\lambda \times_s S}(X_{C^*}) & \xrightarrow{(i_{C^*, S})^*} & D_{H_\lambda}(f_{C^*}^{-1}(0)) \\ \downarrow \text{equiv. } i_{\xi_0, \eta}^* & & \downarrow \text{equiv. } i_{\xi_0, S}^* & & \downarrow \text{equiv. } i_{\xi_0, s}^* \\ D_{Z_{H_\lambda}(\xi_0) \times_s \eta}(f_{\xi_0}^{-1}(\eta)) & \xrightarrow{(j_{\xi_0, S})^*} & D_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}) & \xrightarrow{(i_{\xi_0, S})^*} & D_{Z_{H_\lambda}(\xi_0)_s}(f_{\xi_0}^{-1}(0)). \end{array}$$

When combined with base change along $\bar{S} \rightarrow S$, it follows that

$$\mathbf{R}\Psi_{f_{C^*}} i_{\xi_0, \eta}^* = i_{\xi_0, s}^* \mathbf{R}\Psi_{f_{\xi_0}}.$$

We find this at the heart of the following commuting diagram, where $(x_0, \xi_0) \in X_{C^*, s}$.

$$\begin{array}{ccc}
D_{H_\lambda}(V_\lambda) & \xrightarrow{p_{C^*}^*} & D_{H_\lambda}(V_\lambda \times C^*) \\
\downarrow & & \downarrow \\
D_{Z_{H_\lambda}(\xi_0) \times_s \eta}(f_{\xi_0}^{-1}(\eta)) & \xleftarrow[i_{\xi_0, \eta}^*]{\text{equiv.}} & D_{H_\lambda \times_s \eta}(f_{C^*}^{-1}(\eta)) \\
\downarrow \text{R}\Psi_{f_{\xi_0}} & & \downarrow \text{R}\Psi_{f_{C^*}} \\
D_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0)) & \xleftarrow[i_{\xi_0, s}^*]{\text{equiv.}} & D_{H_\lambda}(f_{C^*}^{-1}(0)) \\
\downarrow & & \downarrow \\
D_{Z_{H_\lambda}(x_0, \xi_0)}(\{(x_0, \xi_0)\}) & \xleftarrow{\quad} & D_H(T_C^*(V_\lambda)_{\text{sreg}})
\end{array}$$

Thus,

$$(85) \quad (\text{R}\Psi_{f_{C^*}}(\mathcal{F} \boxtimes \mathbb{1}_{C^*}))_{(x_0, \xi_0)} \cong (\text{R}\Psi_{f_{\xi_0}} \mathcal{F})_{x_0},$$

compatible with $Z_{H_\lambda}(x_0, \xi_0)$ -actions. On the other hand,

$$(i_{X_{C^*, s}}^*(\mathcal{F} \boxtimes \mathbb{1}_{C^*}))_{(x_0, \xi_0)} = (\mathcal{F} \boxtimes \mathbb{1}_{C^*})_{(x_0, \xi_0)} = (i_{\xi_0}^* \mathcal{F})_{x_0},$$

as $Z_{H_\lambda}(x_0, \xi_0)$ -spaces. Using (73), it follows that

$$(86) \quad (\text{R}\Phi_{f_{C^*}}(\mathcal{F} \boxtimes \mathbb{1}_{C^*}))_{(x_0, \xi_0)} \cong (\text{R}\Phi_{f_{\xi_0}} \mathcal{F})_{x_0},$$

compatible with the natural $Z_{H_\lambda}(x_0, \xi_0)$ -action. □

Lemma 6.4.2. — *For every $\xi_0 \in B$*

$$X_B \cong (H_\lambda \times_s S) \times_{(Z_{H_\lambda}(\xi_0) \times_s S)} X_{\xi_0}$$

in S -schemes and the closed embedding

$$\begin{aligned}
i_{\xi_0} : X_{\xi_0} &\rightarrow X_B \\
x &\mapsto (x, \xi_0)
\end{aligned}$$

induces an equivalence

$$i_{\xi_0}^* : D_{H_\lambda \times_s S}(X_B) \rightarrow D_{Z_{H_\lambda}(\xi_0) \times_s S}(X_{\xi_0}).$$

Proof. — First we must show that $(H_\lambda \times_s S) \times_{(Z_{H_\lambda}(\xi_0) \times_s S)} X_{\xi_0}$ exists in S -schemes.

To do that, it will be helpful to prove: for every $\delta \in \mathbb{A}^1$ and $\xi_0 \in B$ there is an H_λ -isomorphism

$$f_B^{-1}(\delta) \cong H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$$

in s -schemes, where $H_\lambda \times f_{\xi_0}^{-1}(\delta) \rightarrow H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$ is an $Z_{H_\lambda}(\xi_0)$ -torsor in \mathbb{C} -varieties. Since $Z_{H_\lambda}(\xi_0)$ is a closed subgroup of H_λ , the quotient $H_\lambda \rightarrow H_\lambda/Z_{H_\lambda}(\xi_0)$ exists in \mathbb{C} -varieties. Consider the monomorphism

$$H_\lambda \times f_{\xi_0}^{-1}(\delta) \rightarrow (H_\lambda/Z_{H_\lambda}(\xi_0)) \times T^*(V_\lambda)$$

given by $(h, x) \mapsto (hZ_{H_\lambda}(\xi_0), h \cdot (x, \xi_0))$. Note that $f_{\xi_0}^{-1}(\delta)$ is a closed subvariety of V_λ . The promised $Z_{H_\lambda}(\xi_0)$ -quotient $H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$ is this morphism restricted to the image:

$$H_\lambda \times f_{\xi_0}^{-1}(\delta) \rightarrow \{(hZ_{H_\lambda}(\xi_0), h \cdot (x, \xi_0)) \in (H_\lambda/Z_{H_\lambda}(\xi_0)) \times V_\lambda \times B \mid h^{-1} \cdot x \in f_{\xi_0}^{-1}(\delta)\}.$$

Following standard practice, we use the notation $(h, x) \mapsto [h, x]_{Z_{H_\lambda}(\xi_0)}$ for this map. Now, projection to the second coordinate

$$H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta) \rightarrow f_B^{-1}(\delta)$$

is given by $[h, x]_{Z_{H_\lambda}(\xi_0)} \mapsto h \cdot (x, \xi_0)$, which is the promised isomorphism. This shows that the $Z_{H_\lambda}(\xi_0)$ -torsor $H_\lambda \times f_{\xi_0}^{-1}(\delta) \rightarrow H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta)$ exists in s -schemes and also that the map

$$H_\lambda \times_{Z_{H_\lambda}(\xi_0)} f_{\xi_0}^{-1}(\delta) \rightarrow T^*(V_\lambda), \quad [h, x]_{Z_{H_\lambda}(\xi_0)} \mapsto h \cdot (x, \xi_0),$$

is an H_λ -isomorphism onto $f_B^{-1}(\delta) \subseteq T^*(V_\lambda)$.

Applying pull-back along the flat morphism $S \rightarrow s$ to $Z_{H_\lambda}(\xi_0) \rightarrow H_\lambda \rightarrow H_\lambda/Z_{H_\lambda}(\xi_0)$ determines the cokernel of $Z_{H_\lambda}(\xi_0) \times_s S \rightarrow H_\lambda \times_s S$ and also shows that the local trivialization of $H_\lambda \rightarrow H_\lambda/Z_{H_\lambda}(\xi_0)$ determines a local trivialization of $H_\lambda \times_s S \rightarrow (H_\lambda \times_s S)/(Z_{H_\lambda}(\xi_0) \times_s S)$. Now we may argue as above to see that $(H_\lambda \times_s S) \times_{(Z_{H_\lambda}(\xi_0) \times_s S)} X_{\xi_0} \rightarrow T^*(V_\lambda) \times_s S$, defined by $[h, x]_{Z_{H_\lambda}(\xi_0) \times_s S} \mapsto h \cdot (x, \xi_0)$, is an isomorphism onto X_B over S .

The last part of the lemma now follows immediately by equivariant descent, arguing as in [BL94, Section 2.6.2], for instance. \square

6.5. Support. —

Proposition 6.5.1. — *Let $C \subseteq V_\lambda$ be an H_λ -orbit. If $\mathcal{F} \in \mathcal{D}_{H_\lambda}(V_\lambda)$ then $\mathrm{Ev}_C \mathcal{F} = 0$ unless $C \subseteq \mathrm{supp} \mathcal{F}$.*

Proof. — Set $F = \mathrm{supp} \mathcal{F}$ and note that F is a union of H_λ -orbits. Let $i_F : F \hookrightarrow V_\lambda$ be inclusion. Then

$$\mathcal{F} = (i_F)_!(i_F)^* \mathcal{F}.$$

Since i_F is proper, we may apply Lemma 6.5.2, below, to this case with $W = F$ and $\pi = i_F$ and $g_{C^*} = f|_{F \times C^*}$. Then $\pi' = i_F \times \mathrm{id}_{C^*}$ and

$$g_{C^*}^{-1}(0) = \{(x, \xi) \in F \times C^* \mid (x \mid \xi) = 0\}$$

and

$$(W \times C^*)_{\pi\text{-reg}} = (F \times C^*) \cap T_C^*(V)_{\mathrm{reg}} = T_C^*(V)_{\mathrm{reg}}.$$

Thus,

$$\begin{aligned} \mathrm{Ev}_C \mathcal{F} &= \mathrm{Ev}_C (i_F)_! (i_F)^* \mathcal{F} \\ &= (\mathrm{R}\Phi_{g_{C^*}}((i_F)^* \mathcal{F} \boxtimes \mathbb{1}_{C^*}))|_{T_C^*(V_\lambda)_{\mathrm{reg}}}, \end{aligned}$$

by Lemma 6.5.2. The support of $(i_F)^* \mathcal{F} \boxtimes \mathbb{1}_{C^*}$ is contained in $F \times C^*$, so the support of

$$\mathrm{R}\Phi_{g_{C^*}}((i_F)^* \mathcal{F} \boxtimes \mathbb{1}_{C^*})$$

is contained in $g_{C^*}^{-1}(0) \cap (F \times C^*)$ so the support of $\mathrm{Ev}_C \mathcal{F}$ is contained in

$$T_C^*(V_\lambda)_{\mathrm{reg}} \cap (F \times C^*).$$

Since $T_C^*(V_\lambda)_{\mathrm{reg}} \subseteq C \times C^*$, this is empty unless $C \subseteq F$. \square

Besides its use in Proposition 6.5.1, above, the following result is key to many of the calculations in Part II.

Lemma 6.5.2. — *Suppose $\pi : W \rightarrow V_\lambda$ is proper with fibres of dimension n . Suppose H_λ acts on W and $\pi : W \rightarrow V_\lambda$ is equivariant. Then*

$$\mathrm{Ev}_C \pi_! \mathcal{E} = (\pi_s'')_! \left((\mathrm{R}\Phi_{g_{C^*}} \mathcal{E} \boxtimes \mathbb{1}_{C^*})|_{(W \times C^*)_{\pi\text{-reg}}} \right),$$

where $\pi' := \pi \times \mathrm{id}_{C^*}$, π'_s is its restriction to special fibres, $g_{C^*} := f_{C^*} \circ \pi'$, and π_s'' and $(W \times C^*)_{\pi\text{-reg}}$ are defined by the cartesian diagrams below.

$$\begin{array}{ccc} W & \xleftarrow{p_{C^*}'} & W \times C^* \\ \pi \downarrow & & \pi' \downarrow \quad \curvearrowright \quad g_{C^*} \\ V_\lambda & \xleftarrow{p_{C^*}} & V_\lambda \times C^* \\ & & f_{C^*} \downarrow \\ & & S \end{array} \qquad \begin{array}{ccc} g_{C^*}^{-1}(0) & \xleftarrow{\quad} & (W \times C^*)_{\pi\text{-reg}} \\ \pi_s' \downarrow & & \downarrow \pi_s'' \\ f_{C^*}^{-1}(0) & \xleftarrow{\quad} & T_C^*(V_\lambda)_{\mathrm{reg}} \\ \downarrow & & \downarrow \\ s & & s \end{array}$$

Proof. — Suppose $\mathcal{E} \in \mathrm{D}_{H_\lambda}(W)$. Then $\pi_! \mathcal{E} \in \mathrm{D}_{H_\lambda}(V_\lambda)$. Let $p_{C^*} : V_\lambda \times C^* \rightarrow V_\lambda$ be projection. Then, by repeated application of proper base change [DK73, Exposé XIV, 2.1.7.1],

$$\begin{aligned} \mathrm{Ev}_C \pi_! \mathcal{E} &= (\mathrm{R}\Phi_{f_{C^*}} p_{C^*}^* \pi_! \mathcal{E})|_{T_C^*(V_\lambda)_{\mathrm{reg}}} \\ &= (\mathrm{R}\Phi_{f_{C^*}} (\pi')_! (p_{C^*}')^* \mathcal{E})|_{T_C^*(V_\lambda)_{\mathrm{reg}}} \\ &= ((\pi_s')_! \mathrm{R}\Phi_{g_{C^*}} (p_{C^*}')^* \mathcal{E})|_{T_C^*(V_\lambda)_{\mathrm{reg}}} \\ &= (\pi_s'')_! \left((\mathrm{R}\Phi_{g_{C^*}} (\mathcal{E} \boxtimes \mathbb{1}_{C^*}))|_{(W \times C^*)_{\pi\text{-reg}}} \right). \end{aligned} \quad \square$$

6.6. Open orbit. —

Lemma 6.6.1. — *For every H -orbit $C \subseteq V$ and every H -equivariant local system \mathcal{L} on C ,*

$$\mathrm{Ev}_C \mathcal{I}(C, \mathcal{L}) = (\mathrm{R}\Phi_{f|_{C \times C^*}} (\mathcal{L} \boxtimes \mathbb{1}_{C^*}))_{T_C^*(V)_{\mathrm{reg}}} [\dim C].$$

Proof. — By the definition of Ev_C given in (83),

$$\mathrm{Ev}_C \mathcal{I}(C, \mathcal{L}) = (\mathrm{R}\Phi_{f_C}^* (\mathcal{I}(C, \mathcal{L}) \boxtimes \mathbb{1}_{C^*}))_{T_C^*(V)_{\mathrm{reg}}}.$$

Using Proposition 5.4.3 and proper base change for $\bar{C} \hookrightarrow V$ as in Lemma 6.5.2 gives

$$(\mathrm{R}\Phi_{f_C}^* (\mathcal{I}(C, \mathcal{L}) \boxtimes \mathbb{1}_{C^*}))_{T_C^*(V)_{\mathrm{reg}}} = \left(\mathrm{R}\Phi_{f|_{\bar{C} \times C^*}} (\mathcal{L}^\sharp[\dim C] \boxtimes \mathbb{1}_{C^*}) \right)_{T_C^*(V)_{\mathrm{reg}}},$$

using the notation $\mathcal{L}^\sharp = \mathcal{IC}(C, \mathcal{L})|_{\bar{C}}[-\dim C]$. Since $C \subseteq V$ is locally closed, it is relatively open in its closure. By smooth base change for $C \hookrightarrow \bar{C}$,

$$\left(\mathbf{R}\Phi_{f|_{C \times C^*}}(\mathcal{L}^\sharp \boxtimes \mathbb{1}_{C^*}) \right)_{T_C^*(V)_{\text{reg}}} = \left(\mathbf{R}\Phi_{f|_{C \times C^*}}(\mathcal{L} \boxtimes \mathbb{1}_{C^*}) \right)_{T_C^*(V)_{\text{reg}}}.$$

This proves the lemma. \square

Proposition 6.6.2. — *Let $C \subseteq V$ be an H -orbit.*

(a) *For any $(x, \xi) \in T_C^*(V)_{\text{reg}}$,*

$$(\text{Ev}_C \mathcal{IC}(C))_{(x, \xi)} = (\mathbf{R}\Phi_{\xi|_C} \mathbb{1}_C)_x [\dim C],$$

as representations of $Z_H(x, \xi)$.

(b) *For any H -equivariant local system \mathcal{L} on C ,*

$$\text{Ev}_C \mathcal{IC}(C, \mathcal{L}) = \text{Ev}_C \mathcal{IC}(C) \otimes (\mathcal{L} \boxtimes \mathbb{1}_{C^*})|_{T_C^*(V)_{\text{reg}}}.$$

Proof. — By Lemma 6.6.1,

$$\text{Ev}_C \mathcal{IC}(C, \mathcal{L}) = (\mathbf{R}\Phi_{f|_{C \times C^*}}(\mathcal{L} \boxtimes \mathbb{1}_{C^*}))_{T_C^*(V)_{\text{reg}}} [\dim C].$$

Taking the case $\mathcal{L} = \mathbb{1}_C$ and passing to stalks using Proposition 6.4.1 gives

$$(\text{Ev}_C \mathcal{IC}(C))_{(x, \xi)} = (\mathbf{R}\Phi_{\xi|_C} \mathbb{1}_C)_x [\dim C],$$

for every $(x, \xi) \in T_C^*(V)_{\text{reg}}$. This proves (a).

To simplify notation slightly, set $\mathcal{E} := \mathcal{L} \boxtimes \mathbb{1}_{C^*}$. It follows from Lemma 6.6.3, below, that

$$\mathbf{R}\Psi_{f|_{C \times C^*}} \mathcal{E} = \mathcal{E}|_{f|_{C \times C^*}^{-1}(0)} \otimes \mathbf{R}\Psi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*}.$$

To see this, let $\bar{X} = X \times_S \bar{S}$ play the role of X in Lemma 6.6.3, let $X_{\bar{\eta}}$ play the role of U , so \bar{j} plays the role of j , take $\mathcal{H} = \mathbb{1}_{\bar{X}}$ and $\mathcal{G} = \mathcal{E}$. It now follows from (73) that

$$\mathbf{R}\Phi_{f|_{C \times C^*}}(\mathcal{E}) = \mathcal{E}|_{f|_{C \times C^*}^{-1}(0)} \otimes \mathbf{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*},$$

since

$$\mathcal{E}|_{f|_{C \times C^*}^{-1}(0)} = \mathcal{E}|_{f|_{C \times C^*}^{-1}(0)} \otimes \mathbb{1}_{f|_{C \times C^*}^{-1}(0)}.$$

Using Proposition 5.4.3 again and restricting from $f|_{C \times C^*}^{-1}(0)$ to $T_C^*(V)_{\text{reg}}$ now gives

$$(\mathbf{R}\Phi_{f|_{C \times C^*}} \mathcal{E})|_{T_C^*(V)_{\text{reg}}} = \mathcal{E}|_{T_C^*(V)_{\text{reg}}} \otimes (\mathbf{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*})|_{T_C^*(V)_{\text{reg}}}$$

Using Lemma 6.6.1 again, this proves (b). \square

Lemma 6.6.3. — *Let $j : U \hookrightarrow X$ be an open immersion. Let \mathcal{G} and \mathcal{H} be local systems on X trivialized by a finite étale cover of X . $\mathcal{F} = j^* \mathcal{H}$. Then the canonical morphism*

$$j_* \mathcal{F} \otimes \mathcal{G} \rightarrow j_*(\mathcal{F} \otimes j^* \mathcal{G})$$

is an isomorphism.

Proof. — The canonical morphism above comes from the unit $1 \rightarrow j_* j^*$ of the adjunction for the pair (j^*, j_*) the exactness of j^* , and the co-unit $j^* j_* \rightarrow 1$:

$$j_* \mathcal{F} \otimes \mathcal{G} \rightarrow j_* j^*(j_* \mathcal{F} \otimes \mathcal{G}) \cong j_*(j^* j_* \mathcal{F} \otimes j^* \mathcal{G}) \rightarrow j_*(\mathcal{F} \otimes j^* \mathcal{G}).$$

To show that this is an isomorphism it is sufficient to show that this induces an isomorphism on stalks. Note that any sheaf homomorphism obtained from the unit $1 \rightarrow j_*j^*$ is a monomorphism while any sheaf homomorphism obtained from the co-unit $j^*j_* \rightarrow 1$ is an isomorphism, so the canonical morphism is injective.

Without loss of generality, we may assume X is connected.

First we show that the lemma is true for constant sheaves. Set $\mathcal{G} = \mathbb{1}_X^m$ and $\mathcal{H} = \mathbb{1}_X^n$ so $\mathcal{F} = \mathbb{1}_U^n$. The stalks of the domain of the canonical morphism at $x \in U$ are $(j_*\mathbb{1}_U^n \otimes \mathbb{1}_X^m)_x = (j_*\mathbb{1}_U^n)_x \otimes \mathbb{1}^m = H^\bullet(j^{-1}(x), \mathbb{1}^n) \otimes \mathbb{1}^m = \mathbb{1}^{nm}[0]$ while the stalks of the codomain are $(j_*(\mathbb{1}_U^n \otimes j^*\mathbb{1}_X^m))_x = (j_*(\mathbb{1}_U^n \otimes \mathbb{1}_U^m))_x = H^\bullet(j^{-1}(x), \mathbb{1}^{nm}) = \mathbb{1}^{nm}[0]$. The case $x \notin U$ is trivial.

Now let $\pi : \tilde{X} \rightarrow X$ be a finite etale cover that trivializes \mathcal{G} . Then the canonical morphism $j_*\mathcal{F} \otimes \mathcal{G} \rightarrow j_*(\mathcal{F} \otimes j^*\mathcal{G})$ induces

$$(87) \quad \pi^*(j_*\mathcal{F} \otimes \mathcal{G}) \rightarrow \pi^*j_*(\mathcal{F} \otimes j^*\mathcal{G}).$$

Note that (87) is injective on stalks. Let $j_\pi : \tilde{U} \rightarrow \tilde{X}$ be the pullback of j along π and let $\pi_j : \tilde{U} \rightarrow U$ be the pullback of π along j . Note that j_π is an open immersion and π_j is again a finite etale cover. The local system $j^*\mathcal{G}$ is trivialized by π_j since

$$\pi_j^*j^*\mathcal{G} = (j \circ \pi_j)^*\mathcal{G} = (\pi \circ j_\pi)^*\mathcal{G} = (j_\pi)^*\pi^*\mathcal{G} = (j_\pi)^*\mathbb{1}_X^m = \mathbb{1}_{\tilde{U}}^m,$$

where $m = \text{rank } \mathcal{G}$; likewise \mathcal{F} is trivialized by π_j . By the exactness of π^* and proper base change,

$$\pi^*(j_*\mathcal{F} \otimes \mathcal{G}) = \pi^*j_*\mathcal{F} \otimes \pi^*\mathcal{G} = (j_\pi)_*\pi_j^*\mathcal{F} \otimes \mathbb{1}_{\tilde{X}}^m = (j_\pi)_*\mathbb{1}_{\tilde{X}}^n \otimes \mathbb{1}_{\tilde{X}}^m$$

where $n = \text{rank } \mathcal{F}$. Likewise,

$$\pi^*j_*(\mathcal{F} \otimes j^*\mathcal{G}) = (j_\pi)_*\pi_j^*(\mathcal{F} \otimes j^*\mathcal{G}) = (j_\pi)_*(\mathbb{1}_{\tilde{X}}^n \otimes j_\pi^*\mathbb{1}_{\tilde{U}}^m).$$

This reduces the general case to the case of constant sheaves, already proved. \square

6.7. Purity and Rank on the open orbit. — In this section we show that, for every H -orbit $C \subseteq V$, the object $\text{Ev}_C \mathcal{IC}(C)$ in $D_H(T_C^*(V)_{\text{reg}})$ is cohomologically concentrated in dimension $\dim C^* - \dim V - 1$, where it is a rank-1 constructible sheaf; see Theorem 6.7.5.

Recall $f : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ from Section 6.3. Having fixed C , in this section we use the notation $\Lambda_C := T_C^*(V) \cap (C \times C^*)$ and set $g := f|_{C \times C^*}$. We also set $e = e_C = \dim C + \dim C^* - \dim V$, the codimension of Λ_C in $C \times C^*$ and refer to this as the *eccentricity* of C .

Lemma 6.7.1. — *The singular locus of $f|_{C \times C^*} : C \times C^* \rightarrow \mathbb{A}^1$ contains Λ_C .*

Proof. — Suppose $(x, \xi) \in \Lambda_C$; we must show that $dg_{(x, \xi)} : T_{(x, \xi)}(C \times C^*) \rightarrow T_0(\mathbb{A}^1)$ is trivial, where $g : C \times C^* \rightarrow \mathbb{A}^1$ is the restriction of f to $C \times C^*$. By Lemma 5.3.2, $\xi \in T_{C, x}^*(V)$, we have $\langle y, \xi \rangle_x = 0$ for all $y \in T_x(C)$, where $\langle \cdot, \cdot \rangle_x$ is the pairing on $T_x(V) \times T_x^*(V)$. Similarly, $x \in T_{C^*, \xi}^*(V^*)$, so $\langle x, \nu \rangle_\xi = 0$ for any $\nu \in T_\xi(C^*)$, where $\langle \cdot, \cdot \rangle_\xi$ is the pairing on $T_\xi^*(V^*) \times T_\xi(V^*)$. Since $dg_{(x, \xi)}(y, \nu) = \langle y, \xi \rangle_x + \langle x, \nu \rangle_\xi = 0$, it now follows that $dg_{(x, \xi)} : T_{(x, \xi)}(C \times C^*) \rightarrow T_0(\mathbb{A}^1)$ is trivial. \square

We now make a study of $f|_{C \times C^*} \rightarrow \mathbb{A}^1$ at regular points $(x, \xi) \in T_C^*(V)_{\text{reg}}$. Let \mathcal{I} be the ideal sheaf for the closed subvariety Λ_C in $C \times C^*$. Using the regularity of $(x, \xi) \in T_C^*(V)_{\text{reg}}$, we may choose an open affine neighbourhood U of (x, ξ) in $C \times C^*$ such that $U \cap \Lambda_C$ is an open affine neighbourhood of (x, ξ) in Λ_C . The sequence

$$0 \rightarrow \mathcal{I}(U) \rightarrow \mathcal{O}_{C \times C^*}(U) \rightarrow \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C) \rightarrow 0$$

is exact. Let $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U)$ be the completion of $\mathcal{O}_{C \times C^*}(U)$ with respect to the ideal $\mathcal{I}(U)$; then

$$(88) \quad \widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U) \cong \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[z_1, \dots, z_e]]$$

where e is the codimension of Λ_C in $C \times C^*$; see for instance, [AM69, Theorem 11.22 and Remark 2]. Recall $g := f|_{C \times C^*}$. We denote the image of g in $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U)$ by \hat{g}_U . Using multindex notation, \hat{g}_U may be written in the form

$$\hat{g}_U = \sum_I a_I z^I,$$

with $I = (i_1, \dots, i_e)$ where $i_j \in \mathbb{N}$; here, $a_I = a_I \in \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)$.

Again using the regularity of (x, ξ) , the completion of \mathcal{O}_{Λ_C} at (x, ξ) is

$$\widehat{\mathcal{O}}_{\Lambda_C, (x, \xi)} \cong \mathbb{C}[[y_1, \dots, y_d]]$$

where $d = \dim \Lambda_C$ and

$$(89) \quad \widehat{\mathcal{O}}_{C \times C^*, (x, \xi)} \cong \widehat{\mathcal{O}}_{\Lambda_C, (x, \xi)}[[z_1, \dots, z_e]].$$

Note that this defines a splitting of the exact sequence

$$0 \rightarrow \widehat{\mathcal{I}}_{(x, \xi)} \rightarrow \widehat{\mathcal{O}}_{C \times C^*, (x, \xi)} \rightarrow \widehat{\mathcal{O}}_{\Lambda_C, (x, \xi)} \rightarrow 0.$$

Let $\hat{g}_{(x, \xi)}$ be the image of $f|_{C \times C^*}$ in $\widehat{\mathcal{O}}_{C \times C^*, (x, \xi)}$. Then

$$(90) \quad \hat{g}_{(x, \xi)} = \sum_{|I| \geq 2} \hat{a}_I(y) z^I$$

where $\hat{a}_I(y) \in \widehat{\mathcal{O}}_{\Lambda_C, (x, \xi)}$ is the image of a_I under $\mathcal{O}_{\Lambda_C}(U \cap \Lambda_C) \rightarrow \widehat{\mathcal{O}}_{\Lambda_C, (x, \xi)}$.

Set $|I| := i_1 + \dots + i_e$. By Lemma 5.3.2, $f|_{C \times C^*} : C \times C^* \rightarrow \mathbb{A}^1$ vanishes on Λ_C so $\hat{a}_I(y) = 0$ for $|I| = 0$. By Lemma 6.7.1, $f|_{C \times C^*} : C \times C^* \rightarrow \mathbb{A}^1$ is singular along Λ_C so $\hat{a}_I(y) = 0$ for all $|I| = 1$. Consequently,

$$(91) \quad \hat{g}_{(x, \xi)} = \sum_{|I| \geq 2} \hat{a}_I(y) z^I.$$

Lemma 6.7.2. — *If $(x, \xi) \in T_C^*(V)_{\text{reg}}$ then the rank of the Hessian for the function $f|_{C \times C^*} : C \times C^* \rightarrow S$ at (x, ξ) is most $\dim C + \dim C^* - \dim V$.*

Proof. — As above, set $g := f|_{C \times C^*}$ and $e := \dim C + \dim C^* - \dim V$. First, observe that the Hessian for g at (x, ξ) is determined by the image $\hat{g}_{(x, \xi)}$ of g under the map $\mathcal{O}_{C \times C^*}(C \times C^*) \rightarrow \widehat{\mathcal{O}}_{C \times C^*, (x, \xi)}$:

$$\mathcal{H}(g)_{(x, \xi)} = \mathcal{H}(\hat{g}_{(x, \xi)})_0.$$

Recall from (90) that we may write $\hat{g}_{(x,\xi)}$ in the form

$$\hat{g}_{(x,\xi)} = \sum_{|I| \geq 2} \hat{a}_I(y) z^I.$$

We now break the Hessian for $\hat{g}_{(x,\xi)}$ at 0 into blocks:

$$\mathcal{H}(\hat{g}_{(x,\xi)})_0 = \begin{pmatrix} A & B \\ {}^t B & A' \end{pmatrix},$$

where A and A' are the matrices of partial derivatives

$$A_{ij} = \frac{\partial^2}{\partial y_i \partial y_j} \left(\sum_{|I| \geq 2} \hat{a}_I(y) z^I \right) \Big|_{(0,0)},$$

for $1 \leq i, j \leq d$, and

$$A'_{ij} = \frac{\partial^2}{\partial z_i \partial z_j} \left(\sum_{|I| \geq 2} \hat{a}_I(y) z^I \right) \Big|_{(0,0)},$$

for $1 \leq i, j \leq e$, and where B is the matrix of mixed partial derivatives

$$B_{ij} = \frac{\partial^2}{\partial y_i \partial z_j} \left(\sum_{|I| \geq 2} \hat{a}_I(y) z^I \right) \Big|_{(0,0)}$$

for $1 \leq i \leq d$ and $1 \leq j \leq e$. Now

$$A_{ij} = \sum_{|I| \geq 2} \left(\frac{\partial^2}{\partial y_i \partial y_j} \hat{a}_I(y) \right) z^I \Big|_{(0,0)} = 0$$

because $z^I|_0 = 0$ for all $|I| \geq 2$, and

$$B_{ij} = \sum_{|I| \geq 2} \left(\frac{\partial \hat{a}_I(y)}{\partial y_i} \Big|_0 \right) \left(\frac{\partial z^I}{\partial z_j} \Big|_0 \right) = 0,$$

because $\frac{\partial z^I}{\partial z_j} \Big|_0 = 0$ for all $|I| \geq 2$. Therefore,

$$\text{rank } \mathcal{H}(\hat{g}_{(x,\xi)})_0 = \text{rank } A'.$$

Since

$$\text{rank } A' \leq e = \dim C + \dim C^* - \dim V,$$

this concludes the proof of Lemma 6.7.2. \square

Lemma 6.7.3. — *If $(x, \xi) \in T_C^*(V)_{\text{reg}}$ then the rank of the Hessian for the function $f|_{C \times C^*} : C \times C^* \rightarrow \mathbb{A}^1$ at (x, ξ) is least $\dim C + \dim C^* - \dim V$.*

Proof. — In this proof we use the analytic site. Consider the covering

$$\begin{aligned} H \times H &\rightarrow C \times C^* \\ (h, h') &\mapsto (\text{Ad}(h)(x), \text{Ad}(h')(\xi)) \end{aligned}$$

After passing to a neighbourhood of $(1, 1)$, pullback through the exponential map $\exp : \mathfrak{h} \rightarrow H$ to define

$$\begin{aligned} G : \mathfrak{h} \times \mathfrak{h} &\rightarrow C \times C^* \\ (z, z') &\mapsto (\text{Ad}(\exp(z))(x), \text{Ad}(\exp(z'))(x)). \end{aligned}$$

Then

$$\text{rank } \mathcal{H}(g)_{(x, \xi)} = \text{rank } \mathcal{H}(\hat{g}_{(x, \xi)})_0 = \text{rank } \mathcal{H}(G)_{(0, 0)},$$

where $\mathcal{H}(G)_{(0, 0)}$ is the Hessian of G at $(0, 0)$. Recall that

$$\text{Ad}(\exp(z))(x) = x + [z, x] + \frac{1}{2}[z, [z, x]] + \cdots + \frac{1}{n!}[z, [z, \dots, [z, x] \cdots]] + \cdots$$

in the formal neighbourhood of $0 \in \mathfrak{h}$; likewise for $\text{Ad}(\exp(z'))(\xi)$. Define $Z(z)$ and $Z'(z')$ by

$$\text{Ad}(\exp(z))(x) = x + [z, x] + Z(z) \quad \text{and} \quad \text{Ad}(\exp(z'))(\xi) = \xi + [z', \xi] + Z'(z').$$

Then

$$\begin{aligned} G(z, z') &= (x | [z', \xi]) + ([z, x] | \xi) \\ &\quad + (Z(z) | \xi) + ([z, x] | [z', \xi]) + (x | Z'(z')) \\ &\quad + ([z, x] | Z'(z')) + (Z(z) | Z'(z')) + (Z(z) | [z', \xi]) \end{aligned}$$

Since the second-order part of $G(z, z')$ is $(Z(z) | \xi) + ([z, x] | [z', \xi]) + (x | Z'(z'))$, the Hessian of G at $(0, 0)$ takes the form

$$\mathcal{H}(G)_{(0, 0)} = \begin{pmatrix} M & N \\ {}^t N & M' \end{pmatrix},$$

where M and M' are the matrices of partial derivatives

$$M_{ij} = \frac{\partial^2}{\partial z_i \partial z_j} (Z(z) | \xi) \Big|_{(0, 0)}, \quad \text{and} \quad M'_{ij} = \frac{\partial^2}{\partial z'_i \partial z'_j} (x | Z'(z')) \Big|_{(0, 0)},$$

and where N is the matrix of mixed partial derivatives

$$N_{ij} = \frac{\partial^2}{\partial z_i \partial z'_j} ([z, x] | [z', \xi]) \Big|_{(0, 0)}.$$

Thus,

$$\text{rank } \mathcal{H}(G)_{(0, 0)} \geq \text{rank } N.$$

In fact, the matrix N is the matrix for the bilinear form

$$\begin{aligned} \mathfrak{h} \times \mathfrak{h} &\rightarrow \mathbb{A}^1 \\ (z, z') &\mapsto ([z, x] | [z', \xi]). \end{aligned}$$

Since

$$([z, x] | [z', \xi]) = ([\xi, [z, x]] | z'),$$

the rank of N is $\dim[\xi, [\mathfrak{h}, x]]$.

We now show that $\dim[\xi, [\mathfrak{h}, x]]$ is $\dim C + \dim C^* - \dim V$. First, note that

$$[\mathfrak{h}, x] = T_x(C) \quad \text{and} \quad \ker[\xi, \cdot]_V = T_{C^*, \xi}^*(V^*),$$

so

$$\dim[\xi, [\mathfrak{h}, x]] = \dim T_x(C) - \dim T_{C^*, \xi}^*(V^*) = \dim C - \dim T_{C^*, \xi}^*(V^*).$$

Since (x, ξ) is regular, $T_{C^*, \xi}^*(V^*) \cap (C \times \{\xi\}) = \{(y, \xi) \in C \times \{\xi\} \mid [y, \xi] = 0\}$ contains an open neighbourhood of (x, ξ) in $T_{C^*, \xi}^*(V^*) = \{(y, \xi) \in V \times \{\xi\} \mid [y, \xi] = 0\}$. Hence $T_{C^*, \xi}^*(V^*) \subseteq T_x(C)$ and

$$\dim T_{C^*, \xi}^*(V^*) = \dim V - \dim T_\xi(C^*) = \dim V - \dim C^*.$$

Therefore,

$$\dim[\xi, [\mathfrak{h}, x]] = \dim C - (\dim V - \dim C^*),$$

which concludes the proof of Lemma 6.7.3. \square

We remark that $[\xi, [\mathfrak{h}, x]]$ is also the image of the map $T_{(x, \xi)}(C \times C^*) \rightarrow \mathfrak{h}$ given by $(y, \nu) \mapsto [x, \nu] + [y, \xi]$, so the proof of Lemma 6.7.3 also shows that

$$T_{(x, \xi)}(\Lambda_C) = \{(y, \nu) \in T_{(x, \xi)}(C \times C^*) \mid [x, \nu] + [y, \xi] = 0\}$$

for $(x, \xi) \in T_C^*(V)_{\text{reg}}$.

Lemma 6.7.4. — Recall the definition of $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U)$ from (88). The open affine $U \subset C \times C^*$ may be chosen so that there is an isomorphism

$$\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U) \cong \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[x_1, \dots, x_e]],$$

such that the image of $f|_{C \times C^*} : C \times C^* \rightarrow \mathbb{A}^1$ in $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U)$ is

$$u_1 x_1^2 + \dots + u_e x_e^2$$

for $u_1, \dots, u_e \in \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)$. Here, $e = \dim C + \dim C^* - \dim V$.

Proof. — We have seen that we may choose U and arrange so that the image of the function $g : C \times C^* \rightarrow \mathbb{A}^1$ in $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U) = \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[z_1, \dots, z_e]]$ will have the form $\hat{g}_U = \sum_{|I| \geq 2} a_I z^I$. Set $A = \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[z_2, \dots, z_e]]$ and let \mathfrak{m} be the ideal in A generated by z_2, \dots, z_e . Write $g_{(x, \xi)} = \sum_{n=0}^{\infty} b_n z_1^n$. Then $b_1 \in \mathfrak{m}$ and $b_2 \in A^*$. Make the substitution $z_1 = w_1 + x$ to give $\hat{g}_{(x, \xi)} = \sum_{n=0}^{\infty} c_n w_1^n$ for $c_n \in A[[x]]$. Importantly, $c_1(x) = b_1 + 2b_2x + \dots$ so $c'_1(x) \in A^*$ (formal derivative). Since we know that $c_1(x) \equiv 2b_2(x) + \dots \pmod{\mathfrak{m}}$, we know $c_1(x)$ has a root $x + \mathfrak{m}$ in A/\mathfrak{m} . By the extension of Hensel's lemma to formal power series, $c_1(x) = 0$ has a solution in A , call it $h_1 \in A$. Now the linear substitution $z_1 \mapsto x_1 + h_1$ sends $\mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[z_1, \dots, z_e]]$ to $\mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)[[x_1, z_2, \dots, z_e]]$ so that the image of $\hat{g}_{(x, \xi)}$ takes the form $b_0 + b_2 x_1^2 + b_3 x_1^3 + \dots = b_0 + u_1 x_1^2$. As an element of $A[[x_1]]$, now $g_{(x, \xi)}$ has no linear term in x_1 and $u_1 \in A[[x_1]]^*$. Continuing inductively concludes the proof of Lemma 6.7.4. \square

Theorem 6.7.5. — For every H_λ -orbit $C \subseteq V_\lambda$ and for all $(x, \xi) \in T_C^*(V)_{\text{reg}}$,

$$(\text{Ev}_C \mathcal{I}(C))_{(x, \xi)} \cong \mathcal{L}_{(x, \xi)}[\dim C + 1 - e_C]$$

where $e_C = \dim C + \dim C^* - \dim V$ and $\mathcal{L}_{(x, \xi)}$ the stalk of a local system for a quadratic character, described in the proof, of an étale neighbourhood of (x, ξ) in $T_C^*(V)_{\text{reg}}$. In particular

$$\text{rank Ev}_C \mathcal{I}(C) = 1.$$

Proof. — By Lemma 6.6.1,

$$\mathrm{Ev}_C \mathcal{IC}(C) = (\mathrm{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*})_{T_C^*(V)_{\mathrm{reg}}} [\dim C].$$

Let U be an open affine neighbourhood of (x, ξ) in $C \times C^*$ and recall the definition $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U)$ from (88). Recall $u_1, \dots, u_e \in \mathcal{O}_{\Lambda_C}(U \cap \Lambda_C)$ from Lemma 6.7.4. Set $U' = U_{u_1 \dots u_e}$; this is again an open affine neighbourhood of (x, ξ) in $C \times C^*$. Let

$$j : \mathrm{Spec}(\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U')) \rightarrow C \times C^*$$

be the map induced by $\mathcal{O}_{C \times C^*}(U) \rightarrow \widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U')$. Then

$$(\mathrm{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*})_{(x, \xi)} = j^* (\mathrm{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*})_0$$

By Lemma 6.7.4, there is an isomorphism

$$\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U') \cong \mathcal{O}_{\Lambda_C}(U' \cap \Lambda_C)[[x_1, \dots, x_e]]$$

where $d = \dim V$ and $e = \dim C + \dim C^* - \dim V$, such that the image of $f|_{C \times C^*}$ in $\widehat{\mathcal{O}}_{C \times C^* / \Lambda_C}(U')$ is $u_1 x_1^2 + \dots + u_e x_e^2$. Note that by our choice of U' , we now have $u_1, \dots, u_e \in \mathcal{O}_{\Lambda_C}(U' \cap \Lambda_C)^*$. By smooth base change,

$$j^* (\mathrm{R}\Phi_{f|_{C \times C^*}} \mathbb{1}_{C \times C^*})_0 = \left(\mathrm{R}\Phi_{u_1 x_1^2 + \dots + u_e x_e^2} \mathbb{1} \right)_0.$$

By definition,

$$\mathrm{R}\Phi_{u_1 x_1^2 + \dots + u_e x_e^2} \mathbb{1} = \mathrm{R}\Phi_X \mathbb{1}_X$$

where

$$X = \mathrm{Spec}(\mathcal{O}_{\Lambda_C}(U' \cap \Lambda_C)[[t]][x_1, \dots, x_e] / (u_1 x_1^2 + \dots + u_e x_e^2 - t)).$$

Write $X = Y \times_S Z$ with

$$Y = \mathrm{Spec}(\mathcal{O}_{\Lambda_C}(U' \cap \Lambda_C)[[t]])$$

and

$$Z = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_e, u_1, \dots, u_e]_{u_1 \dots u_e}[[t]] / (u_1 x_1^2 + \dots + u_e x_e^2 - t)).$$

Now, by smooth base change, the Sebastiani-Thom isomorphism, and Lemma 6.2.1,

$$\mathrm{R}\Phi_X \mathbb{1}_X = \mathbb{1}_{\bar{Y}_s} \boxtimes \mathrm{R}\Phi_Z \mathbb{1}_Z.$$

Using Proposition 6.2.5, it now follows that

$$(\mathrm{Ev}_C \mathcal{IC}(C))_{(x, \xi)} = (\mathrm{R}\Phi_Z \mathbb{1}_X)_0 [\dim C] \cong \mathcal{L}_0[\dim C + 1 - e]$$

where \mathcal{L} is the rank-1 local system described in Proposition 6.2.5. This concludes the proof of Theorem 6.7.5. \square

6.8. Perversity. — We now show that when shifted by the appropriate degree, the functor Ev_C takes perverse sheaves to perverse sheaves.

Proposition 6.8.1. — *Let $C \subseteq V_\lambda$ be an H_λ -orbit. If $\mathcal{P} \in \mathrm{Per}_{H_\lambda}(V_\lambda)$ then*

$$\mathrm{Ev}_C \mathcal{P}[\dim C^* - 1] \in \mathrm{Per}_{H_\lambda}(T_C^*(V)_{\mathrm{reg}}).$$

Proof. — From (83) recall that $\mathrm{Ev}[\dim C^* - 1] : \mathrm{D}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{D}_H(T_C^*(V)_{\mathrm{reg}})$ is defined by

$$\mathrm{Ev}[\dim C^* - 1]\mathcal{P} = (\mathrm{R}\Phi_{f_{C^*}}[-1](\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*]))|_{T_C^*(V)_{\mathrm{reg}}}.$$

Since C^* is smooth, $\mathbb{1}_{C^*}[\dim C^*]$ is perverse, so $\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*]$ is a perverse sheaf on $V_\lambda \times C^*$; see also [BBD82, 4.2.4]. The restriction of $\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*]$ from $V_\lambda \times C^*$ to the open $(\cdot| \cdot)^{-1}(\mathbb{A}_\lambda^1)$ is also perverse. It follows from [BBD82, Proposition 4.4.2] that $\mathrm{R}\Phi_{f_{C^*}}[-1](\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*])$ is perverse; see also [Bry86, Théorème 1.2]. It is also H_λ -equivariant by transport of structure. The functor

$$\mathrm{R}\Phi_{f_{C^*}}[-1](\cdot \boxtimes \mathbb{1}_{C^*}[\dim C^*]) : \mathrm{D}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{D}_H(f_{C^*}^{-1}(0))$$

takes equivariant perverse sheaves to equivariant perverse sheaves.

By Lemma 6.8.2 for every $\mathcal{P} \in \mathrm{Per}_H(V)$ the support of $\mathrm{R}\Phi_{f_{C^*}}(\mathcal{P} \boxtimes \mathbb{1}_{C^*})$ is contained in

$$\{(x, \xi) \in V \times C^* \mid [x, \xi] = 0\}.$$

Thus, the restriction of the perverse sheaf $\mathrm{R}\Phi_{f_{C^*}}[-1](\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*])$ from

$$\{(x, \xi) \in V \times C^* \mid (x| \xi) = 0\}$$

to

$$\{(x, \xi) \in V \times C^* \mid [x, \xi] = 0\}$$

is again perverse. Since $T_C^*(V_\lambda)_{\mathrm{reg}}$ is open in $\{(x, \xi) \in V \times C^* \mid [x, \xi] = 0\}$, it follows from [BBD82, Section 1.4] that the restriction

$$\mathrm{Ev}_C \mathcal{P}[\dim C^* - 1] = (\mathrm{R}\Phi_{f_{C^*}}[-1](\mathcal{P} \boxtimes \mathbb{1}_{C^*}[\dim C^*]))|_{T_C^*(V_\lambda)_{\mathrm{reg}}}$$

is perverse. This proves 6.8.1. \square

Lemma 6.8.2. — *For every $\mathcal{P} \in \mathrm{Per}_H(V)$, the support of $\mathrm{R}\Phi_{f_{C^*}}(\mathcal{P} \boxtimes \mathbb{1}_{C^*})$ is contained in $\{(x, \xi) \in V \times C^* \mid [x, \xi] = 0\}$.*

Proof. — Since $\mathrm{R}\Phi_{f_{C^*}}$ is exact, we may assume \mathcal{P} is simple: set $\mathcal{P} = \mathcal{IC}(C_0, \mathcal{L}_0)$, where $C_0 \subseteq V$ is an H -orbit and \mathcal{L}_0 is a simple H -equivariant local system on C_0 . By Proposition 6.5.1, the support of $\mathrm{R}\Phi_{f_{C^*}}(\mathcal{P} \boxtimes \mathbb{1}_{C^*})$ is contained in $\overline{C_0}$. Let $\pi : \widetilde{C_0} \rightarrow \overline{C_0}$ be a proper morphism with $\widetilde{C_0}$ smooth over s such that $\mathcal{IC}(C_0, \mathcal{L}_0)$ appears, up to shift, in $\pi_* \mathcal{IC}(\widetilde{C_0})$. Now, define $\pi' := \pi \times \mathrm{id}_{C^*} : \widetilde{C_0} \times C^* \rightarrow \overline{C_0} \times C^*$ and $g := f_{C^*} \circ \pi' : \widetilde{C_0} \times C^* \rightarrow S$. To simplify notation somewhat, we set $Y = \widetilde{C_0} \times C^*$ for the remainder of this proof. Arguing as in the proof of Lemma 6.5.2 using proper base change [DK73, Exposé XIV, 2.1.7.1], it follows that

$$\pi'_* \mathrm{R}\Phi_g \mathbb{1}_Y = \mathrm{R}\Phi_{f_{C^*}} \pi_* \mathbb{1}_Y.$$

Then $\mathrm{R}\Phi_{f_{C^*}}(\mathcal{IC}(C_0, \mathcal{L}_0) \boxtimes \mathbb{1}_{C^*})$ is a summand of this sheaf, up to shift.

It follows from Lemma 6.2.2, that the support of $\mathrm{R}\Phi_g \mathbb{1}_Y$ is contained in the singular locus of Y . Thus, to prove the lemma it is sufficient to show that the singular locus of Y is contained in $\{(\tilde{x}, \xi) \in \widetilde{C_0} \times C^* \mid [\pi(\tilde{x}), \xi] = 0\}$. Accordingly, suppose $y = (\tilde{x}, \xi) \in Y$ is singular. Since Y is smooth over s , there exists an open neighbourhood $U \subset Y$ containing y and a closed embedding $U \rightarrow \mathbb{A}^n$ such that $d\dot{g}_y \in T_{U,y}^*(\mathbb{A}^n)$ for an extension \dot{g} of $g|_U$ to an open neighbourhood of U in \mathbb{A}^n ; see [Gai, Theorem 3.1.2], for instance. Observe that the stalk $T_{U,y}^*(\mathbb{A}^n)$ of the

conormal bundle $T_U^*(\mathbb{A}^n)$ is precisely the complex vector space of dh_y for $h \in I(U)$. Without loss of generality, we may take the embedding $U \rightarrow \mathbb{A}^n$ to be of the form $y \mapsto (z, \pi'(y))$, or equivalently $(\tilde{x}, \xi) \mapsto (z, \pi(\tilde{x}), \xi)$, where $\tilde{x} \mapsto (z, \pi(\tilde{x}))$ is an affine embedding of \tilde{C} . Observe that $C^* \subseteq V^*$ comes with an affine embedding. Now $I(Y) = I(\tilde{C}_0 \times C^*) \cong I(\tilde{C}_0) \oplus I(C^*)$ in the coordinate ring of \mathbb{A}^n and the projection of $d\dot{g}_y \in T_y^*(\mathbb{A}^n)$ onto $I(C^*)$ is $d(\pi(\tilde{x}) | \cdot)_\xi$. Thus, $d(\pi(\tilde{x}) | \cdot)_\xi \in T_{C^*, \xi}^*(V^*)$. Identifying the dual of V^* with V , as in Section 5.4, gives

$$T_{C^*, \xi}^*(V^*) \cong \{x \in V \mid [x, \xi] = 0\}.$$

Thus, the singular locus of Y is contained in

$$\pi'^{-1}(\{(x, \xi) \in \tilde{C}_0 \times C^* \mid [x, \xi] = 0\}) = \{(\tilde{x}, \xi) \in \tilde{C}_0 \times C^* \mid [\pi(\tilde{x}), \xi] = 0\}.$$

Thus, the support of $R\Phi_g \mathbb{1}_Y$ is contained in this variety. Since

$$(\pi'_* R\Phi_g \mathbb{1}_Y)_{(x, \xi)} = H^\bullet(\pi^{-1}(x) \times \{\xi\}, R\Phi_g \mathbb{1}_Y),$$

it now follows that the support of $\pi'_* R\Phi_g \mathbb{1}_Y = R\Phi_{f_{C^*}} \pi_* \mathbb{1}_Y$ is contained in the subset $\{(x, \xi) \in C_0 \times C^* \mid [x, \xi] = 0\}$. Since $R\Phi_{f_{C^*}}(\mathcal{IC}(C_0, \mathcal{L}_0) \boxtimes \mathbb{1}_{C^*})$ is a summand of $R\Phi_{f_{C^*}} \pi_* \mathbb{1}_Y$, its support too is contained in $\{(x, \xi) \in C_0 \times C^* \mid [x, \xi] = 0\}$, as claimed. \square

Using Proposition 6.8.1, we now define

$$(92) \quad {}^p\mathrm{Ev}_C := \mathrm{Ev}_C[\dim C^* - 1] : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T_C^*(V)_{\mathrm{reg}}).$$

Also define

$$(93) \quad {}^p\mathrm{Ev} : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}})$$

by

$$({}^p\mathrm{Ev} \mathcal{P})|_{T_C^*(V)_{\mathrm{reg}}} = {}^p\mathrm{Ev}_C \mathcal{P}.$$

6.9. Local systems on the strongly regular conormal bundle. — Recall the definition of $T_C^*(V)_{\mathrm{sreg}} \subseteq T_C^*(V)_{\mathrm{reg}}$ from Section 5.5.

Proposition 6.9.1. — *Suppose $T_C^*(V)_{\mathrm{sreg}}$ is non-empty. If $\mathcal{P} \in \mathrm{Per}_H(V)$ then the restriction of ${}^p\mathrm{Ev}_C \mathcal{P}$ to $T_C^*(V)_{\mathrm{sreg}}$ is a local system concentrated in degree $\dim V$.*

Proof. — By Proposition 6.8.1, we have that ${}^p\mathrm{Ev}_C \mathcal{P}$ is an H -equivariant perverse sheaf on $T_C^*(V)_{\mathrm{sreg}}$. From Section 5.5 we see that $T_C^*(V)_{\mathrm{sreg}} \subseteq T_C^*(V)_{\mathrm{reg}}$ is open. Then the restriction of ${}^p\mathrm{Ev}_C \mathcal{P}$ to $T_C^*(V)_{\mathrm{sreg}}$ is an H -equivariant perverse sheaf, by [BBD82, Proposition 4.4.2], and thus a direct sum of simple perverse sheaves. Simple perverse sheaves on $T_C^*(V)_{\mathrm{sreg}}$ are perverse extensions of simple H -equivariant local systems on H -orbits in $T_C^*(V)_{\mathrm{sreg}}$. By Proposition 5.5.1, $T_C^*(V)_{\mathrm{sreg}}$ is a single H -orbit, so the restriction of ${}^p\mathrm{Ev}_C \mathcal{P}$ to $T_C^*(V)_{\mathrm{sreg}}$ is an H -equivariant local system on $T_C^*(V)_{\mathrm{sreg}}$ in degree $\dim T_C^*(V)_{\mathrm{sreg}}$. Since we assume $T_C^*(V)_{\mathrm{sreg}}$ is non-empty, $\dim T_C^*(V)_{\mathrm{sreg}} = \dim T_C^*(V)_{\mathrm{reg}}$. Thus, the restriction of ${}^p\mathrm{Ev}_C \mathcal{P}$ to $T_C^*(V)_{\mathrm{sreg}}$ is concentrated in degree $\dim T_C^*(V)_{\mathrm{reg}} = \dim T_C^*(V) = \dim V$. \square

Using Proposition 6.9.1, we now have an exact functor

$$(94) \quad \mathrm{Ev}_C := {}^p\mathrm{Ev}_C[-\dim V]|_{T_C^*(V)_{\mathrm{sreg}}} : \mathrm{Per}_H(V) \rightarrow \mathrm{Loc}_H(T_C^*(V)_{\mathrm{sreg}}),$$

for each H -orbit $C \subseteq V$. Putting these together defines

$$(95) \quad \mathbf{Evs} : \mathrm{Per}_H(V) \rightarrow \mathrm{Loc}_H(T_H^*(V)_{\mathrm{sreg}})$$

so that

$$\mathbf{Evs} \mathcal{P}|_{T_C^*(V)_{\mathrm{sreg}}} = \mathbf{Evs}_C \mathcal{P}.$$

Any attempt to understand the exact functor

$$(96) \quad \mathbf{Evs} : \mathrm{Per}_H(V) \rightarrow \mathrm{Loc}_H(T_H^*(V)_{\mathrm{sreg}})$$

naturally begins with the local system $\mathcal{T} \in \mathrm{Loc}_H(T_H^*(V)_{\mathrm{sreg}})$ defined by

$$(97) \quad \mathcal{T}|_{T_C^*(V)_{\mathrm{sreg}}} := \mathcal{T}_C := \mathbf{Evs}_C \mathcal{I}(C) \in \mathrm{Loc}_H(T_C^*(V)_{\mathrm{sreg}}).$$

It follows from Proposition 6.6.2, Part (a) that for every $(x, \xi) \in T_C^*(V)_{\mathrm{sreg}}$,

$$\mathcal{T}_{C,(x,\xi)} = (\mathrm{R}\Phi_{\xi|_C}[-1] \mathbb{1}_C)_x [e_C],$$

as representations of $Z_H(x, \xi)$, where

$$(98) \quad e_C := \dim C + \dim C^* - \dim V.$$

By Proposition 6.6.2, Part (b)

$$(99) \quad \mathbf{Evs}_C \mathcal{I}(C, \mathcal{L}) = \mathcal{T}_C \otimes (\mathcal{L} \boxtimes \mathbb{1}_{C^*})|_{T_C^*(V)_{\mathrm{sreg}}},$$

for every H -equivariant local system \mathcal{L} on C .

By Proposition 5.6.1, every Arthur parameter $\psi \in Q_\lambda(LG)$ determines a base point $(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V)_{\mathrm{sreg}}$. By Proposition 5.7.1, the equivariant fundamental group of $T_{C_\psi}^*(V)_{\mathrm{sreg}}$ is A_ψ . Thus, the base point $(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V)_{\mathrm{sreg}}$ determines an equivalence of categories

$$(100) \quad \mathrm{Loc}_H(T_{C_\psi}^*(V)_{\mathrm{sreg}}) \rightarrow \mathrm{Rep}(A_\psi).$$

Combining (94) and (100) defines an exact functor

$$(101) \quad \mathbf{Ev}_\psi : \mathrm{Per}_H(V) \rightarrow \mathrm{Rep}(A_\psi).$$

For instance

$$(102) \quad \mathcal{T}_\psi := \mathbf{Ev}_\psi \mathcal{I}(C_\psi)$$

is the representation of A_ψ corresponding to \mathcal{T}_{C_ψ} , under the equivalence (100), so Proposition 6.6.2, Part (a) shows how to calculate this representation of A_ψ .

More generally, for every stratum $C \subseteq V$, the microlocal fundamental group A_C^{mic} of C is the equivariant fundamental group of a generic H -orbit in $T_C^*(V)_{\mathrm{reg}}$, which may be chosen using [ABV92, Lemma 24.3(f)], for example. In this way, every *sufficiently regular* $(x, \xi) \in T_C^*(V)_{\mathrm{reg}}$ determines an exact functor

$$(103) \quad \mathbf{Ev}_{(x,\xi)} : \mathrm{Per}_H(V) \rightarrow \mathrm{Rep}(A_C^{\mathrm{mic}}).$$

This generalizes (101), since A_ψ is the microlocal fundamental group of C_ψ .

6.10. Normalization of \mathbf{Ev} . — We normalize \mathbf{Ev} by the following definition.

Definition 2. — Let

$$\mathbf{NEv}_C : \mathrm{D}_H(V) \rightarrow \mathrm{D}_H(T_C^*(V)_{\mathrm{reg}}),$$

be the functor defined by

$$\mathbf{NEv}_C := (\mathbf{Ev}_C \mathcal{IC}(C))^\vee \otimes \mathbf{Ev}_C,$$

where $(\mathbf{Ev}_C \mathcal{IC}(C))^\vee = \mathcal{H}\mathrm{om} \left(\mathbf{Ev}_C \mathcal{IC}(C), \mathbb{1}_{T_C^*(V)_{\mathrm{reg}}} \right)$; here and below we use the left derived tensor product. Likewise define

$$\mathbf{NEv} : \mathrm{D}_H(V) \rightarrow \mathrm{D}_H(T_H^*(V)_{\mathrm{reg}})$$

so that

$$\mathbf{NEv}_C \mathcal{F} = (\mathbf{NEv} \mathcal{F})|_{T_C^*(V)_{\mathrm{reg}}}.$$

We refer to \mathbf{NEv} as the *normalised microlocal vanishing cycles functor*.

Supposing $T_C^*(V)_{\mathrm{sreg}}$ is non-empty, we will mainly be concerned with this functor after restriction from $T_C^*(V)_{\mathrm{reg}}$ to $T_C^*(V)_{\mathrm{sreg}}$:

$$(104) \quad \mathbf{NEvs}_C \mathcal{F} := (\mathbf{NEv}_C \mathcal{F})|_{T_C^*(V)_{\mathrm{sreg}}}$$

Then

$$(105) \quad \mathbf{NEvs}_C = \mathcal{T}_C^\vee \otimes \mathbf{Evs}_C$$

where \mathcal{T}_C is given in (97) and \mathcal{T}_C^\vee is its dual local system.

Theorem 6.10.1. — *Let $\lambda : W_F \rightarrow {}^L G$ be an infinitesimal parameter. Suppose $T_C^*(V)_{\mathrm{sreg}}$ is non-empty.*

- (a) *The functor $\mathbf{NEvs}_C : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Loc}_H(T_C^*(V)_{\mathrm{sreg}})$ is exact.*
- (b) *If $\mathcal{P} \in \mathrm{Per}_{H_\lambda}(V_\lambda)$ then $\mathbf{NEvs}_C \mathcal{P} = 0$ unless $C \subseteq \mathrm{supp} \mathcal{P}$.*
- (c) *If $\mathcal{P} \in \mathrm{Per}_{H_\lambda}(V_\lambda)$ then*

$$\mathrm{rank}(\mathbf{NEvs}_C \mathcal{P}) = \mathrm{rank}(\mathrm{R}\Phi_\xi \mathcal{P})_x,$$

for every $(x, \xi) \in T_C^(V)_{\mathrm{sreg}}$.*

- (d) *For every H_λ -equivariant local system \mathcal{L} on C ,*

$$\mathbf{NEvs}_C \mathcal{IC}(C, \mathcal{L}) = (\mathcal{L} \boxtimes \mathbb{1}_{C^*})|_{T_C^*(V_\lambda)_{\mathrm{sreg}}};$$

in particular,

$$\mathrm{rank} \mathbf{NEvs}_C \mathcal{IC}(C, \mathcal{L}) = \mathrm{rank} \mathcal{L}.$$

Proof. — By first part of Proposition 6.4.1, \mathbf{Ev}_C is exact. Since restriction from $T_C^*(V)_{\mathrm{reg}}$ to $T_C^*(V)_{\mathrm{sreg}}$ is also exact, so is \mathbf{Evs}_C . Since \mathbf{NEvs}_C is obtained by tensoring \mathbf{Evs}_C with \mathcal{T}_C^\vee , \mathbf{NEvs}_C is also exact. \mathbf{NEvs}_C produces local systems by Proposition 6.9.1. This proves Part (a). Part (b) is a consequence of Proposition 6.5.1 and the definition of \mathbf{NEvs}_C and the fact that $\mathrm{rank} \mathcal{T}_C^\vee = 1$ by Theorem 6.7.5. Part (c) follows from the second part of Proposition 6.4.1, using Theorem 6.7.5. Part (d) follows from Proposition 6.6.2, Part (b) and the isomorphism $\mathcal{T}_C^\vee \otimes \mathcal{T}_C \cong \mathbb{1}_{T_C^*(V)_{\mathrm{sreg}}}$, again using Theorem 6.7.5. \square

Note that the functor

$$\mathbf{Evs}_C : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Loc}_H(T_C^*(V)_{\mathrm{sreg}})$$

satisfies the conditions appearing in Theorem 6.10.1 except the first part of Theorem 6.10.1, Part (d) since, for every H_λ -equivariant local system \mathcal{L} on C ,

$$\mathrm{Evs}_C \mathcal{I}(C, \mathcal{L}) = \mathcal{T}_C \otimes (\mathcal{L} \boxtimes \mathbb{1}_{C^*})|_{T_C^*(V_\lambda)_{\mathrm{sreg}}}.$$

Moreover, the rank-1 local system \mathcal{T}_C is in general not trivial as a representation of the equivariant fundamental group A_C^{mic} of $T_C^*(V_\lambda)_{\mathrm{sreg}}$.

If $\psi \in Q_\lambda({}^L G)$ is an Arthur parameter with infinitesimal parameter λ , then $T_{C_\psi}^*(V_\lambda)_{\mathrm{sreg}}$ is non-empty by Proposition 5.6.1. In this case we use the equivalence (100) and replace Evs_{C_ψ} with the functor Ev_ψ , the local system \mathcal{T}_{C_ψ} with the representation \mathcal{T}_ψ of A_ψ , and define

$$(106) \quad \mathrm{NEv}_\psi := \mathcal{T}_\psi^\vee \otimes \mathrm{Ev}_\psi.$$

This is the functor appearing in (10).

Corollary 6.0.1 is simply a rephrasing of Theorem 6.10.1 using Proposition 3.6.2.

7. Arthur packets and ABV-packets

In this section we articulate the conjectures which, taken together, lie at the heart of the concept of p -adic ABV-packets. In this section, G is a quasi-split connected reductive linear algebraic group over F . When referring to work of Arthur, we will further assume G is a split symplectic or special orthogonal group.

7.1. ABV-packets. — We fix an admissible homomorphism $\lambda : W_F \rightarrow {}^L G$ and recall the Vogan variety V_λ from Section 3. As above, set $H_\lambda := Z_{\widehat{G}}(\lambda)$.

From Proposition 3.6.2 recall that the local Langlands correspondence for pure rational forms determines a canonical bijection between isomorphism classes of simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ and $\Pi_{\mathrm{pure}, \lambda}(G/F)$:

$$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}} \leftrightarrow \Pi_{\mathrm{pure}, \lambda}(G/F).$$

Recall that we use the notation $\mathcal{P}(\pi, \delta)$ for a simple H_λ -equivariant perverse sheaf on V_λ matching a representation (π, δ) of a pure rational form of G under this correspondence.

Definition 3. — For any H_λ -orbit C in V_λ , the *ABV-packet* for C is

$$(107) \quad \Pi_{\mathrm{pure}, C}^{\mathrm{ABV}}(G/F) := \{[\pi, \delta] \in \Pi_{\mathrm{pure}, \lambda}(G/F) \mid \mathrm{Ev}_C \mathcal{P}(\pi, \delta) \neq 0\}.$$

If $C = C_\phi$ for a Langlands parameter ϕ , we sometimes use the notation

$$\Pi_{\mathrm{pure}, \phi}^{\mathrm{ABV}}(G/F) := \Pi_{\mathrm{pure}, C_\phi}^{\mathrm{ABV}}(G/F).$$

7.2. Virtual representations attached to ABV-packets. — From Section 2.11 recall the definition of the virtual representation

$$\eta_\psi = \sum_{[\pi, \delta] \in \Pi_{\mathrm{pure}, \psi}(G/F)} \langle a_\psi, [\pi, \delta] \rangle_\psi e(\delta) [\pi, \delta],$$

based on Arthur's work.

Definition 4. — Let ψ be an Arthur parameter for G with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$. Consider the virtual representation

$$\eta_\psi^{\text{Ev}} := (-1)^{\dim C_\psi} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, C_\psi}^{\text{ABV}}(G/F)} (-1)^{\dim \text{supp } \mathcal{P}(\pi, \delta)} \text{rank Ev}_\psi \mathcal{P}(\pi, \delta) e(\delta) [\pi, \delta],$$

where $\text{Ev}_\psi : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Rep}(A_\psi)$ is defined in (101). Recall from Section 2.11 that $e(\delta)$ is the Kottwitz sign attached to the pure rational form $\delta \in Z^1(F, G)$.

Let λ be the infinitesimal parameter of ψ . Then, using (92), (101), (98) and Proposition 6.4.1, we have

$$\begin{aligned} & (-1)^{\dim C_\psi - \dim \text{supp } \mathcal{P}(\pi, \delta)} \text{rank Ev}_\psi \mathcal{P}(\pi, \delta) \\ &= \text{rank} (\text{Ev}_\psi \mathcal{P}(\pi, \delta) [\dim C_\psi - \dim C_{\pi, \delta}]) \\ &= \text{rank} ({}^p \text{Ev}_{C_\psi} \mathcal{P}(\pi, \delta) [\dim C_\psi - \dim V_\lambda - \dim C_{\pi, \delta}])_{(x_\psi, \xi_\psi)} \\ &= \text{rank} (\text{Ev}_{C_\psi} \mathcal{P}(\pi, \delta) [-1 + \dim C_\psi^* + \dim C_\psi - \dim V_\lambda - \dim C_{\pi, \delta}])_{(x_\psi, \xi_\psi)} \\ &= \text{rank} (\text{Ev}_{C_\psi} \mathcal{P}(\pi, \delta) [-1 + e_{C_\psi} - \dim C_{\pi, \delta}])_{(x_\psi, \xi_\psi)} \\ &= \text{rank} (\text{R}\Phi_{\xi_\psi} [-1] \mathcal{P}(\pi, \delta) [-\dim C_{\pi, \delta}] [e_{C_\psi}])_{x_\psi} \\ &= \text{rank} \left(\text{R}\Phi_{\xi_\psi} [-1] \mathcal{L}_{\pi, \delta}^\# [e_{C_\psi}] \right)_{x_\psi}, \end{aligned}$$

where, with reference to Proposition 3.6.2, we set

$$\mathcal{L}_{\pi, \delta}^\# = \mathcal{P}(\pi, \delta) [-\dim C_{\pi, \delta}] = \mathcal{I}\mathcal{C}(C_{\pi, \delta}, \mathcal{L}_{\pi, \delta}) [-\dim C_{\pi, \delta}].$$

So, Definition 4 may also be written in the form

$$(108) \quad \eta_\psi^{\text{Ev}} = \sum_{[\pi, \delta] \in \Pi_{\text{pure}, C_\psi}^{\text{ABV}}(G/F)} \text{rank} \left(\text{R}\Phi_{\xi_\psi} [-1] \mathcal{L}_{\pi, \delta}^\# [e_{C_\psi}] \right)_{x_\psi} e(\delta) [\pi, \delta].$$

Definition 5. — Let ψ be an Arthur parameter for G with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$. For any $s \in Z_{\widehat{G}}(\psi)$, consider the virtual representation

$$\eta_{\psi, s}^{\text{NEv}} := (-1)^{\dim C_\psi} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, C_\psi}^{\text{ABV}}(G/F)} (-1)^{\dim \text{supp } \mathcal{P}(\pi, \delta)} \text{trace}_{a_s} \text{NEv}_\psi \mathcal{P}(\pi, \delta) e(\delta) [\pi, \delta],$$

where a_s is the image of s in A_ψ and where $\text{NEv}_\psi : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Rep}(A_\psi)$ is defined in (106). Also set $\eta_\psi^{\text{NEv}} := \eta_{\psi, 1}^{\text{NEv}}$.

7.3. Main conjecture. — Recall the definition of $\Pi_{\text{pure}, \psi}(G/F)$ from Section 2.10. Recall the definitions of η_ψ and $\eta_{\psi, s}$ from Section 2.11.

Conjecture 1. — Let G be a quasi-split symplectic or special orthogonal p -adic group. Let $\psi : L_F \times \text{SL}(2, \mathbb{C}) \rightarrow {}^L G$ be an Arthur parameter. Then

(a) Pure Arthur packets are ABV-packets:

$$\Pi_{\text{pure}, \psi}(G/F) = \Pi_{\text{pure}, \phi_\psi}^{\text{ABV}}(G/F).$$

(b) Arthur's stable distributions are calculated by Ev :

$$\eta_\psi = \eta_\psi^{\text{Ev}}.$$

(c) Endoscopic transfer of Arthur's stable distributions are calculated by NEv :

$$\eta_{\psi,s} = \eta_{\psi,s}^{\text{NEv}},$$

for every semisimple $s \in Z_{\widehat{G}}(\psi)$.

By Proposition 6.4.1, Conjecture 1(a) is equivalent to the claim: for all $[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F)$,

$$[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F) \quad \text{if and only if} \quad (\text{R}\Phi_{\xi_\psi} \mathcal{P}(\pi, \delta))_{x_\psi} \neq 0.$$

Assuming Conjecture 1(a), and with reference to (25), Conjecture 1(b) is equivalent to: for all $[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)$,

$$\langle a_\psi, [\pi, \delta] \rangle_\psi = \text{rank} \left(\text{R}\Phi_{\xi_\psi} [-1] \mathcal{L}_{\pi, \delta}^\# [e_{C_\psi}] \right)_{x_\psi},$$

which is to say,

$$\langle a_\psi, [\pi, \delta] \rangle_\psi = (-1)^{\dim C_\psi - \dim \text{supp } \mathcal{P}(\pi, \delta)} \text{rank Ev}_\psi \mathcal{P}(\pi, \delta).$$

Likewise, assuming Conjecture 1(a), Conjecture 1(c), is equivalent to the claim: for every $[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)$ and for every semisimple $s \in Z_{\widehat{G}}(\psi)$,

$$(109) \quad \langle a_s a_\psi, [\pi, \delta] \rangle_\psi = (-1)^{\dim C_\psi - \dim C_{[\pi, \delta]}} \text{trace}_{a_s} (\text{NEv}_\psi \mathcal{P}(\pi, \delta)),$$

where $a_\psi \in A_\psi$ is defined in Section 2.11 and a_s is the image of s in A_ψ . Thus, Conjecture 1 promises a new way to calculate the character $\langle a_s, [\pi, \delta] \rangle_\psi$ when π is an admissible representation of $G_\delta(F)$ for a pure rational form δ of G , and when the complete Langlands parameter for (π, δ) is known; this fact is illustrated with examples in Part II. Conjecture 1 also suggests how to define the character for Langlands parameters that are not of Arthur type. We also show several examples of that in Part II.

Assuming Conjecture 1(a), it follows that Conjecture 1(c) implies Conjecture 1(b). To see this, recall (106) that $\text{NEv}_\psi = \mathcal{T}_\psi^\vee \otimes \text{Ev}_\psi$, so

$$(110) \quad \eta_{\psi,s}^{\text{NEv}} = \left(\text{trace}_{a_s^{-1}} \mathcal{T}_\psi \right) \eta_{\psi,s}^{\text{Ev}},$$

where

$$\eta_{\psi,s}^{\text{Ev}} := (-1)^{\dim C_\psi} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, C_\psi}^{\text{ABV}}(G/F)} (-1)^{\dim \text{supp } \mathcal{P}(\pi, \delta)} \text{trace}_{a_s} (\text{Ev}_\psi \mathcal{P}(\pi, \delta)) e(\delta) [\pi, \delta].$$

Taking $s = 1$, this gives

$$\eta_\psi^{\text{NEv}} = \eta_{\psi,1}^{\text{NEv}} = (\text{trace}_1 \mathcal{T}_\psi) \eta_{\psi,1}^{\text{Ev}} = (\text{rank } \mathcal{T}_\psi) \eta_\psi^{\text{Ev}}.$$

Using Theorem 6.7.5, this becomes

$$(111) \quad \eta_\psi^{\text{NEv}} = \eta_\psi^{\text{Ev}}.$$

So, Conjecture 1(c) gives $\eta_{\psi,s} = \eta_{\psi,s}^{\text{NEv}}$ which implies $\eta_\psi = \eta_\psi^{\text{NEv}} = \eta_\psi^{\text{Ev}}$, whence Conjecture 1(b).

Conjecture 1 may be expressed using the pairing of Grothendieck groups

$$(112) \quad \langle \cdot, \cdot \rangle : K\Pi_{\text{pure}, \lambda}(G/F) \times K\text{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathbb{Z}$$

introduced in [Vog93, (8.11')(a)] (see also [ABV92, Theorem 1.24]) which is defined on $\Pi_{\text{pure}, \lambda}(G/F)$ and isomorphism classes of simple objects in $\text{Per}_{H_\lambda}(V_\lambda)$ by

$$\langle [\pi, \delta], \mathcal{P} \rangle = \begin{cases} e(\mathcal{P})(-1)^{\dim \text{supp}(\mathcal{P})}, & \text{if } \mathcal{P} = \mathcal{P}(\pi, \delta) \\ 0, & \text{otherwise,} \end{cases}$$

where $e(\mathcal{P})$ is the Kottwitz sign of the group $G_{\delta_{\mathcal{P}}}$ for the pure rational form $\delta_{\mathcal{P}}$ of G determined by \mathcal{P} , as in Section 2.7. Conjecture 1(a) and (b) together are equivalent to:

$$(113) \quad \langle \eta_\psi, \mathcal{P} \rangle = (-1)^{\dim C_\psi} \text{rank Ev}_\psi \mathcal{P},$$

for all $\mathcal{P} \in K\text{Per}_{H_\lambda}(V_\lambda)$. In its entirety, Conjecture 1 is equivalent to:

$$(114) \quad \langle \eta_{\psi, s}, \mathcal{P} \rangle = (-1)^{\dim C_\psi} \text{trace}_{a_s}(\text{NEv}_\psi \mathcal{P}),$$

for every semisimple $s \in Z_{\widehat{G}}(\psi)$ and for every $\mathcal{P} \in \text{Per}_{H_\lambda}(V_\lambda)$.

7.4. Arthur perverse sheaves. — The pairing (112) may be used to attach to ψ an equivariant *perverse* sheaf \mathcal{A}_ψ on V_λ , defined up to isomorphism, from which the virtual representation η_ψ^{Ev} is easily recovered:

$$(115) \quad \mathcal{A}_\psi := \bigoplus_{\mathcal{P} \in \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}} (\text{rank Ev}_\psi \mathcal{P}) \mathcal{P}.$$

Then

$$\eta_\psi^{\text{NEv}} = (-1)^{\dim C_\psi} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F)} \langle [\pi, \delta], \mathcal{A}_\psi \rangle [\pi, \delta].$$

If we assume Conjecture 1(a) for a moment, this gives

$$\eta_\psi = (-1)^{\dim C_\psi} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F)} \langle [\pi, \delta], \mathcal{A}_\psi \rangle [\pi, \delta].$$

By Proposition 6.5.1, the summation appearing in the definition of \mathcal{A}_ψ (115) can be taken the over simple $\mathcal{P} \in \text{Per}_{H_\lambda}(V_\lambda)$ supported by $\overline{C_\psi}$:

$$\mathcal{A}_\psi = \bigoplus_{\mathcal{P} \in \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}, \text{supp}(\mathcal{P}) \subseteq \overline{C_\psi}} (\text{rank Ev}_\psi \mathcal{P}) \mathcal{P}.$$

Taking the cases when $\mathcal{P} = \mathcal{IC}(C_\psi, \mathcal{L})$, consider the summand *pure packet perverse sheaf*

$$(116) \quad \mathcal{B}_\psi := \bigoplus_{\mathcal{L} \in \text{Loc}_{H_\lambda}(C_\psi)_{/\text{iso}}^{\text{simple}}} (\text{rank Ev}_{C_\psi} \mathcal{IC}(C_\psi, \mathcal{L})) \mathcal{IC}(C_\psi, \mathcal{L})$$

where the sum runs over all simple H_λ -equivariant local systems \mathcal{L} on C_ψ . By (99), $\text{rank Ev}_\psi \mathcal{IC}(C_\psi, \mathcal{L}) = \text{rank}(\mathcal{L})$, so

$$\mathcal{B}_\psi = \bigoplus_{\mathcal{L} \in \text{Loc}_{H_\lambda}(C_\psi)_{\text{iso}}^{\text{simple}}} (\text{rank } \mathcal{L}) \mathcal{IC}(C_\psi, \mathcal{L}).$$

The simple perverse sheaves appearing in \mathcal{B}_ψ correspond exactly to the irreducible admissible representations in the pure Langlands packet $\Pi_{\text{pure}, \phi_\psi}(G/F)$, where ϕ_ψ is the Langlands parameter matching ψ under Proposition 3.2.2.

The perverse sheaf

$$(117) \quad \mathcal{C}_\psi := \bigoplus_{\substack{\mathcal{IC}(C, \mathcal{L}) \in \text{Per}_{H_\lambda}(V_\lambda)_{\text{iso}}^{\text{simple}}, \\ C \leq C_\psi}} (\text{rank Ev}_\psi \mathcal{IC}(C, \mathcal{L})) \mathcal{IC}(C, \mathcal{L})$$

is called the *coronal perverse sheaf* for C_ψ , where the sum is taken over all $C \subset \overline{C_\psi}$ with $C \neq C_\psi$ and over all simple H_λ -equivariant local systems \mathcal{L} on C . So

$$(118) \quad \mathcal{A}_\psi = \mathcal{B}_\psi \oplus \mathcal{C}_\psi.$$

7.5. A basis for strongly stable virtual representations. — Definition 3 of $\Pi_{\text{pure}, C}^{\text{ABV}}$ applies to all strata $C \subseteq V_\lambda$, not just those attached to Arthur parameters. But in Section 7.3, we only made conjectures related to ABV-packets of the form $\Pi_{\text{pure}, C_\psi}^{\text{ABV}}$, for some Arthur parameter ψ with infinitesimal parameter λ . However, Definitions 4 and 5 extend from Arthur parameters ψ with infinitesimal parameter λ to all strata C in V_λ as long as $T_C^*(V)_{\text{sreg}}$ is non-empty, as follows.

Definition 6. — Let $\lambda : W_F \rightarrow {}^L G$ be an infinitesimal parameter. Let $C \subseteq V_\lambda$ be an H_λ -orbit such that $T_C^*(V_\lambda)_{\text{sreg}}$ is non-empty. Suppose $s \in Z_H(x, \xi)$ for some $(x, \xi) \in T_C^*(V)_{\text{sreg}}$. Set

$$\eta_{C, s}^{\text{NEv}} := (-1)^{\dim C} \sum_{[\pi, \delta] \in \Pi_{\text{pure}, C}^{\text{ABV}}(G/F)} (-1)^{\dim \text{supp } \mathcal{P}(\pi, \delta)} \text{trace}_{a_s} \text{NEvs}_C \mathcal{P}(\pi, \delta) e(\delta) [\pi, \delta],$$

where a_s is the image of s in $A_{(x, \xi)} = \pi_0(Z_H(x, \xi))$. Likewise define $\eta_{C, s}^{\text{Ev}}$. Set $\eta_C^{\text{NEv}} := \eta_{C, 1}^{\text{NEv}}$ and $\eta_C^{\text{Ev}} := \eta_{C, 1}^{\text{Ev}}$.

It is easy to show that the definition of $\eta_{C, s}^{\text{NEv}}$ is independent of the choice of (x, ξ) .

Conjecture 2, below, is an adaptation of [Vog93, Conjecture 8.15']. It suggests how to extend the definition of Arthur packets from Langlands parameters of Arthur type to all Langlands parameters and also how to find the associated stable distributions.

Conjecture 2. — Let G be a quasi-split connected reductive linear algebraic group over F . For any $\lambda \in \Lambda({}^L G)$ (Section 3.1) and any stratum $C \subseteq V_\lambda$, $T_C^*(V_\lambda)_{\text{sreg}}$ is non-empty and the virtual representation η_C^{NEv} is strongly stable in the sense of [Vog93, 1.6]. Moreover,

$$\{\eta_C^{\text{NEv}} \mid H_\lambda\text{-orbits } C \subseteq V_\lambda\}$$

is a basis for the Grothendieck group of strongly stable virtual representations with infinitesimal character λ .

It should be noted that strongly stable virtual representations of G produce stable virtual representations, and thus stable distributions, of all the groups $G_\delta(F)$ as δ ranges over pure rational forms of G . It should also be noted that in Conjecture 2 we dropped the hypothesis that G is a quasi-split symplectic or special orthogonal p -adic group, which appeared in Conjecture 1, and replaced it with the hypothesis that G is any quasi-split connected reductive linear algebraic group over F . The scope of Conjecture 2 is therefore very broad, as it refers to all pure inner forms of all quasi-split connected reductive p -adic groups.

In Part II we gather evidence for Conjectures 1 and 2 by verifying them for 38 admissible representations of 12 p -adic groups.

7.6. Remarks on stratified Morse theory and microlocalisation. — In the discussion after [ABV92, Theorem 24.8], one finds some words about the relation between stratified Morse theory, microlocalisation and the vanishing cycles functor; these words are clarified considerably in [Sch03]. As our goal in this article was to establish the properties of NEv needed to make precise definitions and testable conjectures about (what we call) ABV-packets and their associated distributions, we did not find it necessary here to discuss the relation between stratified Morse theory, microlocalisation and vanishing cycles in any serious way. Even for the calculations in Part II, that is unnecessary. We expect, however, that progress toward proving the Conjectures in this article and in [Vog93] in full generality would be aided by an ability to pass between these three perspectives, rigourously.

With this in mind, we offer some words of caution. The definitive reference for stratified Morse theory is, of course, [GM88]. Vanishing cycles appear only once in this book, in a remark in an appendix [GM88, 6.A.2]: “Then, the Morse group $A_\xi^i(\mathcal{F})$ is canonically isomorphic to the vanishing cycles $R^i\Phi(\mathcal{F})_p$ of [DK73].” In [GM88, 6.A.1] we see how to calculate the Morse group, using normal Morse data according to the formula $A_\xi^i(\mathcal{F}) = H^i(J, K; \mathcal{F})$, where the pair (J, K) is the normal Morse data corresponding to any smooth function $f : M \rightarrow \mathbb{R}$ such that $df(p) = \xi$. In the proof of [ABV92, Theorem 24.8] we see that the stalks of $Q_C^{\text{mic}}(\mathcal{F})$ are Morse groups, or more precisely, $Q_C^{\text{mic}}(\mathcal{F})_{(x, \xi)}^i = H^{i - \dim C}(J, K; \mathcal{F})$ for $(x, \xi) \in T_C^*(V)_{\text{reg}}$. In this article we show that the stalks of $\text{Ev}_C(\mathcal{F})$ are given by vanishing cycles, or more precisely, $(\text{Ev}_C \mathcal{F})_{(x, \xi)} = (R\Phi_\xi \mathcal{F})_x$ for $(x, \xi) \in T_C^*(V)_{\text{reg}}$. For this reason, one might expect that, after invoking [DK73, Exposé XIV, Théorème 2.8] to pass from the algebraic description of $R\Phi$ based on [DK73, Exposé XIII] to the analytic version of $R\Phi$ given in [DK73, Exposé XIV], perhaps $Q_C^{\text{mic}}(\mathcal{F})$ coincides with $\text{Ev}_C \mathcal{F}[-\dim C]$. But that is false, and not just because something has gone awry with the shifts. The difference between the functor Q_C^{mic} and the appropriately shifted analytic version of Ev is easy to miss, because they do produce sheaves with the same support and rank: $\text{rank } Q_C^{\text{mic}}(\mathcal{F})_{(x, \xi)}^i = \text{rank } \text{Ev}_{(x, \xi)}^{i - \dim C}(\mathcal{F}[e_C - 1])$ for all $(x, \xi) \in T_C^*(V)_{\text{reg}}$ and for all $i \in \mathbb{Z}$. However, as spaces with an action of $Z_H(x, \xi)$, these stalks are not equal, which means that the sheaves produced by Q_C^{mic} and the sheaves produced by $\text{Ev}_C[-1 + e_C - \dim C]$ are different as *equivariant* sheaves. Even using microlocal Euler characteristics, one cannot see this issue.

This discrepancy is entirely responsible for introducing the functor \mathbf{NEv} in this article, in Definition 2. To bring \mathbf{Ev}_C and Q_C^{mic} into alignment, we use an idea from stratified Morse theory. The local system $\mathcal{T}_C := \mathbf{Evs}_C \mathcal{IC}(C)$ is designed play the role of tangential Morse data. Examples in Part II show that the local system \mathcal{T}_C is not trivial in general. Recall that \mathbf{NEvs} is formed by twisting \mathbf{Evs} by the dual of this non-trivial sheaf: $\mathbf{NEvs}_C := \mathcal{T}_C^\vee \otimes \mathbf{Evs}_C$. Since $\text{rank } \mathcal{T}_C = 1$ by Theorem 6.7.5, the stalks of $\mathbf{NEvs}_C \mathcal{F}$ satisfy the relation

$$(119) \quad (\mathbf{Evs}_C \mathcal{F})_{(x,\xi)} = (\mathbf{R}\Phi_\xi[-1] \mathbb{1}_C)_x [e_C] \otimes (\mathbf{NEvs}_C \mathcal{F})_{(x,\xi)}.$$

This relation is designed to mirror [GM88, Section 3.7. The Main Theorem], given colloquially there as

$$\text{Local Morse data} \cong (\text{Tangential Morse data}) \times (\text{Normal Morse data}).$$

Schürmann has shown how to interpret this in the language of vanishing cycles, in certain contexts; see especially [Sch03, Theorem 5.4.1 (5.87)].

We suspect, therefore, that it may be possible to express the stalks of $\mathbf{NEvs}_C \mathcal{F}$ using normal slices, as we now explain. Suppose G is split, the infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ is unramified and $\lambda(\text{Fr}) = s_\lambda \times \text{Fr}$, where $s_\lambda \in \widehat{G}$ is elliptic. Observe that Section 4, especially Theorem 4.1.1, shows how the general case can be reduced to this case. With reference to the exponential function for \mathfrak{j}_λ , set $z := \log s_\lambda$. Then $z \in \mathfrak{j}_{\lambda,0} = \mathfrak{h}_\lambda$. For every $x \in V_\lambda = \mathfrak{j}_{\lambda,2}$, there is a unique $\xi_x \in V_\lambda^* = \mathfrak{j}_{\lambda,-2}$ such that (x, ξ_x, z) is an $\text{SL}(2)$ -triple in \mathfrak{j}_λ . Then $x + \ker \text{ad } \xi_x$ is a transverse slice to the J_λ -orbit of x in \mathfrak{j}_λ [Slo80, Section 7.4] and its intersection with V_λ ,

$$\mathcal{S}_x := x + \{y \in V_\lambda \mid [y, \xi_x] = 0\},$$

is a transverse slice to the H_λ -orbit C of x at x . Suppose $\xi \in T_{C,x}^*(V_\lambda)_{\text{sreg}}$. Then we expect

$$(120) \quad (\mathbf{NEvs}_C \mathcal{F})_{(x,\xi)} = (\mathbf{R}\Phi_{\xi|_{\mathcal{S}_x}}[-1](\mathcal{F}|_{\mathcal{S}_x}))_x [-\dim C],$$

for $\mathcal{F} \in \mathbf{D}_{H_\lambda}(V_\lambda)$, in which case (119) becomes

$$\begin{aligned} & (\mathbf{R}\Phi_\xi[-1]\mathcal{F})_x [e_C - \dim C] \\ &= (\mathbf{R}\Phi_\xi[-1] \mathbb{1}_C)_x [e_C] \otimes (\mathbf{R}\Phi_{\xi|_{\mathcal{S}_x}}[-1](\mathcal{F}|_{\mathcal{S}_x}))_x [-\dim C], \end{aligned}$$

or equivalently,

$$(121) \quad (\mathbf{R}\Phi_\xi[-1]\mathcal{F})_x = (\mathbf{R}\Phi_\xi[-1] \mathbb{1}_C)_x \otimes (\mathbf{R}\Phi_{\xi|_{\mathcal{S}_x}}[-1](\mathcal{F}|_{\mathcal{S}_x}))_x.$$

That is exactly what one finds in [Sch03, Theorem 5.4.1 (5.87)]. Moreover, all the examples in Part II conform to expectation (120). We believe, therefore, that \mathbf{NEvs} coincides with Q^{mic} .

Expectation (120) would also, in principle, allow us to use [Gin86, Proposition 6.19] and [Gin86, Proposition 7.7.1] to identify $\text{rank } \mathbf{NEvs}$ with the microlocal Euler characteristic. However, the proofs of those two results from [Gin86] rely in the general case on [Bry86] and the relevant result there makes use of [Kas83, Théorème 3.2.5]. As we remarked in Section 6.3, [Kas83, Théorème 3.2.5] does not exist in the published version of the original notes, and we have not been able to procure the original notes, so using this approach would oblige us to use a result for which we

cannot find a complete proof in the literature. That is another reason why we have built \mathbf{Ev} from scratch and established its main properties by hand in Section 6.

PART II. EXAMPLES

Each example follows essentially the same four-part plan, explained in some detail in Section 8 and outlined here.

After fixing a connected reductive group G over a p -adic field F and an infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$, we enumerate all admissible representations π of all pure rational forms of G with infinitesimal parameter λ . We partition these admissible representations into L -packets and show how Aubert duality operates on the representations. Then, for each L -packet of Arthur type, we find the Arthur packet that contains it. We calculate a twisting character which measures the difference between Arthur's parametrization of representations in an Arthur packet with Mœglin's parametrization. We find the coefficients in the invariant distributions

$$(122) \quad \Theta_{\psi,s}^G = \sum_{\pi \in \Pi_{\psi}(G(F))} \langle s s_{\psi}, \pi \rangle_{\psi} \operatorname{trace} \pi$$

that arise from stable distributions attached to Arthur packets for endoscopic groups for $G(F)$ in [Art13, Theorem 1.5.1]. We also calculate the virtual representations $\eta_{\psi,s}$ using Arthur's work. See Section 8.1 for more detail on this part of the examples.

In the second part of each example, called *Vanishing cycles of perverse sheaves*, we set up all the tools needed to calculate $\langle s s_{\psi}, \pi \rangle_{\psi}$, and its generalisation to pure rational forms of G , geometrically. We find the stratified variety V_{λ} attached to λ and study the category $\operatorname{Per}_{Z_{\widehat{G}}(\lambda)}(V_{\lambda})$ of equivariant perverse sheaves on V_{λ} . We show how this category decomposes into summand categories, called the cuspidal support decomposition of $\operatorname{Per}_{Z_{\widehat{G}}(\lambda)}(V_{\lambda})$. Then we calculate the functor

$$(123) \quad \mathbf{Ev}_{\psi} : \operatorname{Per}_{Z_{\widehat{G}}(\lambda)}(V_{\lambda}) \rightarrow \operatorname{Rep}(A_{\psi})$$

on simple objects, using properties of vanishing cycles; \mathbf{NEv} is defined in Section 6 and recalled in Section 8.2.6. The results of these calculations – one for each example – are presented in Sections 9.2.3, 10.2.5, 11.2.5, 12.2.5, 13.2.5 and 14.2.6. Section 8.2 includes an overview of how we made these calculations. We also show how the Fourier transform interacts with the functor \mathbf{NEv} .

In the third part we connect the two sides of this story, as treated above. To begin, we find Vogan's bijection between: admissible representations of split p -adic groups and their pure rational forms with fixed infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$, as recalled in Section 8.1; and simple equivariant perverse sheaves on V_{λ} , as recalled in Section 8.2. With this bijection in hand, and the calculation of \mathbf{Ev} from Section 8.2, we easily find the ABV-packets $\Pi_{\text{pure},\phi}^{\text{ABV}}$ and associated virtual representations $\eta_{\phi,s}^{\mathbf{NEv}}$.

By referring back to Section 8.1, we easily see

$$(124) \quad \eta_{\psi,s} = \eta_{\phi_\psi,s}^{\text{NEV}}$$

for all Arthur parameters ψ with infinitesimal parameter λ , thus confirming Conjecture 1 in the examples. This implies (12) and also implies

$$(125) \quad \Pi_{\text{pure},\psi} = \Pi_{\text{pure},\phi_\psi}^{\text{ABV}}$$

for every Arthur parameter with infinitesimal parameter λ . We also verify the Kazhdan-Lusztig conjecture in each example, which allows us to verify Conjecture 2 in our examples. We show how the twisting characters χ_ψ from Section 8.1.5 relate to the twisting local system \mathcal{T}_ψ introduced in Section 6.9 and recalled in Section 8.2.8. While (125) shows that every Arthur packet is an ABV-packet, the converse is not true; in this article we find four examples of ABV-packets that are not Arthur packets. See Section 8.3 for more detail on this part of the examples.

In the fourth part, we show how to calculate endoscopic transfer, geometrically. Specifically, when G admits an elliptic endoscopic group G' and an infinitesimal parameter $\lambda' : W_F \rightarrow {}^L G'$ such that $\lambda = \epsilon \circ \lambda'$ with $\epsilon : {}^L G' \rightarrow {}^L G$, we show how the transfer of stable distributions attached to Arthur parameter for G' to G may be apprehended through the restriction of equivariant perverse sheaves from V_λ to $V_{\lambda'}$. To see this, for each simple $\mathcal{P} \in \text{Per}_{H_\lambda}(V_\lambda)$, we calculate every term in the identity

$$(126) \quad \text{trace}_{a'_s} \text{NEV}_{\psi'} \mathcal{P}|_{V'} = (-1)^{\dim C - \dim C'} \text{trace}_{a_s} \text{NEV}_\psi \mathcal{P},$$

where $\psi' \in T_{C'}^*(V')_{\text{reg}}$ with image $\psi \in T_C^*(V)_{\text{reg}}$, where the semisimple $s \in \widehat{G}$ is part of the endoscopic data of G' , a_s is the image of s in A_ψ and a'_s is the image of s in $A'_{\psi'}$. See Section 8.4 for more detail on this part of the examples.

Although do not show every calculation in every example, in Section 8 we explain the ideas needed and then illustrate them as they appear in the examples.

8. Template for the examples

Here we explain the plan for all the examples. We have tried to make the examples (Sections 9 through 14) as brief as possible, by making repeated reference back to this section.

In each example we begin by choosing G from the following list of split algebraic groups over a p -adic field F : in order, we take G to be $\text{SL}(2)$, $\text{SO}(3)$, $\text{PGL}(4)$, $\text{SO}(5)$, $\text{SO}(5)$ again, and finally, $\text{SO}(7)$. In each case we find $Z^1(F, G)$, and thus all pure rational forms of G , and relate these to the inner forms of G using the maps

$$H^1(F, G) \rightarrow H^1(F, G_{\text{ad}}) \rightarrow H^1(F, \text{Aut}(G)).$$

Every pure rational form $\delta \in Z^1(F, G)$ determines a rational form G_δ of G , often also called a pure rational form of G . The examples that we consider illustrate the fact that the maps above are neither injective nor surjective, in general. In each case we also fix an infinitesimal parameter

$$\lambda : W_F \rightarrow {}^L G.$$

We consider two infinitesimal parameters λ for $\mathrm{SO}(5)$, but otherwise choose one λ for each group in the list, above.

Having fixed G and $\lambda : W_F \rightarrow {}^L G$, we consider the conjectures from Section 7. We prove these conjectures by brute force calculation in these examples. However, our real goal here is to show how to use results from Part I to calculate the stable distributions in Arthur's local result [Art13, Theorem 1.5.1] and also how to calculate the coefficients that appear when these stable distributions are transferred to certain endoscopic groups. As a consequence, we give complete examples of [Art13, Theorem 1.5.1] and explain how to use geometry to make the calculations.

8.1. Arthur packets. — We enumerate all admissible representations π of all pure rational forms δ of G with a shared infinitesimal parameter λ . We show how these representations fall into L-packets, indexed by Langlands parameters ϕ with infinitesimal parameter λ . Then if ϕ is of Arthur type, we find corresponding the Arthur packet. We find the stable distributions attached to these L-packets, and also all the invariant distributions obtained from these representations by endoscopy.

8.1.1. Parameters. — We find all Langlands parameters $\phi : L_F \rightarrow {}^L G$ such that $\phi(w, d_w) = \lambda(w)$, where $d_w \in \mathrm{SL}(2)$ is defined by $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$, as in Section 2.5. As in Section 2.3, we write $P_\lambda({}^L G)$ for these Langlands parameters and $\Phi_\lambda(G/F)$ for the isomorphism classes of these Langlands parameters under $Z_{\widehat{G}}(\lambda)$ -conjugation.

Then we find all Arthur parameters $\psi : L_F \times \mathrm{SL}(2, \mathbb{C}) \rightarrow {}^L G$ such that $\psi(w, d_w, d_w) = \lambda(w)$. As in Section 2.4, the set of Arthur parameters that arise in this way is denoted by $Q_\lambda({}^L G)$. Although the map $Q_\lambda({}^L G) \rightarrow P_\lambda({}^L G)$ is injective, it is not surjective in general.

8.1.2. Admissible representations and their pure L-packets. — Now we can list all representations (π, δ) of all pure rational forms of G , in the sense of [Vog93], with infinitesimal parameter λ . This means that for every pure rational form $\delta \in Z^1(F, G)$, we find all irreducible admissible representations π of the rational form G_δ attached to G , such that the Langlands parameter ϕ for π lies in $P_\lambda({}^L G)$. These representations are not tempered in most of the cases considered in this article. When the pure rational form δ is clear from context, we may write π for (π, δ) .

We arrange these admissible representations into L-packets and into pure L-packets. For this, we must find the component group

$$A_\phi := Z_{\widehat{G}}(\phi) / Z_{\widehat{G}}(\phi)^0,$$

for each $\phi \in P_\lambda({}^L G)$. According to the pure Langlands correspondence [Vog93], equivalence classes of irreducible representations of pure rational forms of G with infinitesimal parameter λ are indexed by the set

$$\Xi_\lambda({}^L G) := \{(\phi, \rho) \mid \phi \in P_\lambda({}^L G) / Z_{\widehat{G}}(\lambda), \rho \in \mathrm{Irrep}(A_\phi)\}.$$

By abuse of notation, we write $\pi(\phi, \rho)$ for an irreducible admissible representation of $G(F)$ corresponding to a pair (ϕ, ρ) above. Each $\rho \in \mathrm{Irrep}(A_\phi)$ determines the class of a pure rational form, denoted by $\delta_\rho \in Z^1(F, G)$, so the L-packet for ϕ and a

rational form G_δ is

$$\Pi_\phi(G_\delta(F)) = \{[\pi(\phi, \rho)] \mid \phi \in P_\lambda({}^L G), \rho \in \text{Irrep}(A_\phi), [\delta_\rho] = [\delta] \in H^1(F, G)\}.$$

We find these L-packets, for all $\phi \in P_\lambda({}^L G)$ and all $\delta \in Z^1(F, G)$, in our examples. We also find the pure L-packets:

$$\Pi_{\text{pure}, \phi}(G/F) = \{[\pi(\phi, \rho), \delta_\rho] \mid \phi \in P_\lambda({}^L G), \rho \in \text{Irrep}(A_\phi)\},$$

for all $\phi \in P_\lambda({}^L G)$. To simplify notation slightly, we often write $\pi(\phi, \rho)$ for the pair $(\pi(\phi, \rho), \delta_\rho)$.

8.1.3. Multiplicity matrix. — To describe the representations with infinitesimal parameter λ we present the multiplicity $m_{\text{rep}}((\phi, \rho), (\phi', \rho'))$ of $\pi(\phi, \rho)$ in the standard module $M(\phi, \rho)$ so that in the Grothendieck group of admissible representations generated by $\Pi_{\text{pure}, \lambda}(G/F)$ we have

$$M(\phi', \rho') \equiv \sum_{(\phi, \rho)} m_{\text{rep}}((\phi, \rho), (\phi', \rho')) \pi(\phi, \rho),$$

where the sum is taken over all $\phi \in P_\lambda({}^L G)$ and all $\rho \in \text{Irrep}(A_\phi)$.

The idea of computing the multiplicities in the standard modules is to compare the Jacquet modules of the standard modules with those of irreducible representations. To be more precise, one can always make some guesses of what should be inside the standard modules by looking at the corresponding inducing representations. Then one can further argue that they are really there. To see there is nothing else, it is enough to show that the Jacquet modules of the standard modules have been exhausted by these representations. We give a sample calculation using this strategy in Section 14.1.3.

8.1.4. Arthur packets. — Recall $Q_\lambda({}^L G)$ from Section 8.1.1. For each $\psi \in Q_\lambda({}^L G)$ we show how the admissible representations above are grouped into Arthur packets

$$\Pi_\psi(G_\delta(F))$$

for rational forms δ of G . Of course, $\Pi_\psi(G_\delta(F))$ contains the L-packet $\Pi_{\phi_\psi}(G_\delta(F))$; our interest is in the representations in $\Pi_\psi(G_\delta(F))$ that are not contained in $\Pi_{\phi_\psi}(G_\delta(F))$; we referred to these as *coronal representations* in Section 7.4. In fact, we further recall the adaptation of Arthur packets to pure rational forms and find the pure Arthur packets

$$\Pi_{\text{pure}, \psi}(G/F)$$

themselves.

Arthur's main local result for quasisplit classical groups is expressed in terms of a map

$$(127) \quad \begin{array}{ccc} \Pi_\psi(G(F)) & \rightarrow & \widehat{\mathcal{S}}_\psi, \\ \pi & \mapsto & \langle \cdot, \pi \rangle_\psi \end{array}$$

where $\mathcal{S}_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0 Z(\widehat{G})^{\Gamma_F}$. As we saw in Section 2, this is easily rephrased in terms of a map

$$(128) \quad \Pi_\psi(G(F)) \rightarrow \text{Irrep}(A_\psi),$$

where

$$A_\psi = Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^0.$$

We find this map in our examples. In fact, using [Art13, Chapter 9], we find the conjectured extension

$$(129) \quad \Pi_{\text{pure}, \psi}(G/F) \rightarrow \text{Irrep}(A_\psi)$$

which includes the non-quasi-split pure rational forms of G , as discussed in Section 2.

8.1.5. Aubert duality. — Aubert involution preserves the infinitesimal parameter λ and so defines an involution on $\text{K}\Pi_\lambda(G_\delta(F))$, for every pure ration form δ for G . For $\pi \in \Pi_\lambda(G_\delta(F))$ we use the notation $\hat{\pi}$ for the admissible representation such that $(-1)^{a(\pi)}\hat{\pi}$ is the Aubert dual of π in $\text{K}\Pi_\lambda(G_\delta(F))$. When restricted to Arthur packets, Aubert duality defines a bijection

$$\begin{aligned} \Pi_\psi(G_\delta(F)) &\rightarrow \Pi_{\hat{\psi}}(G_\delta(F)) \\ \pi &\mapsto \hat{\pi}, \end{aligned}$$

where $\hat{\psi}(w, x, y) := \psi(w, y, x)$. We display this bijection in our examples.

Although the component groups A_ψ and $A_{\hat{\psi}}$ are isomorphic, a comparison of the characters $\langle \cdot, \pi \rangle_\psi$ and $\langle \cdot, \hat{\pi} \rangle_{\hat{\psi}}$ shows that they do not coincide, in general. Accordingly, their ratio defines a character χ_ψ of A_ψ such that

$$(130) \quad \langle s, \hat{\pi} \rangle_{\hat{\psi}} = \chi_\psi(s) \langle s, \pi \rangle_\psi,$$

for $s \in Z_{\widehat{G}}(\psi)$ where, as usual, we use the map $Z_{\widehat{G}}(\psi) \rightarrow A_\psi$. In our examples, this character χ_ψ of A_ψ is given by

$$(131) \quad \chi_\psi = \epsilon_\psi^{M/W} \epsilon_{\hat{\psi}}^{M/W},$$

where $\epsilon_\psi^{M/W}$ is the character of A_ψ appearing in [Xu17, Theorem 8.9]. As explained in [Xu, Introduction], the character $\epsilon_\psi^{M/W}$ measures the difference between Mœglin's parametrization of representations in Π_ψ by A_ψ and Arthur's parametrization of representations in Π_ψ by A_ψ . We compute the character χ_ψ in our examples; it is non-trivial in Sections 13.1.5 and 14.1.5 only.

8.1.6. Stable distributions and endoscopy. — Armed with (128), we easily find the coefficients in the stable invariant distribution

$$(132) \quad \Theta_\psi^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \text{trace } \pi,$$

where s_ψ denotes the image of the non-trivial central element in $\text{SL}(2)$ in A_ψ . Likewise, for $s \in Z_{\widehat{G}}(\psi)$ we compute

$$(133) \quad \Theta_{\psi, s}^G = \sum_{\pi \in \Pi_\psi(G(F))} \langle ss_\psi, \pi \rangle_\psi \text{trace } \pi.$$

Arthur's work shows that $\Theta_{\psi,s}$ is the Langlands-Shelstad transfer of the invariant distribution

$$(134) \quad \Theta_{\psi'}^{G'} = \sum_{\pi' \in \Pi_{\psi'}(G'(F))} \langle s_{\psi'}, \pi' \rangle_{\psi'} \text{trace } \pi',$$

from the endoscopic group G' attached to s , where $\psi' : L_F \times \text{SL}(2) \rightarrow {}^L G'$ factors through ${}^L G' \rightarrow {}^L G$ thus defining $\psi' : L_F \times \text{SL}(2) \rightarrow {}^L G'$. In our examples, we illustrate this fact by choosing a particular $s \in \widehat{G}$ and computing $\Theta_{\psi'}$.

In order to illuminate Conjecture 1 we use (129) to exhibit the virtual representations

$$(135) \quad \eta_{\psi} = \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)} e(\delta) \langle s_{\psi}, [\pi, \delta] \rangle_{\psi} [\pi, \delta]$$

and

$$(136) \quad \eta_{\psi,s} = \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \psi}(G/F)} e(\delta) \langle ss_{\psi}, [\pi, \delta] \rangle_{\psi} [\pi, \delta]$$

for $s \in Z_{\widehat{G}}(\psi)$, as defined in Section 7. Likewise we find

$$(137) \quad \eta_{\psi'} = \sum_{[\pi', \delta'] \in \Pi_{\text{pure}, \psi'}(G'/F)} e(\delta') \langle s_{\psi'}, [\pi', \delta'] \rangle_{\psi'} [\pi', \delta']$$

with s and ψ' as above.

8.2. Vanishing cycles of perverse sheaves. — Having reviewed Arthur packets and transfer coefficients for the chosen G and $\lambda : W_F \rightarrow {}^L G$, we now turn to geometry. In this section we introduce the geometric tools needed to demonstrate Conjecture 1 and calculate the coefficients $\langle ss_{\psi}, [\pi, \delta] \rangle_{\psi}$ appearing above. This is done by a brute force calculation of the exact functor

$${}^p\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{reg}}),$$

defined in Section 6, on simple objects, following a strategy that we now explain.

8.2.1. Vogan variety. — We find the variety $V := V_{\lambda}$ attached to the infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$, the action of $H := H_{\lambda} := Z_{\widehat{G}}(\lambda)$ on V , and the stratification of V into H -orbits. If λ is not unramified, we use Theorem 4.1.1 to replace the action $H_{\lambda} \times V_{\lambda} \rightarrow V_{\lambda}$ with $H_{\lambda_{\text{nr}}} \times V_{\lambda_{\text{nr}}} \rightarrow V_{\lambda_{\text{nr}}}$ where $\lambda_{\text{nr}} : W_F \rightarrow {}^L G_{\lambda}$ is the "unramification" of $\lambda : W_F \rightarrow {}^L G$. We may now assume λ is unramified and $\lambda(\text{Fr})$ is elliptic semisimple in \widehat{G} .

For classical groups, the variety V admits a description which is quite convenient for calculations, as we now explain.

First consider the case $G = \text{GL}(n)$. The variety V can be decomposed as a finite direct product of varieties according to

$$V \cong \text{Hom}(E_0, E_1) \times \text{Hom}(E_1, E_2) \times \cdots \times \text{Hom}(E_{r-1}, E_r),$$

where each E_i is an eigenspace for $\lambda(\text{Fr})$ with eigenvalue λ_i . We may then denote elements of V , i.e., quiver representations, by $v = (v_{i,i+1})_i$, for $v_{i,i+1} \in \text{Hom}(E_i, E_{i+1})$.

Then

$$H \cong \mathrm{GL}(E_0) \times \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_r)$$

acting on $\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{r-1}, E_r)$ by $h_i \cdot v_{i,i+1} = v_{i,i+1} \circ h_i^{-1}$ and $h_i \cdot v_{i-1,i} = h_i \circ v_{i-1,i}$ and $h_i \cdot v_{j,j+1} = v_{j,j+1}$ for $j \neq i, i-1$. The H -orbit of $v \in V$ is fully characterized by the collection of integers

$$r_{ij} := \mathrm{rank}(v_{j-1,j} \circ \cdots \circ v_{i,i+1}).$$

One derives a natural set of inequalities which describes admissible collections of ranks. The partial order of adjacency is identical to the partial ordering on the symbols $(r_{ij})_{ij}$.

We next note, that in general, passing between G , its derived group, its adjoint form or its simply connected form (or effectively any other associated form), has no impact on the variety V nor on the *type* of the group H . It does however tend to alter significantly the center of the group H . Though this will not impact the collection of orbits of H in V , it will tend to have a significant impact on the equivariant fundamental groups, and hence the set of equivariant local systems which must be considered.

Passing from the case when the derived group of G is of type A_n to the classical forms of B_n , C_n or D_n simply results in an identification of the λ_i eigenspace of $\lambda(\mathrm{Fr})$ with the dual of the λ_i^{-1} eigenspace. There are essentially two cases to consider: either $E_i = E_{r-i}^*$ or no two of E_0, \dots, E_r are dual. In the later case, V is isomorphic to one arising from an inclusion of a subgroup of type A_n and one can freely study the variety by passing to this subgroup. In the former case, there are essentially four sub-cases depending on if we are inside an orthogonal or symplectic group and if r is even or odd. In either case the variety we are studying is the one where $v_{i,i+1} = v_{r-i-1,r-i}^t$ and the group acting factors through $h_i = h_{r-i}^t$. These equations impose further, obvious, restrictions on the set of admissible collections of ranks/nullities, but otherwise the collection of strata is still indexed by the set of admissible vectors $(r_{ij})_{ij}$ and the adjacency relations do not change.

For simplicity of exposition one can describe these varieties which occur when G is of type B_n as one of

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{\ell-1}, E_\ell) \times \mathrm{Sym}^2(E_\ell^*)$$

with the group acting being $\mathrm{GL}(E_i)$ at every factor or

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{\ell-1}, E_\ell)$$

Where the group acts by $\mathrm{GL}(E_i)$ on every factor except E_ℓ where the group is $\mathrm{Sp}(E_\ell)$. When G is of type C_n or D_n they are

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{\ell-1}, E_\ell) \times \mathrm{Alt}^2(E_\ell^*)$$

with the group acting being $\mathrm{GL}(E_i)$ at every factor.

$$\mathrm{Hom}(E_0, E_1) \times \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{\ell-1}, E_\ell),$$

where the group acts by $\mathrm{GL}(E_i)$ on every factor except E_ℓ where the group is $\mathrm{O}(E_\ell)$. In all of these cases, ℓ is either $r/2$ or $(r+1)/2$, and the combinatorial data which describes the strata is still the collection of ranks $r_{i,j}$ for $0 \leq i < j \leq r$.

8.2.2. Orbit duality. — As we saw in Section 5, the cotangent bundle $T^*(V)$ is equipped with two important functions: the natural pairing $(\cdot | \cdot) : T^*(V) \rightarrow \mathbb{A}^1$ which coincides with the restriction of the Killing form on \mathfrak{j}_λ ; and $[\cdot, \cdot] : T^*(V) \rightarrow \mathfrak{h}$ which coincides with the restriction of the Lie bracket on \mathfrak{j}_λ . In particular, for every H -orbit C in V ,

$$T_C^*(V) = \{(x, \xi) \in T^*(V) \mid x \in C, [x, \xi] = 0\}.$$

In the examples, we present the duality between H -orbits C in V and H -orbits C^* in V^* , defined by the property that they have isomorphic conormal bundles

$$\overline{T_C^*(V)} \cong \overline{T_{C^*}^*(V^*)}$$

under $T^*(V) \rightarrow T^*(V^*)$ given by $(x, \xi) \mapsto (\xi, x)$, where we identify V^{**} with V using $(\cdot | \cdot)$. In fact, this duality between H -orbits in V and H -orbits in V^* is also characterized by the following statement:

$$(138) \quad T_C^*(V)_{\text{reg}} \subseteq C \times C^*,$$

where

$$T_C^*(V)_{\text{reg}} := T_C^*(V) \setminus \bigcup_{C' \subsetneq C} \overline{T_{C'}^*(V)}.$$

In the examples we present all this information by describing the conormal bundle

$$T_H^*(V) := \bigcup_C T_C^*(V),$$

where the union is taken over all H -orbits C in V and the union is taken in $T^*(V)$. We also describe the regular conormal bundle

$$T_H^*(V)_{\text{reg}} := \bigcup_C T_C^*(V)_{\text{reg}}.$$

From this, one simply restricts the bundle maps $T^*(V) \rightarrow V$ and $T^*(V) \rightarrow V^*$ to $T_C^*(V)_{\text{reg}}$ to recover C and its dual orbit C^* .

8.2.3. Equivariant perverse sheaves. — The next step is to find all simple objects in the category $\text{Per}_H(V)$ of H -equivariant perverse sheaves on V . Again, we use Theorem 4.1.1 to reduce to the case when λ is unramified and $\lambda(\text{Fr})$ is hyperbolic.

It is convenient to begin by enumerating all equivariant local systems \mathcal{L} on all H -orbits C in V . This is done by picking a base point $x \in C$ and computing the equivariant fundamental group

$$A_x := \pi_0(Z_H(x)) \cong \pi_1(C, x)_{Z_H(x)^0}.$$

Since the isomorphism type of this group is independent of the choice of base point, this group is commonly denoted by A_C . For the groups G that we consider here, the fundamental group A_C is always abelian, but this is not true in general. In any case, the choice of $x \in C$ determines an equivalence

$$\text{Rep}(A_C) \rightarrow \text{Loc}_H(C).$$

It is now easy to enumerate all simple objects in category $\text{Per}_H(V)$:

$$\text{Per}_H(V)_{/\text{iso}}^{\text{simple}} = \left\{ \mathcal{IC}(C, \mathcal{L}) \mid H\text{-orbit } C \subseteq V, \mathcal{L} \in \text{Loc}_H(V)_{/\text{iso}}^{\text{simple}} \right\}.$$

We will need to compute the equivariant perverse sheaves $\mathcal{IC}(C, \mathcal{L})$ themselves, or rather, their image in the Grothendieck group

$$\mathrm{Per}_H(V) \rightarrow \mathrm{KPer}_H(V) = \mathrm{KD}_{c,H}^b(V).$$

For every H -orbit C in V and every H -equivariant local system \mathcal{L} on V , consider the shifted standard sheaf

$$\mathcal{S}(C, \mathcal{L}) := j_{C!} \mathcal{L}[\dim C],$$

where $j_C : C \hookrightarrow V_\lambda$ is inclusion. Then, in $\mathrm{KPer}_{H_\lambda}(V_\lambda)$ we have

$$\mathcal{IC}(C, \mathcal{L}) \equiv \sum_{(C', \mathcal{L}')} m_{\mathrm{geo}}((C', \mathcal{L}'), (C, \mathcal{L})) \mathcal{S}(C', \mathcal{L}')$$

and $m_{\mathrm{geo}}((C, \mathcal{L}), (C, \mathcal{L})) = 1$ and $m_{\mathrm{geo}}((C', \mathcal{L}'), (C, \mathcal{L})) = 0$ unless $C' \leq C$. We refer to the matrix m_{geo} as the geometric multiplicity matrix. Set

$$\mathcal{L}^\sharp := \mathcal{IC}(C, \mathcal{L})[-\dim C] \quad \text{and} \quad \mathcal{L}^\natural := \mathcal{S}(C, \mathcal{L})[-\dim C].$$

Then, in $\mathrm{KPer}_{H_\lambda}(V_\lambda)$,

$$\mathcal{L}^\sharp \equiv \sum_{(C', \mathcal{L}')} (-1)^{\dim C - \dim C'} m_{\mathrm{geo}}((C', \mathcal{L}'), (C, \mathcal{L})) \mathcal{L}'^\natural.$$

A purity result of Lusztig shows that \mathcal{L}^\sharp is cohomologically concentrated in even degrees, so

$$m'_{\mathrm{geo}}((C', \mathcal{L}'), (C, \mathcal{L})) := (-1)^{\dim C - \dim C'} m_{\mathrm{geo}}((C', \mathcal{L}'), (C, \mathcal{L}))$$

is a non-negative integer. We refer to the matrix m'_{geo} as the normalised geometric multiplicity matrix.

We compute the normalised geometric multiplicity matrix m'_{geo} in each example in this article. In Sections 9.2.2 and 11.2.2 we use Theorem 4.1.1 to make this calculation. In Sections 10.2.2, 12.2.2, 13.2.3 and 14.2.3 we give examples of the following strategy. For each stratum $C \subseteq V$ and each local system \mathcal{L} on C , we construct a proper cover $\pi : \tilde{C} \rightarrow \overline{C}$ such that \tilde{C} is smooth and $\mathcal{IC}(C, \mathcal{L})$ appears in $\pi_! \mathbb{1}_{\tilde{C}}[\dim \tilde{C}]$. We can explicitly describe the fibres of π over each stratum in \overline{C} and typically arrange things so that the cover is semi-small, though this is not essential. We then find all the other simple perverse sheaves $\mathcal{IC}(C', \mathcal{L}')$, for $C' \leq C$, appearing in $\pi_! \mathbb{1}_{\tilde{C}}[\dim \tilde{C}]$, using the Decomposition Theorem. By doing this for C and all strata on the boundary of C , we can describe $\mathcal{IC}(C, \mathcal{L})$. Note that this process is performed inductively on $\dim C$, as well as on $\mathrm{rank}(\pi_! \mathbb{1}_{\tilde{C}})|_C$.

8.2.4. Cuspidal support decomposition and Fourier transform. — Category $\mathrm{Per}_H(V)$ decomposes into a direct sum of full subcategories indexed by cuspidal pairs for \widehat{G} , or more correctly, cuspidal local systems on cuspidal pairs [Lus95b, Proposition 8.16]. We refer to this as the *cuspidal support decomposition* of $\mathrm{Per}_H(V)$:

$$\mathrm{Per}_H(V) = \bigoplus_{(L, \mathcal{O}, \mathcal{E})} \mathrm{Per}_H(V)_{L, \mathcal{O}, \mathcal{E}},$$

where the sum is taken over all cuspidal Levi subgroups L of \widehat{G} , and all cuspidal local systems \mathcal{E} on nilpotent orbits $\mathcal{O} \subset \text{Lie } L$, up to \widehat{G} -conjugation. In the cases we consider there is only one $(\mathcal{O}, \mathcal{E})$ for every cuspidal Levi L , so we abbreviate $\text{Per}_H(V)_{L,C,\mathcal{E}}$ to $\text{Per}_H(V)_L$. In each example we partition the simple objects in $\text{Per}_H(V)$ according to this decomposition. Simple objects in $\text{Per}_H(V)_L$ are characterized by the property that they appear in the semisimple complex formed by parabolic induction along Vogan varieties from the cuspidal local system on $\text{Lie } L \cap V$; see [Lus95c].

The cuspidal support decomposition of $\text{Per}_H(V)$ offers insight into the blocks that appearing within the geometric multiplicity matrix. It is also quite helpful for finding the proper covers appearing in Section 8.2.3.

We also compute the Fourier transform

$$\text{Ft} : \text{Per}_H(V) \rightarrow \text{Per}_H(V^*)$$

on all simple objects. This functor is compatible with the cuspidal support decomposition in the sense that Ft restricts to $\text{Per}_H(V)_L \rightarrow \text{Per}_H(V^*)_L$.

8.2.5. Local systems on the regular conormal bundle. — In preparation for the calculation of $\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{reg}})$, we must describe local systems on H -orbits $T_C^*(V)_{\text{sreg}}$ and also show how local systems relate to the pullback of local systems along the bundle maps $T_C^*(V)_{\text{sreg}} \rightarrow C$ and $T_C^*(V)_{\text{sreg}} \rightarrow C^*$. For this we pick a base point $(x, \xi) \in T_C^*(V_\lambda)_{\text{sreg}}$ and compute the equivariant fundamental groups

$$A_{(x,\xi)} = \pi_0(Z_H(x, \xi)) = \pi_1(T_C^*(V)_{\text{sreg}}, (x, \xi))_{Z_H(x, \xi)^0}.$$

The isomorphism type of $A_{(x,\xi)}$ is independent of the choice of base point; it is precisely the microlocal fundamental group of C , denoted by A_C^{mic} . So the choice of base point determines an equivalence

$$\text{Rep}(A_C^{\text{mic}}) \rightarrow \text{Loc}_H(T_C^*(V)_{\text{sreg}}).$$

We use this to enumerate the simple objects in $\text{Loc}_H(T_C^*(V)_{\text{sreg}})$ and then to describe the functors

$$\text{Loc}_H(C) \longrightarrow \text{Loc}_H(T_C^*(V)_{\text{sreg}}) \longleftarrow \text{Loc}_H(C^*)$$

obtained by pullback the along the projections

$$C \longleftarrow T_C^*(V)_{\text{sreg}} \longrightarrow C^*,$$

by way of the induced homomorphisms of equivariant fundamental groups.

$$A_x \longleftarrow A_{(x,\xi)} \longrightarrow A_\xi.$$

8.2.6. Vanishing cycles of perverse sheaves. — Here we present the results of applying the functor

$${}^p\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{sreg}})$$

to simple objects in $\text{Per}_H(V)$. Recall from Section 6 that ${}^p\text{Ev} = \oplus_{C'} {}^p\text{Ev}_{C'}$, where

$${}^p\text{Ev}_{C'} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_{C'}^*(V)_{\text{reg}}).$$

is defined by

$${}^p\text{Ev}_C(\mathcal{F}) = \text{R}\Phi_{(\cdot|\cdot)}[-1](\mathcal{F} \boxtimes \mathbb{1}_{C^*})|_{T_C^*(V)_{\text{reg}}}[\dim C^*],$$

where $(\cdot|\cdot) : T^*(V) \rightarrow \mathbb{A}^1$ appeared in Section 8.2.2. Recall also from Section 6 that

$$({}^p\mathrm{Ev} \mathcal{F})_{(x,\xi)} = (\mathrm{R}\Phi_\xi[-1]\mathcal{F})_x[\dim C^*],$$

for all $(x, \xi) \in T_H^*(V)_{\mathrm{reg}}$.

We present the results of our calculations in a table which offers two perspectives on ${}^p\mathrm{Ev}$. Then recall that if $\mathcal{I}\mathcal{C}(C, \mathcal{L})$ is simple, then ${}^p\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})[-\dim V]$ is a local system on $T_{C'}^*(V)_{\mathrm{reg}}$ and this local system is determined by its restriction $\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ to the H -orbit $T_{C'}^*(V)_{\mathrm{sreg}}$. Our tables record ${}^p\mathrm{Ev} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ in form $\oplus_{C'} \mathcal{I}\mathcal{C}(\mathcal{O}', \mathcal{E}')$, where $\mathcal{O}' := T_{C'}^*(V)_{\mathrm{sreg}}$. To describe each \mathcal{E}' , we use the base points $(x', \xi') \in T_{C'}^*(V)_{\mathrm{sreg}}$ to view $\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ as a representation of the equivariant fundamental group $A_{(x', \xi')}$ of $T_{C'}^*(V)_{\mathrm{sreg}}$. The second part of the table records the characters of the representations $\mathrm{Ev}_{(x', \xi')} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ of $A_{(x', \xi')}$, as C' ranges over all strata in V and as $\mathcal{I}\mathcal{C}(C, \mathcal{L})$ ranges over all simple objects in $\mathrm{Per}_H(V)$.

By Proposition 6.5.1 we know that ${}^p\mathrm{Ev}_{C'} \mathcal{P} = 0$ unless $C' \subseteq \mathrm{supp} \mathcal{I}\mathcal{C}(C, \mathcal{L})$, which is to say, unless $C' \leq C$. Proposition 6.6.2 shows that in the case $C' = C$, we get

$$\mathrm{Ev}_C \mathcal{I}\mathcal{C}(C, \mathcal{L}) = \mathcal{T}_C \otimes (p^* \mathcal{L})|_{T_C^*(V)_{\mathrm{sreg}}},$$

where $p : T_C^*(V) \rightarrow C$ is the restriction of the bundle map $T^*(V) \rightarrow V$ and where \mathcal{T}_C is the local system defined in Section 6.9. The local systems $(p^* \mathcal{L})|_{T_C^*(V)_{\mathrm{sreg}}}$ were described in Section 8.2.5 and they are worked out in the corresponding sections in each example. The work that remains to calculate ${}^p\mathrm{Ev} \mathcal{I}\mathcal{C}(C, \mathcal{L})$, therefore, is the cases $\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ for $C' < C$.

To calculate ${}^p\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ for $C' < C$ we use Lemma 6.5.2. We describe our method in some detail here. From Section 8.2.3 we recall a proper map $\pi : \tilde{C} \rightarrow \overline{C}$ from a smooth variety \tilde{C} chosen so that $\mathcal{I}\mathcal{C}(C, \mathcal{L})$ appears in $\pi_* \mathbb{1}_{\tilde{C}}[\dim \tilde{C}]$. Using proper base change and the exactness of Ev , Proposition 6.4.1, we find $\mathrm{Ev}_{C'} \mathcal{I}\mathcal{C}(C, \mathcal{L})$ by computing

$$(139) \quad \left(\pi_s''^* \mathrm{R}\Phi_{(\cdot|\cdot) \circ (\pi \times \mathrm{id}_{C'^*})}(\mathbb{1}_{\tilde{C} \times C'^*}) \right) |_{T_{C'}^*(V)_{\mathrm{reg}}},$$

where π_s'' is defined in Section 6.5. Since $\tilde{C} \times C'^*$ is smooth and $\mathbb{1}_{\tilde{C} \times C'^*}$ is a local system, the vanishing cycles

$$(140) \quad \mathrm{R}\Phi_{(\cdot|\cdot) \circ (\pi \times \mathrm{id}_{C'^*})}(\mathbb{1}_{\tilde{C} \times C'^*})$$

is a skyscraper sheaf on the singular locus of $(\cdot|\cdot) \circ (\pi \times \mathrm{id}_{C'^*})$ on $\tilde{C} \times C'^*$. This singular locus is easy to find using the Jacobian condition for smoothness, because of the explicit nature of π and because we have already found equations for $\overline{C'^*}$ in V^* . The map π_s'' restricts to a proper map from this singular locus onto $T_{C'}^*(V)$. In fact, this map is finite over $T_{C'}^*(V)_{\mathrm{reg}}$; this is a post-hoc consequence of the fact that the fibres of π_s'' are closed and the stalks of the vanishing cycles functor are concentrated in a single degree. After restricting (140) to the preimage of $T_{C'}^*(V)_{\mathrm{reg}}$ under $\pi \times \mathrm{id}_{C'^*}$, we use the Decomposition Theorem to explicitly describe (139). While it is typically very easy to compute the rank of the resulting local system, determining the representation of the fundamental group that describes the local system is considerably more subtle as it depends on the local structure of the singularities. We give examples of these calculations in Sections 10.2.5, 12.2.5, 13.2.5 and 14.2.6.

We observe that many of these calculations may be simplified considerably by a judicious use of the formula (141) from Section 8.2.8 and formula (154) from Section 8.4.

8.2.7. Normalization of Ev and the twisting local system. — Having calculated ${}^p\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{reg}})$ in Section 8.2.6, here we calculate the normalization of Ev , as given in Definition 2. In the process, we make explicit the rank-one local system \mathcal{T} on $T_H^*(V)_{\text{sreg}}$ defined in (97).

8.2.8. Fourier transform and vanishing cycles. — In this section we predict how the Fourier transform interacts with vanishing cycles, or more precisely, with the functor Ev and its dual $\text{Ev}^* : \text{Per}_H(V^*) \rightarrow \text{Per}_H(T_H^*(V^*)_{\text{reg}})$, where the latter is defined exactly as above but with V replaced by V^* . We believe that there is a local system \mathcal{T}^{ft} on $T_H^*(V)_{\text{reg}}$ such that

$$a_* (\mathcal{T}^{\text{ft}} \otimes {}^p\text{Ev}) = {}^p\text{Ev}^* \text{Ft},$$

where $a : T^*(V) \rightarrow T^*(V^*)$ is the isomorphism $a(x, \xi) = (\xi, -x)$ and where we identify the dual of V^* with V using $(\cdot | \cdot)$. In our examples this local system \mathcal{T}^{ft} coincides with the local system \mathcal{T} which we introduced in (97). We show this by verifying the formula

$$(141) \quad a_* {}^p\text{NEv}_C = {}^p\text{Ev}_{C^*} \text{Ft},$$

for all strata $C \subseteq V$, in our examples. The rank-one local system \mathcal{T} is non-trivial in Sections 13.2.7 and 14.2.8, only.

8.2.9. Arthur sheaves. — In the examples we close each version of Section 8.2 by displaying the Arthur sheaves \mathcal{A}_C that appeared in Section 7.4, for each stratum $C \subseteq V$. These equivariant perverse sheaves are defined, up to isomorphism, by

$$\mathcal{A}_C := \sum_{\mathcal{P} \in \text{Per}_H(V)_{\text{iso}}^{\text{simple}}} (\text{rank Ev}_C \mathcal{P}) \mathcal{P}$$

We also remark that

$$(142) \quad \text{Ft } \mathcal{A}_C = \mathcal{A}_{C^*}.$$

8.3. ABV-packets. — Having calculated the vanishing cycles of perverse sheaves on Vogan varieties in Section 8.2, it is a simple matter now to find the ABV-packets for all Langlands parameters with given infinitesimal parameter. In this section we also see that the Arthur packets described in the examples are indeed ABV-packets. But the real object of the conjectures from Section 7 are the characters $\langle \cdot, \pi \rangle_\psi$ of A_ψ that appear in Arthur's main local result, and their generalisations to pure rational forms of G . In this part of each example we show

$$\langle s, \pi \rangle_\psi = \text{trace}_{a_s} \text{NEv}_{C_\psi} \mathcal{P}(\pi)$$

for $s \in Z_{\widehat{G}}(\psi)$ with image $a_s \in A_\psi$, and verify Conjecture 1 and Conjecture 2.

8.3.1. Admissible representations versus equivariant perverse sheaves. — As shown in Proposition 3.2.2 every Langlands parameter $\phi \in P_\lambda({}^L G)$ determines a point

$x_\phi \in V$ and every $x \in V$ arises in this way. The function $\phi \mapsto x_\phi$ is also H -equivariant, so it induces a bijection between $\Phi_\lambda(LG)$ and the set of H -orbits in V . We write C_ϕ for the H -orbit of x_ϕ . There is a canonical isomorphism of groups

$$(143) \quad A_\phi \cong A_{C_\phi},$$

where $A_\phi = \pi_0(Z_{\widehat{G}}(\phi))$ is the component group appearing in the pure Langlands correspondence. Consequently, there is a natural bijection between pairs (ϕ, ρ) , where ρ is a representation of A_ϕ , and pairs $(C_\phi, \mathcal{L}_\rho)$, where \mathcal{L}_ρ is the equivariant local system matching ρ under the isomorphism above. This, in turn, determines a bijection

$$(144) \quad \begin{aligned} \Pi_{\text{pure}, \lambda}(G/F) &\rightarrow \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}} \\ (\pi, \delta) &\mapsto \mathcal{P}(\pi, \delta) \end{aligned}$$

8.3.2. ABV-packets. — Using this bijection, we determine the ABV-packets for all Langlands parameters with infinitesimal parameter λ , in each example, using the definition

$$(145) \quad \Pi_{\text{pure}, \phi}^{\text{ABV}}(G/F) := \{[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F) \mid \text{Ev}_{C_\phi} \mathcal{P}(\pi, \delta) \neq 0\}.$$

By restricting our attention to Langlands parameters of Arthur type, we readily verify that all Arthur packets for all admissible representations with infinitesimal parameter λ are ABV-packets:

$$(146) \quad \Pi_{\text{pure}, \phi_\psi}^{\text{ABV}}(G/F) = \Pi_{\text{pure}, \psi}(G/F).$$

Having verified (146) in the examples, we turn to Conjecture 1, which begins with the canonical isomorphism

$$A_\psi \cong A_{C_\psi}^{\text{mic}},$$

where ψ is an Arthur parameter and where $C_\psi := C_{\phi_\psi}$. Right away, this isomorphism tells us that the character $\langle \cdot, \pi \rangle_\psi$ of A_ψ appearing in Arthur's main local result may be interpreted as an equivariant local system on $T_{C_\psi}^*(V)_{\text{sreg}}$. How does the admissible representation π of $G(F)$ determine that local system? That question is answered by Conjecture 1: for every $s \in Z_{\widehat{G}}(\psi)$ and for every admissible representation π of $G(F)$,

$$(147) \quad \langle ss_\psi, \pi \rangle_\psi = (-1)^{\dim C_\psi - \dim C_\pi} \text{trace}_{a_s} \text{NEv}_{C_\psi} \mathcal{P}(\pi),$$

where a_s is the image of $s \in Z_{\widehat{G}}(\psi)$ in A_ψ and where C_π is the stratum in V attached to the Langlands parameter of π . In other words, the equivariant local system on $T_{C_\psi}^*(V)_{\text{sreg}}$ determined by the admissible representation π of $G(F)$ is $\text{NEv}_{C_\psi} \mathcal{P}(\pi)$.

Having calculated the left-hand side of (147) in Section 8.1 and right-hand side in Section 8.2, we can prove Conjecture 1 in our examples by simply comparing the results of those calculations. In fact we confirm more in the examples, by showing that

$$(148) \quad \eta_{\psi, s} = \eta_{\psi, s}^{\text{NEv}},$$

for every Arthur parameter ψ with infinitesimal parameter λ and for every $s \in Z_{\widehat{G}}(\psi)$. Here, $\eta_{\psi, s}^{\text{NEv}}$ is defined in Section 7:

$$(149) \quad \eta_{\psi, s}^{\text{NEv}} = \sum_{[\pi, \delta] \in \Pi_{\text{pure}, \lambda}(G/F)} e(\delta) (-1)^{\dim C_\psi - \dim C_\pi} \text{trace}_{a_s} \text{Ev}_{C_\psi} \mathcal{P}(\pi, \delta) [\pi, \delta].$$

8.3.3. Kazhdan-Lusztig conjecture. — Recall in Section 7.3 that we have defined a pairing

$$\langle \cdot, \cdot \rangle : \mathbf{K}\Pi_{\text{pure},\lambda}(G/F) \times \mathbf{K}\text{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathbb{Z}$$

such that for any $(\phi, \rho), (\phi', \rho') \in \Xi_\lambda(LG)$

$$\langle \pi(\phi, \rho), \mathcal{P}(\phi', \rho') \rangle = (-1)^{\dim C_\phi} e(\phi, \rho) \delta_{(\phi, \rho), (\phi', \rho')}$$

where $e(\phi, \rho)$ is the Kottwitz sign of G_δ determined by (ϕ, ρ) . Kazhdan-Lusztig conjecture predicts that

$$\langle M(\phi, \rho), \mathcal{P}(\phi', \rho') \rangle = e(\phi, \rho) \delta_{(\phi, \rho), (\phi', \rho')}$$

for any $(\phi, \rho), (\phi', \rho') \in \Xi_\lambda(LG)$. We verify the Kazhdan-Lusztig conjecture in our examples. This is done by comparing the multiplicity matrix m_{rep} from Section 8.1.3 with the normalised geometric multiplicity matrix m'_{geo} from Section 8.2.3:

$${}^t m_{\text{rep}} = m'_{\text{geo}}.$$

As a consequence, we can verify Conjecture 2 in our examples following the argument below. Let $\mathbf{K}_{\mathbb{C}}\Pi_{\text{pure},\lambda}(G/F)^{\text{st}}$ be the subspace of strongly stable virtual representations in $\mathbf{K}_{\mathbb{C}}\Pi_{\text{pure},\lambda}(G/F) := \mathbf{K}\Pi_{\text{pure},\lambda}(G/F) \otimes_{\mathbb{Z}} \mathbb{C}$. It has a natural basis

$$\eta_\phi := \sum_{\rho: (\phi, \rho) \in \Xi_\lambda(LG)} \dim(\rho) e(\phi, \rho) M(\phi, \rho)$$

parametrized by $\phi \in P_\lambda(LG)/H_\lambda$. After identifying $\mathbf{K}_{\mathbb{C}}\Pi_{\text{pure},\lambda}(G/F)$ with

$$\mathbf{K}_{\mathbb{C}}\text{Per}_{H_\lambda}(V_\lambda)^* = \text{Hom}_{\mathbb{Z}}(\mathbf{K}\text{Per}_{H_\lambda}(V_\lambda), \mathbb{C}),$$

through the pairing above, we would like to characterize $\mathbf{K}_{\mathbb{C}}\Pi_{\text{pure},\lambda}(G/F)^{\text{st}}$ in $\mathbf{K}_{\mathbb{C}}\text{Per}_{H_\lambda}(V_\lambda)^*$. By the Kazhdan-Lusztig conjecture,

$$\langle \eta_\phi, \mathcal{P} \rangle = \chi_{C_\phi}^{\text{loc}}(\mathcal{P}) := \sum_{\rho: (\phi, \rho) \in \Xi_\lambda(LG)} (-1)^{\dim C_\phi} m_{\text{geo}}(\mathcal{S}(C_\phi, \mathcal{L}_\rho), \mathcal{P}),$$

for any $\mathcal{P} \in \mathbf{K}\text{Per}_{H_\lambda}(V_\lambda)$. Therefore, $\mathbf{K}_{\mathbb{C}}\Pi_{\text{pure},\lambda}(G/F)^{\text{st}}$ is spanned by $\chi_{C_\phi}^{\text{loc}}(\cdot)$ for $\phi \in P_\lambda(LG)/H_\lambda$ in $\mathbf{K}_{\mathbb{C}}\text{Per}_{H_\lambda}(V_\lambda)^*$. On the other hand, by Ginzburg, Kashiwara and Dubson [BDK81] [Kas73], we know that for any $\phi \in P_\lambda(LG)$ and $\mathcal{P} \in \mathbf{K}\text{Per}_{H_\lambda}(V_\lambda)$,

$$\chi_{C_\phi}^{\text{mic}}(\mathcal{P}) := \text{rank Ev}_{C_\phi}(\mathcal{P}) = \sum_{\phi' \in P_\lambda(LG)/H_\lambda} c(C_\phi, C_{\phi'}) \chi_{C_{\phi'}}^{\text{loc}}(\mathcal{P}),$$

where $c(C_\phi, C_{\phi'})$ satisfies the following properties: $c(C_\phi, C_\phi) = (-1)^{\dim C_\phi}$; and $c(C_\phi, C_{\phi'}) \neq 0$ only if $\bar{C}_{\phi'} \supseteq C_\phi$. The coefficients $c(C_\phi, C_{\phi'})$ are related to the local Euler obstructions defined by MacPherson. In particular, it measures the singularity of the closure of $C_{\phi'}$ at its boundary stratum C_ϕ . As a consequence, we see the set of $\chi_{C_\phi}^{\text{mic}}(\cdot)$ for $\phi \in P_\lambda(LG)/H_\lambda$ forms another basis for $\mathbf{K}\Pi_{\text{pure},\lambda}(G/F)_{\mathbb{C}}^{\text{st}}$. Finally, it is easy to see that for any $\phi \in P_\lambda(LG)$ and $\mathcal{P} \in \mathbf{K}\text{Per}_{H_\lambda}(V_\lambda)$

$$\langle \eta_{C_\phi}^{\text{NEv}}, \mathcal{P} \rangle = (-1)^{\dim C_\phi} \chi_{C_\phi}^{\text{mic}}(\mathcal{P}).$$

So the set of $\eta_{C_\phi}^{\text{NEv}}$ for $\phi \in P_\lambda(LG)/H_\lambda$ also forms a basis for $\mathbf{K}\Pi_{\text{pure},\lambda}(G/F)_{\mathbb{C}}^{\text{st}}$. This proves Conjecture 2.

8.3.4. Aubert duality and Fourier transform. — In order to compare Aubert duality with the Fourier transform, we equip V with the symmetric bilinear form $(x, y) \mapsto -(x | {}^t y)$, where t refers to transposition in j_λ , and we use this to define an isomorphism $V \rightarrow V^*$. We use the notation $\hat{C} := {}^t C^*$. Let $\vartheta : H \rightarrow H$ be the isomorphism of algebraic groups given by $\vartheta(h) = {}^t h^{-1}$, in which t refers to transposition in J_λ . Then $V \rightarrow V^*$ is equivariant for the usual action of H on V and the twist by ϑ of the usual action of H on V^* . Now, *equivariant* pullback defines an equivalence of categories $\text{Per}_H(V^*) \rightarrow \text{Per}_H(V)$. When pre-composed with the Fourier transform $\text{Ft} : \text{Per}_H(V) \rightarrow \text{Per}_H(V^*)$, this defines a functor denoted by ${}^\wedge : \text{Per}_H(V) \rightarrow \text{Per}_H(V)$. Our examples show

$$(150) \quad \mathcal{P}(\hat{\pi}, \delta) = \widehat{\mathcal{P}(\pi, \delta)}.$$

Using the equivalence $\text{Per}_H(V^*) \rightarrow \text{Per}_H(V)$ described above, (141) may be re-written in form

$$(151) \quad {}^p\text{NEv} \mathcal{P} = {}^p\text{Ev} \hat{\mathcal{P}},$$

Taking traces, and recalling ${}^p\text{NEv} = \mathcal{T}^\vee \otimes {}^p\text{Ev}$, this implies

$$\text{trace}_a \left(\text{NEv}_{\hat{C}} \hat{\mathcal{P}} \right) = \text{trace}_s \mathcal{T}_C \text{trace}_a (\text{NEv}_C \mathcal{P})$$

for every $a \in A_C^{\text{mic}}$. Taking $\mathcal{P} = \mathcal{P}(\pi)$ and $C = C_\psi$ and using (150) and (152), we recover (130).

8.3.5. Normalisation. — Recall the character χ_ψ of A_ψ given by (130). Recall from Section that our examples suggest that this character coincides with $\epsilon_\psi^{M/W} \epsilon_{\hat{\psi}}^{M/W}$ (131). Now recall the local system \mathcal{T} on $T_H^*(V^*)_{\text{reg}}$ appearing in Sections 8.2.6. In this article we see in our examples that

$$(152) \quad \text{trace } \mathcal{T}_\psi = \chi_\psi,$$

where \mathcal{T}_ψ is the restriction of the local system \mathcal{T} on $T_H^*(V)_{\text{sreg}}$ introduced in Section 6.9. Recall also that this local system \mathcal{T} appeared in our study of the Fourier transform, specifically, (141). It seems remarkable to us that the characters χ_ψ , $\epsilon_\psi^{M/W} \epsilon_{\hat{\psi}}^{M/W}$, $\text{trace } \mathcal{T}|_{C_\psi}$ and $\text{trace } \mathcal{T}^{\text{Rt}}|_{C_\psi}$ all coincide in our examples.

8.3.6. ABV-packets that are not pure Arthur packets. — While all pure Arthur packets are ABV-packets in these examples, it is not true that all ABV-packets are pure Arthur packets. In Sections 12.3.7 and 14.3.6 we discuss examples of ABV-packets that are not pure Arthur packets and yet enjoy many of the properties we expect from Arthur packets.

8.4. Endoscopy and equivariant restriction of perverse sheaves. — One of the ingredients in the proof of Conjecture 1 in [CFMX] for unipotent representations of odd orthogonal groups, is the following theorem. Let G' be an endoscopic group for G though which $\lambda : W_F \rightarrow {}^L G$ factors, thus defining $\lambda' : W_F \rightarrow {}^L G'$. Set $V' = V_{\lambda'}$. Let C' be an H' -orbit in V' ; pick $(x', \xi') \in T_{C'}^*(V')_{\text{reg}}$ and let C be the H -orbit in V of the image of x' under $V' \hookrightarrow V$. Suppose that the conormal map $T_{C'}^*(V') \rightarrow T_C^*(V)$ restricts to $T_{C'}^*(V')_{\text{reg}} \rightarrow T_C^*(V)_{\text{reg}}$. Let $(x, \xi) \in T_C^*(V)_{\text{reg}}$ be its

image of $(x', \xi') \in T_{C'}^*(V')_{\text{reg}}$ under that map. Then, for every $\mathcal{P} \in \text{Per}_H(V)$,

$$(153) \quad (-1)^{\dim C'} \text{trace}_{a'_s} (\text{NEv}' \mathcal{P}|_{V'})_{(x', \xi')} = (-1)^{\dim C} \text{trace}_{a_s} (\text{NEv} \mathcal{P})_{(x, \xi)},$$

where a_s is the image of s under $Z_{\widehat{G}}(x, \xi) \rightarrow A_{(x, \xi)}$ and a'_s is the image of s under $Z_{\widehat{G}'}(x', \xi') \rightarrow A_{(x', \xi')}$.

In the examples in this article, we calculate both sides of (153), independently, in order to illustrate how the functor of vanishing cycles Ev interacts with the equivariant restriction functor $D_H(V) \rightarrow D_{H'}(V')$. As explained in [CFMX], it is (154) that allows us to conclude that $\eta_{\phi, s}^{\text{NEv}}$ is the endoscopic transfer of a strongly stable virtual representation on G' .

Although we don't show the calculations here, the same arguments used to prove (153) also show

$$(154) \quad (-1)^{e_{C'} - \dim C'} \text{trace}_{a'_s} (\text{Ev}' \mathcal{P}|_{V'})_{(x', \xi')} = (-1)^{e_C - \dim C} \text{trace}_{a_s} (\text{Ev} \mathcal{P})_{(x, \xi)},$$

under the same hypotheses.

8.4.1. Endoscopic Vogan varieties. — After recalling the endoscopic groups G' and the infinitesimal parameters $\lambda' : W_F \rightarrow {}^L G'$ such that $\lambda = \epsilon \circ \lambda'$ from Section 8.1.6, we describe $V' := V_{\lambda'}$ and its stratification into orbits under the action by $H' := Z_{\widehat{G}'}(\lambda')$. In all cases, $G' = G^{(2)} \times G^{(1)}$ so $\lambda' = (\lambda^{(2)}, \lambda^{(1)})$. Except for Section 9, we have arranged the sequence of examples so that by the time we get to $\lambda' : W_F \rightarrow {}^L G'$, both $\lambda^{(1)} : W_F \rightarrow {}^L G^{(1)}$ and $\lambda^{(2)} : W_F \rightarrow {}^L G^{(2)}$ have already been studied. Since $H' = H^{(2)} \times H^{(1)}$ and $V' = V^{(2)} \times V^{(1)}$, we use the equivalence

$$\text{Per}_{H^{(2)}}(V^{(2)}) \times \text{Per}_{H^{(1)}}(V^{(1)}) \xrightarrow{\boxtimes} \text{Per}_{H'}(V')$$

to answer all questions about $\text{Per}_{H'}(V')$ using earlier work.

The H' -invariant function $(\cdot | \cdot) : T^*(V') \rightarrow \mathbb{A}^1$ is simply the sum of the functions $T^*(V^{(1)}) \rightarrow \mathbb{A}^1$ and $T^*(V^{(2)}) \rightarrow \mathbb{A}^1$ while $[\cdot, \cdot] : T^*(V') \rightarrow \mathfrak{h}'$ is likewise built from the functions $T^*(V^{(1)}) \rightarrow \mathfrak{h}^{(1)}$ and $T^*(V^{(2)}) \rightarrow \mathfrak{h}^{(2)}$. Consequently, the conormal bundle is

$$T_{H'}^*(V') = T_{H^{(2)}}^*(V^{(2)}) \times T_{H^{(1)}}^*(V^{(1)}),$$

so $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$ can be completely described using earlier work.

8.4.2. Vanishing cycles. — It follows from the Thom-Sebastiani Theorem, [III17] and [Mas01], that

$$\text{Ev}' \left(\mathcal{IC}(C^{(2)}, \mathcal{L}^{(2)}) \boxtimes \mathcal{IC}(C^{(1)}, \mathcal{L}^{(1)}) \right) = \left(\text{Ev} \mathcal{IC}(C^{(2)}, \mathcal{L}^{(2)}) \right) \boxtimes \left(\text{Ev} \mathcal{IC}(C^{(1)}, \mathcal{L}^{(1)}) \right).$$

Thus, the functor

$$\text{Ev}' : \text{Per}_{H'}(V') \rightarrow \text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$$

may also be deduced from earlier work.

8.4.3. Restriction. — The equivariant restriction functor

$$(155) \quad \begin{array}{ccc} D_H(V) & \longrightarrow & D_{H'}(V') \\ \mathcal{F} & \mapsto & \mathcal{F}|_{V'} \end{array}$$

does not take perverse sheaves to perverse sheaves. Since we wish to illustrate (154), we compute (155) in each example, after passing to Grothendieck groups.

8.4.4. Restriction and vanishing cycles. — We have now assembled all the pieces needed to illustrate (154). We begin by identifying all $(x', \xi') \in T_{H'}^*(V')_{\text{reg}}$ such that the image of (x', ξ') in $T_H^*(V)$ is regular. This gives us an opportunity to revisit the question of finding all Arthur parameters $\psi : L_F \times \text{SL}(2) \rightarrow {}^L G$ with infinitesimal parameter λ that factor through $\epsilon : {}^L G' \rightarrow {}^L G$ to define Arthur parameters $\psi : L_F \times \text{SL}(2) \rightarrow {}^L G$ with infinitesimal parameter λ' . Finally, for such (x', ξ') we pick a simple perverse sheaf $\mathcal{P} \in \text{Per}_H(V)$ and compute both sides of (154), where s is determined by the elliptic endoscopic group G' .

9. $\text{SL}(2)$ 4-packet of quadratic unipotent representations

Set $G = \text{SL}(2)$ over F ; so $\widehat{G} = \text{PGL}(2, \mathbb{C})$ and ${}^L G = \text{PGL}(2, \mathbb{C}) \times W_F$. Suppose q is odd.

The function $H^1(F, G) \rightarrow H^1(F, G_{\text{ad}})$ is injective but not surjective; indeed, $H^1(F, G)$ is trivial but $H^1(F, G_{\text{ad}}^*) \cong \mu_2$. In other words, $\text{SL}(2)$ has no pure rational forms but it does have an inner rational form.

Let $\varpi \in F$ be a uniformizer and let $u \in F$ be a non-square unit integer. Let E/F be the biquadratic extension $E = F(\sqrt{\varpi}, \sqrt{u})$. Then $\text{Gal}(E/F) = \{1, \sigma, \tau, \sigma\tau\}$ where $\sigma(\sqrt{u}) = -\sqrt{u}$ and $\tau(\sqrt{\varpi}) = -\sqrt{\varpi}$. Define $\varrho : \text{Gal}(E/F) \rightarrow \text{PGL}(2)$ by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\lambda : W_F \rightarrow {}^L G$ be the infinitesimal parameter defined by the composition $L_F \rightarrow W_F \rightarrow \Gamma_F \rightarrow \text{Gal}(E/F)$ followed by $\varrho : \text{Gal}(E/F) \rightarrow \widehat{G}$; thus,

$$\lambda(w) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w \in {}^L G, \quad \text{if } w|_E = \sigma,$$

and

$$\lambda(w) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w \in {}^L G, \quad \text{if } w|_E = \tau.$$

9.1. Arthur packets. —

9.1.1. Parameters. — There is only one Langlands parameter with infinitesimal parameter λ chosen above: $\phi(w, x) = \lambda(w)$. This Langlands parameter is of Arthur type: $\psi(w, x, y) = \lambda(w)$.

9.1.2. L -packets. — With ϕ as above, we have

$$Z_{\widehat{G}}(\phi) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let $A_\phi \cong \mu_2 \times \mu_2$ be the isomorphism determined by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto (-1, +1) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto (+1, -1).$$

Using this isomorphism, the characters of A_ϕ will be denoted by $(++)$, $(+-)$, $(-+)$ and $(--)$. The L-packet $\Pi_\phi(G(F))$ is the unique cardinality-4 L-packet for $\mathrm{SL}(2, F)$:

$$\Pi_\phi(G(F)) = \{\pi(\phi, ++), \pi(\phi, +-), \pi(\phi, -+), \pi(\phi, --)\}.$$

This L-packet, which is described in [She79, Section 11], may be obtained by restricting a supercuspidal representation of $\mathrm{GL}(2, F)$ given in [JL70, Theorem 4.6] to $\mathrm{SL}(2, F)$. Alternately, these depth-zero supercuspidal representations are produced by compact induction from a maximal parahoric (there are two, up to $G(F)$ -conjugation), from (the inflation of) the two cuspidal irreducible representations appearing in the only non-singleton Deligne-Lusztig representation of $\mathrm{SL}(2, \mathbb{F}_q)$. The characters of these representations are described in [ADSS11, Section 15].

Since G has no pure inner forms, the pure packet for the Langlands parameter ϕ is an L-packet:

$$\Pi_{\mathrm{pure}, \phi}(G/F) = \Pi_\phi(G(F)).$$

9.1.3. Arthur packets. — The L-packet $\Pi_\phi(G(F))$ is an Arthur packet:

$$\Pi_{\mathrm{pure}, \psi}(G/F) = \Pi_{\mathrm{pure}, \phi}(G/F).$$

9.1.4. Aubert duality. — Aubert involution fixes all the representations in this example.

9.1.5. Stable distributions and endoscopy. — Since ψ is trivial on $\mathrm{SL}(2)$ in its domain, it follows that $s_\psi = 1$, so the stable invariant distribution (132) attached to ψ is

$$\Theta_\psi = \mathrm{trace} \pi(\phi, ++) + \mathrm{trace} \pi(\phi, +-) + \mathrm{trace} \pi(\phi, -+) + \mathrm{trace} \pi(\phi, --).$$

For any $s \in Z_{\widehat{G}}(\psi)$ (the 4-group $Z_{\widehat{G}}(\phi)$ appearing in Section 9.1.2) the coefficients of $\Theta_{\psi, s}$ appearing in (133) are simply

$$(156) \quad \langle s, \pi(\phi, \pm\pm) \rangle_\psi = (\pm\pm)(s).$$

Besides G itself, three endoscopic groups are relevant to ψ : the unramified torus $\mathrm{U}(1)$ split over $F(\sqrt{u})$, and the two ramified tori split over $F(\sqrt{\varpi})$, and $F(\sqrt{u\varpi})$. More precisely, in the case of the unramified torus, take $s \in \widehat{G}$ to be

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and set

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that

$$s = \psi^\circ(w), \quad \text{if } w|_E = \tau$$

and

$$n = \psi^\circ(w), \quad \text{if } w|_E = \sigma.$$

Let G' be the endoscopic group $\mathrm{U}(1)$ split over $F(\sqrt{u})$ with: $\widehat{G'} = Z_{\widehat{G}}(s)^0$; action of W_F on $\widehat{G'}$ determined by $\pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{u})/F)$; and $\varepsilon : {}^L G' \rightarrow {}^L G$ given by

$$\widehat{G'} = Z_{\widehat{G}}(s)^0 \subset \widehat{G} \text{ and}$$

$$\varepsilon(1 \rtimes w) := nw, \quad \text{if } w|_E = \sigma.$$

Then the Arthur parameter $\psi : L_F \times \mathrm{SL}(2) \rightarrow {}^L G$ factors through $\varepsilon : {}^L G' \rightarrow {}^L G$ to define $\psi' : L_F \times \mathrm{SL}(2) \rightarrow {}^L G'$, so

$$\psi'(w) = s \rtimes w \in {}^L G', \quad \text{if } w|_E = \tau.$$

The representation of $G'(F)$ with Arthur parameter ψ' is the quadratic character attached to the extension $F(\sqrt{u})/F$ by class field theory. Then the endoscopic transfer of the quadratic character from $G'(F)$ to $G(F)$ is

$$\Theta_{\psi,s} = \mathrm{trace} \pi(\phi, ++) - \mathrm{trace} \pi(\phi, +-) + \mathrm{trace} \pi(\phi, -+) - \mathrm{trace} \pi(\phi, --).$$

The set-up is similar for the ramified tori, as we now explain. Take

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and, in the same order, set

$$n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{respectively,} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$s = \psi^\circ(w), \quad \text{if } w|_E = \sigma, \quad \text{respectively, } w|_E = \sigma\tau,$$

and

$$n = \psi^\circ(w), \quad \text{if } w|_E = \sigma\tau, \quad \text{respectively, } w|_E = \tau.$$

Let G' be the endoscopic group $\mathrm{U}(1)$ split over $F(\sqrt{\varpi})$, respectively, $F(\sqrt{u\varpi})$ with: $\widehat{G'} = Z_{\widehat{G}}(s)^0$; action of W_F on $\widehat{G'}$ determined by

$$\pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{\varpi})/F), \quad \text{respectively, } \pi_0(Z_{\widehat{G}}(s)) \cong \mathrm{Gal}(F(\sqrt{u\varpi})/F);$$

and $\varepsilon : {}^L G' \rightarrow {}^L G$ given by $\widehat{G'} = Z_{\widehat{G}}(s)^0 \subset \widehat{G}$ and

$$\varepsilon(1 \rtimes w) := nw, \quad \text{if } w|_E = \sigma\tau, \quad \text{respectively, } w|_E = \tau.$$

Then the Arthur parameter $\psi : L_F \times \mathrm{SL}(2) \rightarrow {}^L G$ factors through $\varepsilon : {}^L G' \rightarrow {}^L G$ to define $\psi' : L_F \times \mathrm{SL}(2) \rightarrow {}^L G'$, so

$$\psi'(w) = s \rtimes w \in {}^L G', \quad \text{if } w|_E = \sigma, \quad \text{respectively, } w|_E = \sigma\tau.$$

The representation of $G'(F)$ with Arthur parameter ψ' is the quadratic character attached to the extension $F(\sqrt{\varpi})/F$, respectively, $F(\sqrt{u\varpi})/F$, by class field theory. Then the endoscopic transfer of the quadratic character from $G'(F)$ to $G(F)$ is $\Theta_{\psi,s}$ which, in order, is

$$\Theta_{\psi,s} = \mathrm{trace} \pi(\phi, ++) + \mathrm{trace} \pi(\phi, +-) - \mathrm{trace} \pi(\phi, -+) - \mathrm{trace} \pi(\phi, --),$$

respectively,

$$\Theta_{\psi,s} = \mathrm{trace} \pi(\phi, ++) - \mathrm{trace} \pi(\phi, +-) - \mathrm{trace} \pi(\phi, -+) + \mathrm{trace} \pi(\phi, --).$$

Together with the stable distribution Θ_ψ , these three $\Theta_{\psi,s}$ form a basis for the vector space spanned by the characters of representations with infinitesimal parameter

λ . These four distributions are expressed in terms of the Fourier transform of regular semisimple orbital integrals, and their endoscopic transfer, in [CG09, Section 6.2].

9.1.6. Jacquet-Langlands. — The L-packet that this example treats also appears in [Art06, Section 4, page 215], alongside the L-packet for the inner form corresponding to a non-trivial cocycle in $Z^1(F, G_{\text{ad}})$, which determines the compact form of G , mentioned at the beginning of this section and now denoted by G_σ . The same Langlands parameter ϕ as above, when viewed as a Langlands parameter for G_σ , produces a singleton L-packet. In this case $\mathcal{S}_{\psi, \text{sc}} = Z_{\widehat{G}_{\text{sc}}}(\psi)$, which is the subgroup of $\widehat{G}_{\text{sc}} = \text{SL}(2)$ isomorphic to Q_8 given by

$$\mathcal{S}_{\psi, \text{sc}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The compact form G_σ of $G = \text{SL}(2)$ carries exactly one admissible representation with infinitesimal parameter λ , and it corresponds to the unique irreducible 2-dimensional representation of this group. We denote this representation by $\pi(\phi, 2)$. Although the theory presented in Part I does include inner rational forms that are not pure, in Section 9.2.7 we will show how to adapt the geometric picture so that it does include $\pi(\phi, 2)$.

9.2. Vanishing cycles of perverse sheaves. —

9.2.1. Vogan variety and orbit duality. — Recall the groups H_λ , J_λ and K_λ from Section 4.3. In the example at hand, these are given by

$$H_\lambda = J_\lambda = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \cong \mu_2 \times \mu_2$$

and $K_\lambda = N_{\widehat{G}^*}(\widehat{T})$. In particular, $G_\lambda = 1$ and $\lambda_{\text{nr}} : W_F \rightarrow {}^L G$ is trivial so $V_{\lambda_{\text{nr}}} = 0$ and $H_{\lambda_{\text{nr}}} = 1$.

9.2.2. Equivariant perverse sheaves. — With reference to Theorem 4.1.1 we have

$$\begin{array}{ccccc} \text{Rep}(A_\lambda) & \xrightarrow{\text{equiv.}} & \text{Per}_{H_\lambda}(V_\lambda) & \xrightleftharpoons[\pi_*]{\pi^*} & \text{Per}_{H_{\lambda_{\text{nr}}}}(0) \\ \parallel & & \parallel & & \parallel \\ \text{Rep}(\mu_2 \times \mu_2) & & \text{Per}_{\mu_2 \times \mu_2}(0) & & \text{Per}_1(0). \end{array}$$

In particular, there are four simple objects in $\text{Per}_{H_\lambda}(V_\lambda)$ corresponding to the four simple H_λ -equivariant local systems on $V_\lambda = \{0\}$, or equivalently, to the four characters of A_λ :

$$\text{Per}_H(V)_{\text{iso}}^{\text{simple}} = \{(+ +)_V, (+ -)_V, (- +)_V, (- -)_V\}.$$

9.2.3. Vanishing cycles of perverse sheaves. — We wish to describe the functor

$$\text{Ev} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}}).$$

We have already seen that $\text{Per}_{H_\lambda}(V_\lambda) = \text{Rep}(A_\lambda)$. In this case we have $T_{H_\lambda}^*(V_\lambda)_{\text{reg}} = \{(0, 0)\}$, so $\text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}}) = \text{Rep}(A_\lambda)$. With these equivalences,

$$\text{Ev} : \text{Rep}(A_\lambda) \rightarrow \text{Rep}(A_\lambda)$$

is the identity functor:

$$(157) \quad \text{trace}_s \text{Ev}_\psi(\pm\pm)_V = (\pm\pm)(s)$$

for every $s \in Z_{\widehat{G}}(\psi)$.

9.2.4. Normalization of Ev and the twisting local system. — Since Ev is trivial in this example, so is \mathcal{T} and NEv ; accordingly, the material of Section 8.2.7 is trivial in this example.

9.2.5. Fourier transform and vanishing cycles. — Since Ev , NEv and Ft are trivial in this example, the material of Section 8.2.8 is trivial.

9.2.6. Arthur sheaves. — Since $V_\lambda = \{0\}$ is a single stratum, there is only one stable perverse sheaf to consider:

$$\mathcal{A}_{C_0} = (++)_V \oplus (+-)_V \oplus (-+)_V \oplus (--)_V.$$

Of course, this is just the regular representation of A_λ .

9.2.7. Jacquet-Langlands. — We now show how to extend the geometric picture to include the admissible representation $\pi(\phi, 2)$ of the inner rational form G_σ of G .

Replace the group action $H_\lambda \times V_\lambda \rightarrow V_\lambda$ with the group action $H_{\lambda, \text{sc}} \times V_\lambda \rightarrow V_\lambda$, where

$$H_{\lambda, \text{sc}} := Z_{\widehat{G}_{\text{sc}}}(\lambda),$$

and where $H_{\lambda, \text{sc}}$ acts on V_λ through $H_{\lambda, \text{sc}} \rightarrow H_\lambda$ induced by the universal cover $\widehat{G}_{\text{sc}} \rightarrow \widehat{G}$. The analysis of Section 3.4 shows that

$$\text{Per}_{H_{\lambda, \text{sc}}}(V_\lambda) \equiv \text{Rep}(A_{\lambda, \text{sc}}),$$

where $A_{\lambda, \text{sc}} := \pi_0(H_{\lambda, \text{sc}})$. Of course, $A_{\lambda, \text{sc}}$ is just the group $\mathcal{S}_{\psi, \text{sc}}$ appearing above. Now $A_{\lambda, \text{sc}}$ has five irreducible representations up to equivalence: four one-dimensional representations obtained by pullback from the four characters of A_λ we have already seen, and one two dimensional representation, denoted by E . Thus, the category $\text{Rep}(A_{\lambda, \text{sc}})$ has exactly five simple objects up to isomorphism, and thence $\text{Per}_{H_{\lambda, \text{sc}}}(V_\lambda)$ has exactly five simple objects up to isomorphism:

$$\text{Per}_{H_{\lambda, \text{sc}}}^{\text{simple}}(V_\lambda)_{/\text{iso}} = \{E_V, (++)_V, (+-)_V, (-+)_V, (--)_V\}.$$

The rest of the story now carries through. For instance, the diagram of functors from Section 9.2.2 becomes the following diagram:

$$\begin{array}{ccccc} \text{Rep}(A_{\lambda, \text{sc}}) & \xrightarrow{\text{equiv.}} & \text{Per}_{H_{\lambda, \text{sc}}}(V_\lambda) & \xrightleftharpoons[\pi_*]{\pi^*} & \text{Per}_{H_{\lambda, \text{sc}}}^0(V_\lambda) \\ \parallel & & \parallel & & \parallel \\ \text{Rep}(Q_8) & & \text{Per}_{Q_8}(0) & & \text{Per}_1(0). \end{array}$$

Also, the functor vanishing cycles, Ev , is again the identity functor $\text{Rep}(A_{\lambda, \text{sc}}) \rightarrow \text{Rep}(A_{\lambda, \text{sc}})$, and the Arthur sheaf is again just the regular representation of $A_{\psi, \text{sc}}$. Thus, simply replacing category $\text{Per}_{H_\lambda}(V_\lambda)$ with $\text{Per}_{H_{\lambda, \text{sc}}}(V_\lambda)$ extends the theory from pure inner twists of G to inner twists of G , allowing us to see the Jacquet-Langlands correspondence from the geometric perspective of Part I.

9.3. ABV-packets. —

9.3.1. Admissible representations versus equivariant perverse sheaves. — The following table displays Vogan’s bijection between $\text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}$ and $\Pi_{\text{pure}, \lambda}(G/F)$, as discussed in Section 8.3.1.

$\text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}$	$\Pi_{\text{pure}, \lambda}(G/F)$
$(++)_V$	$\pi(\phi, ++)$
$(+-)_V$	$\pi(\phi, +-)$
$(-+)_V$	$\pi(\phi, -+)$
$(--)_V$	$\pi(\phi, --)$

9.3.2. ABV-packets. — Using the bijection from Section 9.3.1 and the trivial functor of Ev from Section 9.2.3, it follows directly from definition (145) that

$$\Pi_\psi(G(F)) = \Pi_{\text{pure}, \phi_\psi}^{\text{ABV}}(G/F).$$

With reference to (149) and (157), in this example we find: if $s = 1$ then

$$\eta_{\psi, 1}^{\text{NEv}} = \pi(\phi, ++) - \pi(\phi, +-) + \pi(\phi, -+) - \pi(\phi, --);$$

if $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then

$$\eta_{\psi, s}^{\text{NEv}} = \pi(\phi, ++) - \pi(\phi, +-) + \pi(\phi, -+) - \pi(\phi, --);$$

if $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then

$$\eta_{\psi, s}^{\text{NEv}} = \pi(\phi, ++) + \pi(\phi, +-) - \pi(\phi, -+) - \pi(\phi, --);$$

and if $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then

$$\eta_{\psi, s}^{\text{NEv}} = \pi(\phi, ++) - \pi(\phi, +-) - \pi(\phi, -+) + \pi(\phi, --).$$

Comparing $\eta_{\psi, s}^{\text{NEv}}$ above with $\eta_{\psi, s}$ as calculated in Section 9.1.5, we see that

$$\text{trace } \eta_{\psi, s}^{\text{NEv}} = \Theta_{\psi, s},$$

in all four cases, thus confirming Conjecture 1 in this example.

9.3.3. Kazhdan-Lusztig conjecture. — The material of Section 8.3.3 is trivial in this example.

9.4. Endoscopy and equivariant restriction of perverse sheaves. — In Section 9.1.5 we saw that the Arthur parameter ψ factors through three elliptic endoscopic groups, G' . For each of these G' , the infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ factors through $\varepsilon : {}^L G' \rightarrow {}^L G$ to define $\lambda' : W_F \rightarrow {}^L G'$.

9.4.1. Endoscopic Vogan variety. — For each G' above, $H' := Z_{\widehat{G'}}(\lambda')$ is the subgroup of H generated by s in H' ; see Section 9.1.5 for s . Thus, $\text{Per}_{H'}(V') \equiv \text{Rep}(H')$

has two simple objects, now denoted by $(+)_V$ and $(-)_V$. Now, Vogan's bijection for $\lambda' : W_F \rightarrow {}^L G'$ is given by the following table.

$\text{Per}_{H'}(V')_{/\text{iso}}^{\text{simple}}$	$\Pi_{\text{pure}, \lambda'}(G'/F)$
$(+)_V$	$\pi(\phi', +)$
$(-)_V$	$\pi(\phi', -)$

Then $\pi(\phi', +) = \pi(\phi', -)$ is the quadratic character of $G'(F) = N_{E'/F}^{-1}(1)$ determined by ϕ' .

9.4.2. Vanishing cycles. — Arguing as in Section 9.2.3, we see that

$$\text{NEv}' : \text{Rep}(A_{\lambda'}) \rightarrow \text{Rep}(A_{\lambda'})$$

is trivial.

9.4.3. Restriction. — The restriction functor $\text{Per}_H(V) \rightarrow \text{Per}_{H'}(V')$ is just restriction $\text{Rep}(H) \rightarrow \text{Rep}(H')$ to the subgroup generated by s .

9.4.4. Restriction and vanishing cycles. — We see (154) almost trivially: the left-hand side of (154) is

$$\text{trace}_{a_s} \text{NEv}_{\psi}(\pm\pm)_V = (\pm\pm)(s)$$

while the right-hand side of (154) is

$$(-1)^{\dim C - \dim C'} \text{trace}_{a'_s} (\text{Ev}'_{\psi'}(\pm\pm)_V|_{V'}) = (-1)^{0-0}(\pm\pm)(s).$$

Arguing as in [CFMX], it follows from (154) that $\eta_{\psi, s}^{\text{NEv}}$ is the Langlands-Shelstad lift of $\eta_{\psi'}^{\text{NEv}}$. These lifts are found by considering each case in turn: in order, take $s \in \widehat{G}$ to be

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and then} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

in the same order, the quadratic extension E'/F is

$$E'/F = F(\sqrt{u})/F, \quad F(\sqrt{u\varpi})/F, \quad \text{and then} \quad F(\sqrt{u\varpi})/F.$$

10. $\text{SO}(3)$ unipotent representations, regular parameter

Set $G = \text{SO}(3)$ split over F , so $\widehat{G} = \text{SL}(2, \mathbb{C})$ and ${}^L G = \text{SL}(2, \mathbb{C}) \times W_F$. In this case,

$$H^1(F, G) = H^1(F, G_{\text{ad}}) = H^1(F, \text{Aut}(G)) \cong \mathbb{Z}/2\mathbb{Z},$$

so there are two isomorphism classes of rational forms of G , each pure. We will use the notation G_1 for the non-quasisplit form of $\text{SO}(3)$ given by the quadratic form

$$\begin{pmatrix} -\varepsilon\varpi & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varpi \end{pmatrix}.$$

Let $\lambda : W_F \rightarrow \widehat{G}$ be the parameter defined by

$$\lambda(w) = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

Although this simple example exhibits some interesting geometric phenomena, the Arthur packets in this example are singletons, so there is no interesting endoscopy here. Nevertheless, this example will be important later when we consider other groups for which $\mathrm{SO}(3)$ is an endoscopic group.

10.1. Arthur packets. —

10.1.1. Parameters. — Up to $Z_{\widehat{G}}(\lambda)$ -conjugacy, there are two Langlands parameters $\phi : L_F \rightarrow \widehat{G}$ with infinitesimal parameter λ ; they are given by

$$\phi_0(w, x) = \lambda(w) = \nu_2(d_w) \quad \text{and} \quad \phi_1(w, x) = \nu_2(x),$$

where $\nu_2 : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(2, \mathbb{C})$ is the identity function, thus an irreducible 2-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. So,

$$P_\lambda({}^L G)/Z_{\widehat{G}}(\lambda) = \{\phi_0, \phi_1\}.$$

Both ϕ_0 and ϕ_1 are of Arthur type: define

$$\psi_0(w, x, y) := \nu_2(y) \quad \text{and} \quad \psi_1(w, x, y) := \nu_2(x).$$

Then

$$Q_\lambda({}^L G)/Z_{\widehat{G}}(\lambda) = \{\psi_0, \psi_1\}.$$

Observe that ψ_1 is tempered but ψ_0 is not. Also observe that the Arthur parameters ψ_0 and ψ_1 are Aubert dual to each other.

10.1.2. L-packets. — The component groups for the parameters $\phi \in P_\lambda({}^L G)$ are

$$A_{\phi_0} = \pi_0(Z_{\widehat{G}}(\phi_0)) = \pi_0(\widehat{T}) \cong 1 \quad \text{and} \quad A_{\phi_1} \cong \pi_0(Z_{\widehat{G}}(\phi_1)) = \pi_0(Z(\widehat{G})) \cong \mu_2.$$

Denoting the two characters of μ_2 by $+$ and $-$, the L-packets for these Langlands parameters are:

$$\begin{aligned} \Pi_{\phi_0}(G(F)) &= \{\pi(\phi_0)\}, & \Pi_{\phi_1}(G(F)) &= \{\pi(\phi_1, +)\}, \\ \Pi_{\phi_0}(G_1(F)) &= \emptyset, & \Pi_{\phi_1}(G_1(F)) &= \{\pi(\phi_1, -)\}. \end{aligned}$$

Here we can view these representations as that of $\mathrm{GL}(2, F)$ (resp. multiplicative group of the quaternion algebra D) with trivial central character for $G(F) \cong \mathrm{GL}(2, F)/F^\times$ (resp. $G_1(F) \cong D^\times/F^\times$). Then $\pi(\phi_0)$ (resp. $\pi(\phi_1, +)$) is given by the trivial (resp. Steinberg) representation of $\mathrm{GL}(2, F)$ and $\pi(\phi_1, -)$ is given by the trivial representation of D^\times .

To see how characters ρ of A_ϕ determine pure inner forms of G , pullback ρ along $\pi_0(Z(\widehat{G})) \rightarrow \pi_0(Z_{\widehat{G}}(\phi))$ and then use the Kottwitz isomorphism: the trivial character of A_{ϕ_0} (resp. A_{ϕ_1}) determines the trivial character of $\pi_0(Z(\widehat{G}))$ and therefore the split pure inner form of G ; the non-trivial character $-$ of A_{ϕ_1} determines the non-trivial character of $\pi_0(Z_{\widehat{G}})$ and therefore the non-trivial pure inner form of G . Therefore, the pure L-packets are:

$$\Pi_{\mathrm{pure}, \phi_0}(G/F) = \{[\pi(\phi_0), 0]\}, \quad \Pi_{\mathrm{pure}, \phi_1}(G/F) = \left\{ \begin{array}{l} [\pi(\phi_1, +), 0] \\ [\pi(\phi_1, -), 1] \end{array} \right\}.$$

10.1.3. Multiplicities in standard modules. —

	$\pi(\phi_0)$	$\pi(\phi_1, +)$	$\pi(\phi_1, -)$
$M(\phi_0)$	1	1	0
$M(\phi_1, +)$	0	1	0
$M(\phi_1, -)$	0	0	1

10.1.4. Arthur packets. — The component groups A_{ψ_0} and A_{ψ_1} are both $Z(\widehat{G})$. The Arthur packets for $\psi \in Q_\lambda({}^L G)$ are

$$\begin{aligned} \Pi_{\psi_0}(G(F)) &= \{\pi(\phi_0)\}, & \Pi_{\psi_1}(G(F)) &= \{\pi(\phi_1, +)\}, \\ \Pi_{\psi_0}(G_1(F)) &= \{\pi(\phi_1, -)\}, & \Pi_{\psi_1}(G_1(F)) &= \{\pi(\phi_1, -)\}. \end{aligned}$$

so the pure Arthur packets are

$$\Pi_{\text{pure}, \psi_0}(G/F) = \left\{ \begin{bmatrix} \pi(\phi_0), 0 \\ \pi(\phi_1, -), 1 \end{bmatrix} \right\}, \quad \Pi_{\text{pure}, \psi_1}(G/F) = \left\{ \begin{bmatrix} \pi(\phi_1, +), 0 \\ \pi(\phi_1, -), 1 \end{bmatrix} \right\}.$$

10.1.5. Aubert duality. — Aubert duality for $G(F)$ and $G_1(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_1, +)$
$\pi(\phi_1, -)$	$\pi(\phi_1, -)$

The twisting character χ_{ψ_0} of A_{ψ_0} is trivial; likewise, the twisting character χ_{ψ_1} of A_{ψ_1} .

10.1.6. Stable distributions and endoscopy. — The characters $\langle \cdot, \pi \rangle_\psi$ appearing in the invariant distributions $\Theta_{\psi, s}^G$ (133) are given by the first two rows of the following table. The last row gives the analogous characters for $\Theta_{\psi, s}^{G_1}$.

π	$\langle \cdot, \pi \rangle_{\psi_0}$	$\langle \cdot, \pi \rangle_{\psi_1}$
$\pi(\phi_0)$	+	0
$\pi(\phi_1, +)$	0	+
$\pi(\phi_1, -)$	−	−

Using the notation $s = \text{diag}(s_1, s_1) \in A_\psi = Z(\widehat{G})$, we now have

$$\begin{aligned} \Theta_{\psi_0, s}^G &= \text{trace } \pi(\phi_0), & \Theta_{\psi_0, s}^{G_1} &= -s_1 \text{trace } \pi(\phi_1, -), \\ \Theta_{\psi_1, s}^G &= \text{trace } \pi(\phi_1, +), & \Theta_{\psi_1, s}^{G_1} &= s_1 \text{trace } \pi(\phi_1, -). \end{aligned}$$

Therefore, in this example, the virtual representations $\eta_{\psi, s}$ (32) are:

$$\begin{aligned} \eta_{\psi_0, s} &= \pi(\phi_0) + s_1 \pi(\phi_1, -), \\ \eta_{\psi_1, s} &= \pi(\phi_1, +) - s_1 \pi(\phi_1, -). \end{aligned}$$

Since $A_\psi = Z(\widehat{G})$, the only endoscopic group relevant to these parameters is G itself.

10.2. Vanishing cycles of perverse sheaves. —

10.2.1. Vogan variety and orbit duality. — Since $\lambda : W_F \rightarrow {}^L G$ is unramified and $\lambda(\text{Fr})$ is elliptic and G is split, we have $\lambda_{\text{nr}} = \lambda$.

The Vogan variety for λ is

$$V_\lambda = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \mid y \right\} \cong \mathbb{A}^1,$$

with $H_\lambda := Z_{\widehat{G}}(\lambda)$ -action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & t^2 y \\ 0 & 0 \end{pmatrix}$$

so V_λ is stratified into H_λ -orbits

$$C_0 := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad C_y := \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \mid y \neq 0 \right\}.$$

The dual Vogan variety V_λ^* is given by

$$V_\lambda^* = \left\{ \begin{pmatrix} 0 & 0 \\ y' & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \mid y' \right\} \cong \mathbb{A}^1,$$

with H_λ -action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ y' & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ t^{-2} y' & 0 \end{pmatrix},$$

so V_λ^* is stratified into H_λ -orbits

$$C_0^t := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad C_y^t := \left\{ \begin{pmatrix} 0 & 0 \\ y' & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} \mid y' \neq 0 \right\}$$

The H_λ -invariant function $[\cdot, \cdot] : T^*(V_\lambda) \rightarrow \mathfrak{h}_\lambda$ is given by

$$\begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix} \mapsto yy' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From this, dual orbits are easily found.

$$\begin{array}{ccc} C_y = \widehat{C}_0 & \dim = 1 & C_0^* = C_y^t \\ \uparrow & & \uparrow \\ C_0 = \widehat{C}_y & \dim = 0 & C_y^* = C_0^t \end{array}$$

10.2.2. Equivariant perverse sheaves. — On the closed stratum C_0 there is one simple local system $\mathbb{1}_{C_0}$ and its perverse extension $\mathcal{IC}(\mathbb{1}_{C_0})$ is the rank-one skyscraper sheaf at C_0 . The open stratum C_y carries two simple local systems: $\mathbb{1}_{C_y}$ and the non-trivial \mathcal{E}_{C_y} corresponding, respectively, to the trivial and non-trivial characters of the equivariant fundamental group of C_y . Therefore, the irreducible shifted standard sheaves on V are:

$$\begin{aligned} \mathcal{S}(\mathbb{1}_{C_0}) &= j_{C_0!} \mathbb{1}_{C_0}[0], \\ \mathcal{S}(\mathbb{1}_{C_y}) &= j_{C_y!} \mathbb{1}_{C_y}[1], \quad \text{and} \quad \mathcal{S}(\mathcal{E}_{C_y}) = j_{C_y!} \mathcal{E}_{C_y}[1]. \end{aligned}$$

There are three simple objects in $\text{Per}_{H_\lambda}(V_\lambda) = \text{Per}_{\mathbb{G}_m}(\mathbb{A}^1)$ up to isomorphism:

$$\text{Per}_{H_\lambda}(V_\lambda)_{\text{iso}}^{\text{simple}} = \{ \mathcal{IC}(\mathbb{1}_{C_0}), \mathcal{IC}(\mathbb{1}_{C_y}), \mathcal{IC}(\mathcal{E}_{C_y}) \}.$$

The perverse extension of $\mathbb{1}_{C_y}$ is the constant sheaf $\mathbb{1}_{V_\lambda}[1] = \mathcal{IC}(\mathbb{1}_{C_y})$ while the perverse extension $\mathcal{IC}(\mathcal{E}_{C_y})$ of \mathcal{E}_{C_y} is the standard sheaf obtained by extension by zero from $\mathcal{E}_{C_y}[1]$.

\mathcal{P}	$\mathcal{P} _{C_0}$	$\mathcal{P} _{C_1}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	$\mathbb{1}_{C_0}[1]$	$\mathbb{1}_{C_y}[1]$
$\mathcal{IC}(\mathcal{E}_{C_y})$	0	$\mathcal{E}_{C_y}[1]$

The first two row of this table are clear since $\overline{C_0}$ and $\overline{C_y}$ are smooth. To see the third row, let $\pi : V \rightarrow V$ be the proper double cover given by $y \mapsto y^2$ and note that

$$\pi_*(\mathbb{1}_V[1]) = \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y}),$$

by the Decomposition Theorem. Since both of $\pi_*(\mathbb{1}_V[1])|_{C_0}$ and $\mathcal{IC}(\mathbb{1}_{C_y})|_{C_0}$ are rank one, it follows that $\mathcal{IC}(\mathcal{E}_{C_y})|_{C_0} = 0$.

Thus, the geometric multiplicity matrix is

	$\mathcal{S}(\mathbb{1}_{C_0})$	$\mathcal{S}(\mathbb{1}_{C_y})$	$\mathcal{S}(\mathcal{E}_{C_y})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	1	0	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	-1	1	0
$\mathcal{IC}(\mathcal{E}_{C_y})$	0	0	1

and the normalised geometric multiplicity matrix is

	$\mathbb{1}_{C_0}^\natural$	$\mathbb{1}_{C_y}^\natural$	$\mathcal{E}_{C_y}^\natural$
$\mathbb{1}_{C_0}^\natural$	1	0	0
$\mathbb{1}_{C_y}^\natural$	1	1	0
$\mathcal{E}_{C_y}^\natural$	0	0	1

10.2.3. Cuspidal support decomposition and Fourier transform. — Up to conjugation, $\widehat{G} = \mathrm{SL}(2, \mathbb{C})$ admits exactly two cuspidal Levi subgroups: \widehat{G} itself and $\widehat{T} = \mathrm{GL}(1)$. Thus,

$$\mathrm{Per}_{H_\lambda}(V_\lambda) = \mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}}.$$

Simple objects in these two subcategories are listed below.

$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}/\mathrm{iso}}^{\mathrm{simple}}$	$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}/\mathrm{iso}}^{\mathrm{simple}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	
$\mathcal{IC}(\mathbb{1}_{C_y})$	$\mathcal{IC}(\mathcal{E}_{C_y})$

The Fourier transform is given on simply objects by:

$$\begin{aligned} \mathrm{Ft} : \mathrm{Per}_{H_\lambda}(V_\lambda) &\longrightarrow \mathrm{Per}_{H_\lambda}(V_\lambda^*) \\ \mathcal{IC}(\mathbb{1}_{C_0}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_0^*}) = \mathcal{IC}(\mathbb{1}_{C_y^t}) \\ \mathcal{IC}(\mathbb{1}_{C_y}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_y^*}) = \mathcal{IC}(\mathbb{1}_{C_0^t}) \\ \mathcal{IC}(\mathcal{E}_{C_y}) &\mapsto \mathcal{IC}(\mathcal{E}_{C_0^*}) = \mathcal{IC}(\mathcal{E}_{C_y^t}) \end{aligned}$$

TABLE 10.2.1. ${}^p\mathrm{Ev} : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 10.

$$\begin{array}{ccc} \mathrm{Per}_H(V) & \xrightarrow{{}^p\mathrm{Ev}} & \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}}) \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ \mathcal{IC}(\mathbb{1}_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}) \\ \mathcal{IC}(\mathcal{E}_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \end{array}$$

10.2.4. Equivariant local systems on the regular conormal bundle. — The regular conormal bundle $T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}}$ decomposes into two H_λ orbits

$$T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}} = T_{C_0}^*(V_\lambda)_{\mathrm{reg}} \bigsqcup T_{C_y}^*(V_\lambda)_{\mathrm{reg}}$$

given by

$$T_{C_0}^*(V_\lambda)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix} \mid \begin{array}{l} y = 0 \\ y' \neq 0 \end{array} \right\}, \quad T_{C_y}^*(V_\lambda)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix} \mid \begin{array}{l} y \neq 0 \\ y' = 0 \end{array} \right\}.$$

We remark that

$$T_{C_0}^*(V_\lambda)_{\mathrm{reg}} = T_{C_0}^*(V_\lambda)_{\mathrm{sreg}} = C_0 \times C_0^* \quad \text{and} \quad T_{C_y}^*(V_\lambda)_{\mathrm{reg}} = T_{C_y}^*(V_\lambda)_{\mathrm{sreg}} = C_y \times C_y^*.$$

These components are H_λ -orbits, so every H -equivariant perverse sheaf on $T_H^*(V)_{\mathrm{reg}}$ is a standard sheaf shifted to degree 1. The equivariant fundamental groups are both given by

$$A_C^{\mathrm{mic}} = \pi_1(T_C^*(V_\lambda), (x, \xi))_{Z_{H_\lambda}(x, \xi)^0} = \pi_0(Z_{H_\lambda}(x, \xi)) = Z(\widehat{G}) \cong \{\pm 1\}.$$

Let $\mathbb{1}_{\mathcal{O}_\psi}$ be the constant local system on $T_{C_\psi}^*(V_\lambda)_{\mathrm{sreg}}$ and let $\mathcal{E}_{\mathcal{O}_\psi}$ be the non-trivial H -equivariant local system on $T_{C_\psi}^*(V_\lambda)_{\mathrm{sreg}}$. Then

$$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_j}) = \mathcal{S}(\mathbb{1}_{\mathcal{O}_j}) \quad \text{and} \quad \mathcal{IC}(\mathcal{E}_{\mathcal{O}_j}) = \mathcal{S}(\mathcal{E}_{\mathcal{O}_j}).$$

In summary,

$$\mathrm{Loc}_H(T_{C_0}^*(V)_{\mathrm{sreg}})_{/\mathrm{iso}}^{\mathrm{simple}} = \{\mathbb{1}_{\mathcal{O}_0}, \mathcal{E}_{\mathcal{O}_0}\}$$

and

$$\mathrm{Loc}_H(T_{C_y}^*(V)_{\mathrm{sreg}})_{/\mathrm{iso}}^{\mathrm{simple}} = \{\mathbb{1}_{\mathcal{O}_y}, \mathcal{E}_{\mathcal{O}_y}\}.$$

10.2.5. Vanishing cycles of perverse sheaves. — The functor ${}^p\mathrm{Ev} : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}})$ is given on simple objects in Table 10.2.1. The lower part uses the identification of local systems on the regular conormal with representations of the corresponding equivariant fundamental groups, so each $\mathbf{Evs}_C \mathcal{P}$ is given as a character of A_C^{mic} .

We now explain the computations behind Tables 10.2.1 and 10.2.2.

(a) Using Lemma 6.2.1 we find

$$\begin{array}{ll} {}^p\mathrm{Ev}_{C_y} \mathcal{IC}(\mathbb{1}_{C_y}) &= \mathbb{1}_{\mathcal{O}_y}[1] & {}^p\mathrm{Ev}_{C_y} \mathcal{IC}(\mathcal{E}_{C_y}) &= \mathcal{E}_{\mathcal{O}_y}[1] \\ {}^p\mathrm{Ev}_{C_y} \mathcal{IC}(\mathbb{1}_{C_0}) &= 0 & {}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_0}) &= \mathbb{1}_{\mathcal{O}_0}[0]. \end{array}$$

It only remains, therefore, to determine ${}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_y})$ and ${}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathcal{E}_{C_y})$.

TABLE 10.2.2. $\mathbf{Evs} : \mathbf{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 10.

\mathcal{P}	$\mathbf{Evs}_{C_0} \mathcal{P}$	$\mathbf{Evs}_{C_y} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	+	0
$\mathcal{IC}(\mathbb{1}_{C_y})$	0	+
$\mathcal{IC}(\mathcal{E}_{C_y})$	−	−

(b) Since $\mathcal{IC}(\mathbb{1}_{C_y}) = \mathbb{1}_V[1]$, we have

$$\mathbf{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_y}) = R\Phi_{yy'}(\mathbb{1}_V[1] \boxtimes \mathbb{1}_{C_0^*})|_{T_C^*(V)_{\text{reg}}}.$$

As $\mathbb{1}_V \boxtimes \mathbb{1}_{C_0^*} = \mathbb{1}_{V \times C_0^*}$ is a local system and the function $(y, y') \mapsto yy'$ is smooth on $V \times C_0^*$, it follows Lemma 6.2.2 that

$$\mathbf{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_y}) = 0.$$

Note that C_0^* specifically excludes the locus $y' = 0$, which is where the singularities would be.

(c) We now consider the case of $\mathcal{IC}(\mathcal{E}_{C_y})$, using the proper double cover $\pi : V \rightarrow V$, already used in Section 10.2.2. Recall that

$$\pi_*(\mathbb{1}_V[1]) = \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y}).$$

Since \mathbf{Ev} is exact by Proposition 6.4.1,

$$\mathbf{Ev}_{C_0} \pi_*(\mathbb{1}_V[1]) = \mathbf{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathbf{Ev}_{C_0} \mathcal{IC}(\mathcal{E}_{C_y}).$$

We have just seen that $\mathbf{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_y}) = 0$, so

$$\mathbf{Ev}_{C_0} \mathcal{IC}(\mathcal{E}_{C_y}) = \mathbf{Ev}_{C_0} \pi_*(\mathbb{1}_V[1]).$$

By Lemma 6.5.2,

$$\mathbf{Ev}_{C_0} \pi_*(\mathbb{1}_V[1]) = \pi! \left(R\Phi_{y^2 y'}(\mathbb{1}_{V \times C_0^*}[1])|_{T_C^*(V)_{\pi\text{-reg}}} \right).$$

Since π is an isomorphism on $T_C^*(V)_{\pi\text{-reg}}$,

$$\mathbf{Ev}_{C_0} \pi_*(\mathbb{1}_V[1]) = R\Phi_{y^2 y'}(\mathbb{1}_{V \times C_0^*}[1])|_{T_C^*(V)_{\text{reg}}}.$$

Now,

$$R\Phi_{y^2 y'}(\mathbb{1}_{V \times C_0^*}[1]) = \pi'_! \mathbb{1}_{C_0 \times C_0^*}[1],$$

where $\pi' : C_0^* \rightarrow C_0^*$ is the double cover $y' \mapsto y'^2$. Note that

$$\pi'_! \mathbb{1}_{C_0 \times C_0^*}[1] = \pi'_! \mathbb{1}_{\mathcal{O}_0}[1].$$

By the Decomposition Theorem,

$$\pi'_! \mathbb{1}_{\mathcal{O}_0}[1] = \mathbb{1}_{\mathcal{O}_0}[1] \oplus \mathcal{E}_{\mathcal{O}_0}[1],$$

where $\mathcal{E}_{\mathcal{O}_0}$ is the non-trivial equivariant local system on \mathcal{O}_0 introduced in Section 10.2.4, which is the associated to the double cover arising from taking $\sqrt{y'}$ over \mathcal{O}_0 . Therefore,

$${}^p\mathbf{Ev}_{C_0} \mathcal{IC}(\mathcal{E}_{C_y}) = \mathcal{E}_{\mathcal{O}_0}[1].$$

This completes the calculation of ${}^p\mathrm{Ev} : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T^*(V)_{\mathrm{reg}})$ on simple objects, as displayed in Table 10.2.1.

10.2.6. Normalization of Ev and the twisting local system. — From Table 10.2.1 we see that the twisting local system \mathcal{T} is trivial in this case, so ${}^p\mathrm{NEv} = {}^p\mathrm{Ev}$.

10.2.7. Fourier transform and vanishing cycles. — Having computed the values of the functor ${}^p\mathrm{Ev} : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}})$ on simple objects, we also know the values of ${}^p\mathrm{Ev}^* : \mathrm{Per}_H(V^*) \rightarrow \mathrm{Per}_H(T_H^*(V^*)_{\mathrm{reg}})$. We use this and the coincidence of ${}^p\mathrm{Ev}$ with ${}^p\mathrm{NEv}$, in the table below.

$$\begin{array}{ccccccc}
\mathrm{Per}_H(V) & \xrightarrow{{}^p\mathrm{Ev}} & \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}}) & \xrightarrow{a_*} & \mathrm{Per}_H(T_H^*(V^*)_{\mathrm{reg}}) & \xleftarrow{\mathrm{Ev}^*} & \mathrm{Per}_H(V^*) \\
\mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*}) & \leftarrow & \mathcal{IC}(\mathbb{1}_{C_0^*}) \\
\mathcal{IC}(\mathbb{1}_{C_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y^*}) & \leftarrow & \mathcal{IC}(\mathbb{1}_{C_y^*}) \\
\mathcal{IC}(\mathcal{E}_{C_y}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y^*}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0^*}) & \leftarrow & \mathcal{IC}(\mathcal{E}_{C_0^*})
\end{array}$$

Since the map from the first to the fourth column is the Fourier transform, this verifies (141).

10.2.8. Arthur sheaves. —

$$\begin{array}{c|cc}
\text{Arthur sheaf} & \text{packet sheaves} & \text{coronal sheaves} \\
\hline
\mathcal{A}_{C_0} & \mathcal{IC}(\mathbb{1}_{C_0}) \oplus & \mathcal{IC}(\mathcal{E}_{C_y}) \\
\mathcal{A}_{C_y} & \mathcal{IC}(\mathbb{1}_{C_y}) \oplus \mathcal{IC}(\mathcal{E}_{C_y}) &
\end{array}$$

10.3. ABV-packets. —

10.3.1. Admissible representations versus equivariant perverse sheaves. — Vogan's bijection for $\lambda : W_F \rightarrow {}^L G$ chosen at the beginning of Section 10 is given by the following table:

$$\begin{array}{c|c}
\mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}} & \Pi_{\mathrm{pure}, \lambda}(G/F) \\
\hline
\mathcal{IC}(\mathbb{1}_{C_0}) & (\pi(\phi_0), 0) \\
\mathcal{IC}(\mathbb{1}_{C_y}) & (\pi(\phi_1), 0) \\
\mathcal{IC}(\mathcal{E}_{C_y}) & (\pi(\phi_1, -), 1)
\end{array}$$

The base points for H -orbits in $T_H^*(V)_{\mathrm{reg}}$ determined by the Arthur parameters ψ_0 and ψ_1 are:

$$(x_{\psi_0}, \xi_{\psi_0}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in T_{C_0}^*(V_\lambda)_{\mathrm{reg}}, \quad (x_{\psi_1}, \xi_{\psi_1}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_{C_y}^*(V_\lambda)_{\mathrm{reg}}.$$

10.3.2. ABV-packets. — Using the bijection of Section 10.3.1, the vanishing cycles calculations of Section 10.2.5, and the definition of ABV-packets from Section 7.1, we find ABV-packets for G for representations with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ from Section 10.1.1:

$$\Pi_{\mathrm{pure}, \psi_0}^{\mathrm{ABV}}(G/F) = \left\{ \begin{array}{c} [\pi(\phi_0), 0] \\ [\pi(\phi_1, -), 1] \end{array} \right\}, \quad \Pi_{\mathrm{pure}, \psi_1}^{\mathrm{ABV}}(G/F) = \left\{ \begin{array}{c} [\pi(\phi_1, +), 0] \\ [\pi(\phi_1, -), 1] \end{array} \right\}.$$

We see that all pure Arthur packets are ABV-packets simply by comparing this with Section 10.1.4. In this example, all the strata in V are of Arthur type, so all ABV-packets are Arthur packets.

10.3.3. Stable invariant distributions and their endoscopic transfer. — We recalled in Section 10.1.6 the coefficient appearing in the invariant distributions $\eta_{\psi,s}$ attached to $\psi \in Q_\lambda({}^L G)$ and $s \in Z_{\widehat{G}}(\psi)$. Using Section 10.2.5, compare $\langle ss_\psi, [\pi, \delta] \rangle_\psi$ with trace $\mathrm{Ev}_\psi \mathcal{P}(\pi, \delta)(ss_\psi)$. This proves (147) and therefore establishes Conjecture 1, in this case:

$$\eta_{\psi,s} = \eta_{\psi,s}^{\mathrm{NEv}},$$

for $\psi \in Q_\lambda({}^L G)$ and $s \in Z_{\widehat{G}}(\psi)$.

Also recall from Section 10.1.6 that the only endoscopic group relevant to ψ_0 and ψ_1 is G .

10.3.4. Kazhdan-Lusztig conjecture. — Using the bijection of Section 10.3.1 we may compare the multiplicity matrix from Section 10.1.3 with the normalised geometric multiplicity matrix from Section 10.2.2:

$$m_{\mathrm{rep}} = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \quad m'_{\mathrm{geo}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Since ${}^t m_{\mathrm{rep}} = m'_{\mathrm{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ given in Section 10.1.1.

10.3.5. Aubert duality and Fourier transform. — By using Vogan's bijection from Section 10.3.1 to compare Aubert duality from Section 10.1.5 with the Fourier transform from Section 10.2.3 one readily verifies (150).

10.3.6. Normalisation. — A comparison of the twisting characters χ_ψ of A_ψ from Section 10.1.5 with the restriction \mathcal{T}_ψ to $T_{C_\psi}^*(V)_{\mathrm{reg}}$ of the local system \mathcal{T}_ψ from Section 10.2.7 verifies (152).

10.4. Endoscopy and equivariant restriction of perverse sheaves. — The material of Section 8.4 is trivial in this example, since $Z_{\widehat{G}}(\psi) = Z(\widehat{G})$.

11. $\mathrm{PGL}(4)$ shallow representations

This example illustrates the utility of Theorem 4.1.1 and the significance of the decomposition of $\lambda(\mathrm{Fr})$ into hyperbolic and elliptic parts. Here, the calculation of the Arthur packets for certain non-tempered representations of $\mathrm{PGL}(4)$ is reduced to the calculation of certain unipotent representations of $\mathrm{SL}(2)$. This example also demonstrates a case when $H^1(F, G_{\mathrm{ad}}) \rightarrow H^1(F, \mathrm{Aut}(G))$ is surjective but not injective.

Set $G = \mathrm{PGL}(4)$ over F and suppose q is odd. So, $\widehat{G} = \mathrm{SL}(4)$ and ${}^L G = \mathrm{SL}(4) \times W_F$. In this case, $H^1(F, G) = H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Irrep}(\mu_4)$, so there are four isomorphism classes of inner forms of G , each one pure. However, G has only three forms, up to isomorphism: the split group G itself, an anisotropic form G_1 , and a non-quasi-split form G_2 with a proper minimal Levi. In fact, the outer automorphism of G induces an action of order 2 on $H^1(F, G)$, and the orbits of this action correspond exactly to the image of $H^1(F, G)$ in $H^1(F, \mathrm{Aut}(G))$. The map $H^1(F, G_{\mathrm{ad}}) \rightarrow H^1(F, \mathrm{Aut}(G))$ from isomorphism classes of inner form of G to isomorphism classes of forms of G is

given by: $0 \mapsto G$, $1 \mapsto G_1$, $2 \mapsto G_2$ and $3 \mapsto G_1$, where the notation refers to an identification of $\text{Irrep}(\mu_4)$ with $\mathbb{Z}/4\mathbb{Z}$.

Let E be the Galois closure of the ramified extension $F(\sqrt[q+1]{\varpi})$. Then E is the compositum of an unramified quadratic extension of F and the totally ramified extension $F(\sqrt[q+1]{\varpi})$; now $\text{Gal}(E/F)$ is the dihedral group with generators σ, τ , where σ has order 2 and τ has order $q+1$ and $\sigma\tau\sigma = \tau^{-1} = \tau^q$. Consider the representation $\varrho : \text{Gal}(E/F) \rightarrow \text{SL}(2, \mathbb{C})$ defined by

$$\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

where $\zeta \in \mathbb{C}$ is a fixed primitive $q+1$ -th root of unity. Let $\rho : W_F \rightarrow \text{SL}(2, \mathbb{C})$ be the composition of $W_F \rightarrow \Gamma_F \rightarrow \text{Gal}(E/F)$ with ϱ . Define $\lambda : W_F \rightarrow {}^L G = \text{SL}(4) \times W_F$ by

$$\lambda(w) := \rho(w) \otimes \nu_2(d_w).$$

Thus, if $w|_E = \sigma$ then

$$\lambda(w) = \begin{pmatrix} 0 & 0 & |w|^{1/2} & 0 \\ 0 & 0 & 0 & |w|^{-1/2} \\ -|w|^{1/2} & 0 & 0 & 0 \\ 0 & -|w|^{-1/2} & 0 & 0 \end{pmatrix}$$

while if $w|_E = \tau$ then

$$\lambda(w) = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta^{-1} \end{pmatrix}.$$

11.1. Arthur packets. —

11.1.1. Parameters. — There are two Langlands parameters with infinitesimal parameter λ , each of Arthur type:

$$\begin{aligned} \phi_0(w, x) &:= \rho(w) \otimes \nu_2(d_w), & \phi_1(w, x) &:= \rho(w) \otimes \nu_2(x) \\ \psi_0(w, x, y) &:= \rho(w) \otimes \nu_2(y), & \psi_1(w, x, y) &:= \rho(w) \otimes \nu_2(x). \end{aligned}$$

Note that ψ_0 and ψ_1 are Aubert dual.

11.1.2. L -packets. — There are 5 admissible representations of the three forms G , G_1 and G_2 , with infinitesimal parameter λ . In order to list them, we start with the component groups of $\phi \in P_\lambda({}^L G)$. First, note that

$$Z_{\widehat{G}}(\lambda) = \left\{ \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & s_2 \end{pmatrix} \mid s_1 s_2 = \pm 1 \right\} \cong \text{GL}(1) \times \mu_2,$$

under the isomorphism $s \mapsto (s_1, s_1 s_2)$. Then

$$A_{\phi_0} = \pi_0(Z_{\widehat{G}}(\phi_0)) = \pi_0(Z_{\widehat{G}}(\lambda)) \cong \mu_2 \quad \text{and} \quad A_{\phi_1} = \pi_0(Z_{\widehat{G}}(\phi_1)) = \pi_0(Z(\widehat{G})) \cong \mu_4.$$

Following our convention, we write $+$ and $-$ for the trivial and non-trivial characters of μ_2 , respectively; the characters of μ_4 will be labeled by $+1, -1, +i$ and $-i$. The admissible representations for the Langlands parameters ϕ_0 and ϕ_1 fall into L-packets for the three forms of G (up to isomorphism) as follows:

$$\begin{aligned} \Pi_{\phi_0}(G(F)) &= \{\pi(\phi_0, +)\} & \Pi_{\phi_1}(G(F)) &= \{\pi(\phi_1, +1)\} \\ \Pi_{\phi_0}(G_1(F)) &= \emptyset & \Pi_{\phi_1}(G_1(F)) &= \{\pi(\phi_1, +i)\} \\ & & &= \{\pi(\phi_1, -i)\} \\ \Pi_{\phi_0}(G_2(F)) &= \{\pi(\phi_0, -)\} & \Pi_{\phi_1}(G_2(F)) &= \{\pi(\phi_1, -1)\}. \end{aligned}$$

However, $\Pi_{\text{pure}, \lambda}(G/F)$ consists of 6 representations of 4 pure rational forms of G :

$$\Pi_{\text{pure}, \phi_0}(G/F) = \{ [\pi(\phi_0, +), 0], [\pi(\phi_0, -), 2] \},$$

and

$$\Pi_{\text{pure}, \phi_1}(G/F) = \{ [\pi(\phi_1, +1), 0], [\pi(\phi_1, +i), 1], [\pi(\phi_1, -1), 2], [\pi(\phi_1, -i), 3] \}.$$

In other words, when passing from the four equivalence classes of pure rational forms $[\delta] \in H^1(F, G)$ to the three isomorphism classes of forms of G , two representations collapse to one, namely, $[\pi(\phi_1, +i), 1]$ and $[\pi(\phi_1, -i), 3]$ map to the same admissible representation of $G_1(F)$.

11.1.3. Multiplicities in standard modules. —

	$\pi(\phi_0, +)$	$\pi(\phi_0, -)$	$\pi(\phi_1, +1)$	$\pi(\phi_1, -1)$	$\pi(\phi_1, +i)$	$\pi(\phi_1, -i)$
$M(\phi_0, +1)$	1	0	1	0	0	0
$M(\phi_0, -1)$	0	1	0	1	0	0
$M(\phi_1, +1)$	0	0	1	0	0	0
$M(\phi_1, -1)$	0	0	0	1	0	0
$M(\phi_1, +i)$	0	0	0	0	1	0
$M(\phi_1, -i)$	0	0	0	0	0	1

11.1.4. Arthur packets. — The component groups A_{ψ_0} and A_{ψ_1} are both $Z(\widehat{G})$, canonically. Arthur packets for rational forms G, G_1 and G_2 of G are

$$\begin{aligned} \Pi_{\psi_0}(G(F)) &= \{\pi(\phi_0, +)\} & \Pi_{\psi_1}(G(F)) &= \{\pi(\phi_1, +1)\} \\ \Pi_{\psi_0}(G_1(F)) &= \{\pi(\phi_1, +i)\} & \Pi_{\psi_1}(G_1(F)) &= \{\pi(\phi_1, +i)\} \\ &= \{\pi(\phi_1, -i)\} & &= \{\pi(\phi_1, -i)\} \\ \Pi_{\psi_0}(G_2(F)) &= \{\pi(\phi_0, -)\} & \Pi_{\psi_1}(G_2(F)) &= \{\pi(\phi_1, -1)\} \end{aligned}$$

The pure Arthur packets for ψ_0 and ψ_1 are

$$\Pi_{\text{pure}, \psi_0}(G/F) = \{ [\pi(\phi_0, +), 0], [\pi(\phi_0, -), 2], [\pi(\phi_1, +i), 1], [\pi(\phi_1, -i), 3] \},$$

and

$$\Pi_{\text{pure}, \psi_1}(G/F) = \{ [\pi(\phi_1, +1), 0], [\pi(\phi_1, +i), 1], [\pi(\phi_1, -1), 2], [\pi(\phi_1, -i), 3] \}.$$

For later reference, we break these pure Arthur packets apart into packet and coronal representations:

pure Arthur packets	pure L-packet representations	coronal representations
$\Pi_{\text{pure}, \psi_0}(G/F)$	$[\pi(\phi_0, +), 0], [\pi(\phi_0, -), 2]$	$[\pi(\phi_1, +i), 1], [\pi(\phi_i, -i), 3]$
$\Pi_{\text{pure}, \psi_1}(G/F)$	$[\pi(\phi_1, +1), 0], [\pi(\phi_1, +i), 1]$ $[\pi(\phi_1, -i), 3], [\pi(\phi_1, -1), 2]$	

11.1.5. Aubert duality. — Aubert duality for admissible representations of $G(F)$ with infinitesimal parameter λ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0, +)$	$\pi(\phi_1, +1)$
$\pi(\phi_1, +1)$	$\pi(\phi_0, +)$

Aubert duality for $G_1(F) = G_3(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_1, +i) = \pi(\phi_1, -i)$	$\pi(\phi_1, +i) = \pi(\phi_1, -i)$

Aubert duality for $G_2(F)$ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0, -)$	$\pi(\phi_1, -1)$
$\pi(\phi_1, -1)$	$\pi(\phi_0, -)$

The twisting characters χ_{ψ_0} and χ_{ψ_1} are trivial.

11.1.6. Stable distributions and endoscopy. — The coefficients $\langle a_s a_\psi, (\pi, \delta) \rangle_{\psi}$ appearing in the invariant distributions $\eta_{\psi, s}$ (32) are given by the following list, in which $s \in A_\psi = Z(\hat{G}) \cong \mu_4$.

$$\begin{aligned}
\eta_{\psi_0} = \eta_{\psi_0, 1} &= [\pi(\phi_0, +), 0] + [\pi(\phi_0, -), 2] + [\pi(\phi_1, +i), 1] + [\pi(\phi_1, -i), 3] \\
\eta_{\psi_0, -1} &= [\pi(\phi_0, +), 0] + [\pi(\phi_0, -), 2] - [\pi(\phi_1, +i), 1] - [\pi(\phi_1, -i), 3] \\
\eta_{\psi_0, i} &= [\pi(\phi_0, +), 0] - [\pi(\phi_0, -), 2] + i[\pi(\phi_1, +i), 1] - i[\pi(\phi_1, -i), 3] \\
\eta_{\psi_0, -i} &= [\pi(\phi_0, +), 0] - [\pi(\phi_0, -), 2] - i[\pi(\phi_1, +i), 1] + i[\pi(\phi_1, -i), 3]
\end{aligned}$$

and

$$\begin{aligned}
\eta_{\psi_1} = \eta_{\psi_1, 1} &= [\pi(\phi_1, +1), 0] - [\pi(\phi_1, -i), 1] + [\pi(\phi_1, -1), 2] - [\pi(\phi_1, -i), 3] \\
\eta_{\psi_1, -1} &= [\pi(\phi_1, +1), 0] - [\pi(\phi_1, -i), 1] - [\pi(\phi_1, -1), 2] + [\pi(\phi_1, -i), 3] \\
\eta_{\psi_1, i} &= [\pi(\phi_1, +1), 0] + [\pi(\phi_1, -i), 1] + i[\pi(\phi_1, -1), 2] + i[\pi(\phi_1, -i), 3] \\
\eta_{\psi_1, -i} &= [\pi(\phi_1, +1), 0] + [\pi(\phi_1, -i), 1] - i[\pi(\phi_1, -1), 2] - i[\pi(\phi_1, -i), 3].
\end{aligned}$$

Since $A_{\psi_0} = Z(\hat{G})$ and $A_{\psi_1} = Z(\hat{G})$, the only endoscopic groups relevant to these Arthur parameters are $G = G$, G_1 and G_2 .

11.2. Vanishing cycles of perverse sheaves. —

11.2.1. Vogan variety and orbit duality. — The Vogan variety V_λ and its dual V_λ^* may both be deduced from the conormal bundle

$$T_H^*(V) = \left\{ \begin{pmatrix} 0 & y & 0 & 0 \\ y' & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & y' & 0 \end{pmatrix} \mid yy' = 0 \right\}$$

on which $H := Z_{\widehat{G}}(\lambda) \cong \mathrm{GL}(1) \times \nu_2$ acts by

$$\begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & s_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & y & 0 & 0 \\ y' & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & y' & 0 \end{pmatrix} = \begin{pmatrix} 0 & s_1 s_2^{-1} y & 0 & 0 \\ s_1^{-1} s_2 y' & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 s_2^{-1} y \\ 0 & 0 & s_1^{-1} s_2 y' & 0 \end{pmatrix}.$$

Recall that $s_1 s_2 = \pm 1$, so $s_1 s_2^{-1} = \pm s_1^2$. From this we see the stratification of V into H -orbits and the duality on those orbits is exactly as in Section 10.2.1.

We now use Theorem 4.1.1 to replace $\lambda : W_F \rightarrow {}^L G$ with an unramified infinitesimal parameter $\lambda_{\mathrm{nr}} : W_F \rightarrow {}^L G_\lambda$ of a split group G_λ such that $\lambda_{\mathrm{nr}}(\mathrm{Fr})$ is hyperbolic. The hyperbolic part of $\lambda(\mathrm{Fr})$ is $s_\lambda \times 1$ with

$$s_\lambda = \rho(1) \otimes \nu_2(\mathrm{Fr}) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$$

while the elliptic part of $\lambda(\mathrm{Fr})$ is $t_\lambda \times \mathrm{Fr}$ with

$$t_\lambda = \rho(\mathrm{Fr}) \otimes \nu_2(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$J_\lambda := Z_{\widehat{G}}(\lambda|_{I_F}, s_\lambda) = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \mid \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1 \right\} \cong \mathrm{SL}(2) \times \mu_2$$

under the isomorphism $\mathrm{diag}(h, h) \mapsto (h', \det h)$ where $h' = h$ if $\det h = 1$ and $h' = ih$ if $\det h = -1$. Therefore, $G_\lambda = \mathrm{PGL}(2)$ and $\lambda_{\mathrm{nr}} : W_F \rightarrow {}^L G_\lambda$ is given by

$$\lambda_{\mathrm{nr}}(w) = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}.$$

Now

$$H_{\lambda_{\mathrm{nr}}} = Z_{\widehat{G}_\lambda}(\lambda_{\mathrm{nr}}) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \neq 0 \right\} \cong \mathrm{GL}(1)$$

and

$$V_{\lambda_{\mathrm{nr}}} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \right\} \cong \mathbb{A}^1$$

with $H_{\lambda_{\text{nr}}}$ -action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & t^2 y \\ 0 & 0 \end{pmatrix}.$$

This brings us back to Section 10.2.1. We will freely use notation from there, below. The H_λ -action on $V_{\lambda_{\text{nr}}}$ is given by

$$(t, \pm 1) : \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \pm t^2 y \\ 0 & 0 \end{pmatrix}.$$

From this we see that every H_λ -orbit in $V_{\lambda_{\text{nr}}}$ coincides with a $H_{\lambda_{\text{nr}}}$ orbit in $V_{\lambda_{\text{nr}}}$.

11.2.2. Equivariant perverse sheaves on Vogan variety. — With reference to Theorem 4.1.1 we have

$$\begin{array}{ccccc} \text{Rep}(A_\lambda) & \longrightarrow & \text{Per}_{H_\lambda}(V_\lambda) & \xrightleftharpoons[\pi_*]{\pi^*} & \text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}}) \\ \parallel & & \parallel & & \parallel \\ \text{Rep}(\mu_2) & & \text{Per}_{\text{GL}(1) \times \mu_2}(\mathbb{A}^1) & & \text{Per}_{\text{GL}(1)}(\mathbb{A}^1) \end{array}$$

The image of the trivial representation $+$ of μ_2 under the functor $\text{Rep}(A_\lambda) \rightarrow \text{Per}_{H_\lambda}(V_\lambda)$ is the trivial local system on V , denoted here by $(+)_V$ to emphasise its genesis; image of the non-trivial irreducible representation $-$ of μ_2 under the functor $\text{Rep}(A_\lambda) \rightarrow \text{Per}_{H_\lambda}(V_\lambda)$ will likewise be denoted by $(-)_V$.

To find the simple objects in $\text{Per}_H(V)$, we begin with the equivariant perverse sheaves on H -orbits in V .

C_0 : The equivariant fundamental group of C_0 is $A_{C_0} = \pi_0(H) \cong \mu_2$. Let us write $\mathbb{1}_{C_0}^+$ and $\mathbb{1}_{C_0}^-$ for the local systems corresponding to the trivial and non-trivial representations of A_{C_0} , respectively. Note that, under the forgetful functor $\text{Loc}_H(C_0) \rightarrow \text{Loc}_{H_{\text{nr}}}(C_0)$, these both map to $\mathbb{1}_{C_0}$, the constant sheaf on C_0 .

C_y : The equivariant fundamental group of C_y is $A_{C_y} = Z(\widehat{G}) \cong \mu_4$. Let us write $\mathbb{1}_{C_y}^+$ and $\mathbb{1}_{C_y}^-$ for the equivariant local systems on C_y that correspond to the trivial $+1$ and order-2 characters -1 of A_{C_y} , respectively; these both map to $\mathbb{1}_{C_y}$ under $\text{Loc}_H(C_y) \rightarrow \text{Loc}_{H_{\text{nr}}}(C_y)$. We write $\mathcal{E}_{C_y}^+$ and $\mathcal{E}_{C_y}^-$ for the equivariant local systems on C_y that correspond to the order-4 characters $+i$ and $-i$, respectively, of A_{C_y} ; these both map to \mathcal{E}_{C_y} under $\text{Loc}_H(C_y) \rightarrow \text{Loc}_{H_{\text{nr}}}(C_y)$.

Therefore, the six simple objects in $\text{Per}_H(V)$ are given by:

$$\text{Per}_H(V)_{\text{iso}}^{\text{simple}} = \left\{ \begin{array}{ccc} \mathcal{IC}(\mathbb{1}_{C_0}^+), & \mathcal{IC}(\mathbb{1}_{C_y}^+), & \mathcal{IC}(\mathcal{E}_{C_y}^+) \\ \mathcal{IC}(\mathbb{1}_{C_0}^-), & \mathcal{IC}(\mathbb{1}_{C_y}^-), & \mathcal{IC}(\mathcal{E}_{C_y}^-) \end{array} \right\}.$$

On simple objects, the functor $\text{Rep}(A_\lambda) \rightarrow \text{Per}_{H_\lambda}(V)$ is given by

$$\begin{array}{ccc} \text{Rep}(A_\lambda) & \rightarrow & \text{Per}_{H_\lambda}(V) \\ (+)_V & \mapsto & \mathcal{IC}(\mathbb{1}_{C_y}^+) \\ (-)_V & \mapsto & \mathcal{IC}(\mathbb{1}_{C_y}^-) \end{array}$$

while the functor $\text{Per}_{H_\lambda}(V) \rightarrow \text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}})$ is given by

$$\begin{aligned} \text{Per}_{H_\lambda}(V) &\rightarrow \text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}}) \\ \mathcal{IC}(\mathbb{1}_{C_0}^\pm) &\mapsto \mathcal{IC}(\mathbb{1}_{C_0}) \\ \mathcal{IC}(\mathbb{1}_{C_y}^\pm) &\mapsto \mathcal{IC}(\mathbb{1}_{C_y})\mathcal{IC}(\mathcal{E}_{C_y}^\pm) \mapsto \mathcal{IC}(\mathcal{E}_{C_y}) \end{aligned}$$

and the functor $\text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}}) \rightarrow \text{Per}_{H_\lambda}(V)$ is given by

$$\begin{aligned} \text{Per}_{H_{\lambda_{\text{nr}}}}(V_{\lambda_{\text{nr}}}) &\rightarrow \text{Per}_{H_\lambda}(V) \\ \mathcal{IC}(\mathbb{1}_{C_0}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_0}^+) \oplus \mathcal{IC}(\mathbb{1}_{C_0}^-) \\ \mathcal{IC}(\mathcal{E}_{C_y}) &\mapsto \mathcal{IC}(\mathcal{E}_{C_y}^+) \oplus \mathcal{IC}(\mathcal{E}_{C_y}^-) \end{aligned}$$

From this we find the stalks of the simple objects in $\text{Per}_H(V)$.

\mathcal{P}	$\mathcal{P} _{C_0}$	$\mathcal{P} _{C_{+1}}$
$\mathcal{IC}(\mathbb{1}_{C_0}^+)$	$\mathbb{1}_{C_0}^+[0]$	0
$\mathcal{IC}(\mathbb{1}_{C_0}^-)$	$\mathbb{1}_{C_0}^-[0]$	0
$\mathcal{IC}(\mathbb{1}_{C_{+1}}^+)$	$\mathbb{1}_{C_0}^+[1]$	$\mathbb{1}_{C_{+1}}^+[1]$
$\mathcal{IC}(\mathbb{1}_{C_{+1}}^-)$	$\mathbb{1}_{C_0}^-[1]$	$\mathbb{1}_{C_{+1}}^-[1]$
$\mathcal{IC}(\mathcal{E}_{C_{+1}}^+)$	0	$\mathcal{E}_{C_{+1}}^+[1]$
$\mathcal{IC}(\mathcal{E}_{C_{+1}}^-)$	0	$\mathcal{E}_{C_{+1}}^-[1]$

This gives us the normalised geometric multiplicity matrix:

	$(\mathbb{1}_{C_0}^+)^\sharp$	$(\mathbb{1}_{C_0}^-)^\sharp$	$(\mathbb{1}_{C_1}^+)^\sharp$	$(\mathbb{1}_{C_1}^-)^\sharp$	$(\mathcal{E}_{C_1}^+)^\sharp$	$(\mathcal{E}_{C_1}^-)^\sharp$
$(\mathbb{1}_{C_0}^+)^\sharp$	1	0	0	0	0	0
$(\mathbb{1}_{C_0}^-)^\sharp$	0	1	0	0	0	0
$(\mathbb{1}_{C_1}^+)^\sharp$	1	0	1	0	0	0
$(\mathbb{1}_{C_1}^-)^\sharp$	0	1	0	1	0	0
$(\mathcal{E}_{C_1}^+)^\sharp$	0	0	0	0	1	0
$(\mathcal{E}_{C_1}^-)^\sharp$	0	0	0	0	0	1

11.2.3. Cuspidal support decomposition and Fourier transform. — The cuspidal support decomposition respects the functors appearing in Theorem 4.1.1, so the results here follow from Section 10.2.3. Specifically, we have

$$\text{Per}_{H_\lambda}(V_\lambda) = \text{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \text{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}},$$

where the simple objects in these summand categories are given here.

$\text{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}/\text{iso}}^{\text{simple}}$	$\text{Per}_{H_\lambda}(V_\lambda)_{\widehat{G}/\text{iso}}^{\text{simple}}$
$\mathcal{IC}(\mathbb{1}_{C_0}^+)$	
$\mathcal{IC}(\mathbb{1}_{C_0}^-)$	
$\mathcal{IC}(\mathbb{1}_{C_y}^+)$	$\mathcal{IC}(\mathcal{E}_{C_y}^+)$
$\mathcal{IC}(\mathbb{1}_{C_y}^-)$	$\mathcal{IC}(\mathcal{E}_{C_y}^-)$

Since the diagram

$$\begin{array}{ccccc}
 \mathrm{Rep}(A_\lambda) & \longrightarrow & \mathrm{Per}_{H_\lambda}(V_\lambda) & \xrightleftharpoons[\pi_*]{\pi^*} & \mathrm{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}}) \\
 \downarrow \mathrm{id} & & \downarrow \mathrm{Ft} & & \downarrow \mathrm{Ft} \\
 \mathrm{Rep}(A_\lambda) & \longrightarrow & \mathrm{Per}_{H_\lambda}(V_\lambda^*) & \xrightleftharpoons[\pi_*]{\pi^*} & \mathrm{Per}_{H_{\lambda_{\mathrm{nr}}}}(V_{\lambda_{\mathrm{nr}}}^*)
 \end{array}$$

commutes, the Fourier transform is given on simple objects as follows.

$$\begin{array}{ll}
 \mathrm{Ft} : \mathrm{Per}_{H_\lambda}(V_\lambda) & \longrightarrow \mathrm{Per}_{H_\lambda}(V_\lambda^*) \\
 \mathcal{IC}(\mathbb{1}_{C_0}^+) & \mapsto \mathcal{IC}(\mathbb{1}_{C_0^*}^+) = \mathcal{IC}(\mathbb{1}_{C_1^t}^+) \\
 \mathcal{IC}(\mathbb{1}_{C_0}^-) & \mapsto \mathcal{IC}(\mathbb{1}_{C_0^*}^-) = \mathcal{IC}(\mathbb{1}_{C_1^t}^-) \\
 \mathcal{IC}(\mathbb{1}_{C_y}^+) & \mapsto \mathcal{IC}(\mathbb{1}_{C_y^*}^+) = \mathcal{IC}(\mathbb{1}_{C_0^t}^+) \\
 \mathcal{IC}(\mathbb{1}_{C_y}^-) & \mapsto \mathcal{IC}(\mathbb{1}_{C_y^*}^-) = \mathcal{IC}(\mathbb{1}_{C_0^t}^-) \\
 \mathcal{IC}(\mathcal{E}_{C_y}^+) & \mapsto \mathcal{IC}(\mathcal{E}_{C_0^*}^+) = \mathcal{IC}(\mathcal{E}_{C_y^t}^+) \\
 \mathcal{IC}(\mathcal{E}_{C_y}^-) & \mapsto \mathcal{IC}(\mathcal{E}_{C_0^*}^-) = \mathcal{IC}(\mathcal{E}_{C_y^t}^-)
 \end{array}$$

11.2.4. Equivariant perverse sheaves on the regular conormal bundle. — Recall that H_λ orbits coincide with $H_{\lambda_{\mathrm{nr}}}$ -orbits. The following diagram commutes:

$$\begin{array}{ccccc}
 \mathrm{Rep}(A_\lambda) & \longrightarrow & \mathrm{Per}_{H_\lambda}(C^*) & \xrightleftharpoons[\pi_*]{\pi^*} & \mathrm{Per}_{H_{\lambda_{\mathrm{nr}}}}(C^*) \\
 \parallel & & \downarrow & & \downarrow \\
 \mathrm{Rep}(A_\lambda) & \longrightarrow & \mathrm{Per}_{H_\lambda}(T_C^*(V_\lambda)_{\mathrm{sreg}}) & \xrightleftharpoons[\pi_*]{\pi^*} & \mathrm{Per}_{H_{\lambda_{\mathrm{nr}}}}(T_C^*(V_{\lambda_{\mathrm{nr}}})_{\mathrm{sreg}}) \\
 \parallel & & \uparrow & & \uparrow \\
 \mathrm{Rep}(A_\lambda) & \longrightarrow & \mathrm{Per}_{H_\lambda}(C) & \xrightleftharpoons[\pi_*]{\pi^*} & \mathrm{Per}_{H_{\lambda_{\mathrm{nr}}}}(C)
 \end{array}$$

We now describe the fundamental groups and associated equivariant local systems on the strongly regular conormal bundle $T_H^*(V)_{\mathrm{sreg}}$. For the computation of the functor $\mathrm{Ev} : \mathrm{Per}_H(V) \rightarrow \mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}})$ in Section 11.2.5 we will need to know the effect of pullback along the bundle map $T_H^*(V)_{\mathrm{reg}} \rightarrow V$, so we also give that information below.

C_0 : We choose a base point for $T_{C_0}^*(V)_{\mathrm{sreg}}$:

$$(x_0, \xi_0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $A_{(x_0, \xi_0)} = Z(\widehat{G}) \cong \mu_4$ and the bundle maps induce the following homomorphisms of fundamental groups:

$$\mu_2 \cong A_{x_0} \hookleftarrow A_{(x_0, \xi_0)} \xrightarrow{\mathrm{id}} A_{\xi_0} \cong \mu_4.$$

Now label local systems on $T_{C_0}^*(V)_{\mathrm{sreg}}$ according to the following chart, which lists the corresponding characters of $A_{(x_0, \xi_0)}$ using the convention for characters

of μ_4 from Section 11.1.2.

$$\begin{array}{c} \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) : \quad \mathbb{1}_{\mathcal{O}_0}^+ \quad \mathbb{1}_{\mathcal{O}_0}^- \quad \mathcal{E}_{\mathcal{O}_0}^+ \quad \mathcal{E}_{\mathcal{O}_0}^- \\ \text{Rep}(A_{(x_0, \xi_0)}) : \quad +1 \quad -1 \quad +i \quad -i \end{array}$$

Pullback of equivariant local systems along the bundle map $T_{C_0}^*(V)_{\text{sreg}} \rightarrow C_0$ is given on simple objects by:

$$\begin{array}{ccc} \text{Loc}_H(C_0) & \rightarrow & \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_0}^\pm & \mapsto & \mathbb{1}_{\mathcal{O}_0}^\pm \\ & & \mathcal{E}_{\mathcal{O}_0}^\pm. \end{array}$$

C_y : We choose a base point for $T_{C_y}^*(V)_{\text{sreg}}$:

$$(x_1, \xi_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A_{(x_1, \xi_1)} = Z(\widehat{G}) \cong \mu_4$ and the bundle maps induce the following homomorphisms of fundamental groups:

$$\mu_4 \cong A_{x_1} \xleftarrow{\text{id}} A_{(x_1, \xi_1)} \longrightarrow A_{\xi_1} \cong \mu_2.$$

Now label local systems on $T_{C_y}^*(V)_{\text{sreg}}$ according to the following chart, which lists the corresponding characters of $A_{(x_1, \xi_1)}$ using the convention for characters of μ_4 from Section 11.1.2.

$$\begin{array}{c} \text{Loc}_H(T_{C_y}^*(V)_{\text{sreg}}) : \quad \mathbb{1}_{\mathcal{O}_y}^+ \quad \mathbb{1}_{\mathcal{O}_y}^- \quad \mathcal{E}_{\mathcal{O}_y}^+ \quad \mathcal{E}_{\mathcal{O}_y}^- \\ \text{Rep}(A_{(x_1, \xi_1)}) : \quad +1 \quad -1 \quad +i \quad -i \end{array}$$

Pullback of equivariant local systems along the bundle map $T_{C_y}^*(V)_{\text{sreg}} \rightarrow C_y$ is given on simple objects by:

$$\begin{array}{ccc} \text{Loc}_H(C_y) & \rightarrow & \text{Loc}_H(T_{C_y}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_y}^\pm & \mapsto & \mathbb{1}_{\mathcal{O}_y}^\pm \\ \mathcal{E}_{C_y}^\pm & \mapsto & \mathcal{E}_{\mathcal{O}_y}^\pm. \end{array}$$

11.2.5. Vanishing cycles of perverse sheaves. — Table 11.2.1 gives the functor $\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{reg}})$ on simple objects. These calculations follow from Table 10.2.1.

11.2.6. Normalization of Ev and the twisting local system. — From Table 11.2.1 we see that the twisting local system \mathcal{T} is trivial in this case, so ${}^p\text{NEv} = {}^p\text{Ev}$.

11.2.7. Vanishing cycles and Fourier transform. — Comparing the table below with $\text{Ft} : \text{Per}_H(V) \rightarrow \text{Per}_H(V^*)$ from Section 11.2.3 verifies (141) in this example.

$$\begin{array}{ccccccc} \text{Per}_H(V) & \xrightarrow{{}^p\text{Ev}} & \text{Per}_H(T_{H_\lambda}^*(V)_{\text{reg}}) & \xrightarrow{a_*} & \text{Per}_H(T_{H_\lambda}^*(V^*)_{\text{reg}}) & \xleftarrow{\text{Ev}^*} & \text{Per}_H(V^*) \\ \mathcal{IC}(\mathbb{1}_{C_0}^\pm) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}^\pm) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*}^\pm) & \mapleftarrow & \mathcal{IC}(\mathbb{1}_{C_0^*}^\pm) \\ \mathcal{IC}(\mathbb{1}_{C_y}^\pm) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}^\pm) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_y^*}^\pm) & \mapleftarrow & \mathcal{IC}(\mathbb{1}_{C_y^*}^\pm) \\ \mathcal{IC}(\mathcal{E}_{C_y}^\pm) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}^\pm) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}^\pm) & \mapsto & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y^*}^\pm) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0^*}^\pm) & \mapleftarrow & \mathcal{IC}(\mathcal{E}_{C_0^*}^\pm) \end{array}$$

TABLE 11.2.1. ${}^p\mathrm{Ev} : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 11.

$\mathrm{Per}_H(V)$	$\xrightarrow{{}^p\mathrm{Ev}}$	$\mathrm{Per}_H(T_{H_\lambda}^*(V)_{\mathrm{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0}^+)$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}^+)$
$\mathcal{IC}(\mathbb{1}_{C_0}^-)$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}^-)$
$\mathcal{IC}(\mathbb{1}_{C_y}^+)$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}^+)$
$\mathcal{IC}(\mathbb{1}_{C_y}^-)$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_y}^-)$
$\mathcal{IC}(\mathcal{E}_{C_y}^+)$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}^+) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}^+)$
$\mathcal{IC}(\mathcal{E}_{C_y}^-)$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}^-) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}^-)$

TABLE 11.2.2. $\mathrm{Evs} : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 11.

\mathcal{P}	\parallel	$\mathrm{Evs}_{C_0} \mathcal{P}$	$\mathrm{Evs}_{C_1} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0}^+)$	\parallel	+1	0
$\mathcal{IC}(\mathbb{1}_{C_0}^-)$	\parallel	-1	0
$\mathcal{IC}(\mathbb{1}_{C_y}^+)$	\parallel	0	+1
$\mathcal{IC}(\mathbb{1}_{C_y}^-)$	\parallel	0	-1
$\mathcal{IC}(\mathcal{E}_{C_y}^+)$	\parallel	+i	+i
$\mathcal{IC}(\mathcal{E}_{C_y}^-)$	\parallel	-i	-i

11.2.8. Arthur sheaves. —

Arthur sheaf	\parallel	packet sheaves	\parallel	coronal sheaves
\mathcal{A}_{C_0}	\parallel	$\mathcal{IC}(\mathbb{1}_{C_0}^+) \oplus \mathcal{IC}(\mathbb{1}_{C_0}^-) \oplus$	\parallel	$\mathcal{IC}(\mathcal{E}_{C_y}^+) \oplus \mathcal{IC}(\mathcal{E}_{C_y}^-)$
\mathcal{A}_{C_y}	\parallel	$\mathcal{IC}(\mathbb{1}_{C_y}^+) \oplus \mathcal{IC}(\mathbb{1}_{C_y}^-) \oplus \mathcal{IC}(\mathcal{E}_{C_y}^+) \oplus \mathcal{IC}(\mathcal{E}_{C_y}^-)$	\parallel	

11.3. ABV-packets. —

11.3.1. Admissible representations versus perverse sheaves. —

$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}}$	\parallel	$\Pi_{\mathrm{pure}, \lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0}^+)$	\parallel	$(\pi(\phi_0, +), 0)$
$\mathcal{IC}(\mathbb{1}_{C_0}^-)$	\parallel	$(\pi(\phi_0, -), 2)$
$\mathcal{IC}(\mathbb{1}_{C_y}^+)$	\parallel	$(\pi(\phi_1, +1), 0)$
$\mathcal{IC}(\mathbb{1}_{C_y}^-)$	\parallel	$(\pi(\phi_1, -1), 2)$
$\mathcal{IC}(\mathcal{E}_{C_y}^+)$	\parallel	$(\pi(\phi_1, +i), 1)$
$\mathcal{IC}(\mathcal{E}_{C_y}^-)$	\parallel	$(\pi(\phi_1, -i), 3)$

11.3.2. ABV-packets. —

ABV-packets	pure L-packet representations	coronal representations
$\Pi_{\text{pure}, \phi_0}^{\text{ABV}}(G/F) :$	$[\pi(\phi_0, +), 0], [\pi(\phi_0, -), 2]$	$[\pi(\phi_1, +i), 1], [\pi(\phi_1, -i), 3]$
$\Pi_{\text{pure}, \phi_1}^{\text{ABV}}(G/F) :$	$[\pi(\phi_1, +1), 0], [\pi(\phi_1, +i), 1]$ $[\pi(\phi_1, -1), 2], [\pi(\phi_1, -i), 3]$	

11.3.3. Stable distributions and endoscopic transfer. —

$$\begin{aligned}
 \eta_{\psi_0}^{\text{NEV}} = \eta_{\psi_0, 1}^{\text{NEV}} &= [\pi(\phi_0, +), 0] + [\pi(\phi_0, -), 2] + [\pi(\phi_1, +i), 1] + [\pi(\phi_1, -i), 3] \\
 \eta_{\psi_0, -1}^{\text{NEV}} &= [\pi(\phi_0, +), 0] + [\pi(\phi_0, -), 2] - [\pi(\phi_1, +i), 1] - [\pi(\phi_1, -i), 3] \\
 \eta_{\psi_0, i}^{\text{NEV}} &= [\pi(\phi_0, +), 0] - [\pi(\phi_0, -), 2] + i[\pi(\phi_1, +i), 1] - i[\pi(\phi_1, -i), 3] \\
 \eta_{\psi_0, -i}^{\text{NEV}} &= [\pi(\phi_0, +), 0] - [\pi(\phi_0, -), 2] - i[\pi(\phi_1, +i), 1] + i[\pi(\phi_1, -i), 3] \\
 \\
 \eta_{\psi_1}^{\text{NEV}} = \eta_{\psi_1, 1}^{\text{NEV}} &= [\pi(\phi_1, 1), 0] - [\pi(\phi_1, i), 1] + [\pi(\phi_1, -1), 2] - [\pi(\phi_1, -i), 3] \\
 \eta_{\psi_1, -1}^{\text{NEV}} &= [\pi(\phi_1, 1), 0] - [\pi(\phi_1, i), 1] - [\pi(\phi_1, -1), 2] + [\pi(\phi_1, -i), 3] \\
 \eta_{\psi_1, i}^{\text{NEV}} &= [\pi(\phi_1, 1), 0] + [\pi(\phi_1, i), 1] + i[\pi(\phi_1, -1), 2] + i[\pi(\phi_1, -i), 3] \\
 \eta_{\psi_1, -i}^{\text{NEV}} &= [\pi(\phi_1, 1), 0] + [\pi(\phi_1, i), 1] - i[\pi(\phi_1, -1), 2] - i[\pi(\phi_1, -i), 3]
 \end{aligned}$$

Comparing with Section 11.1.6 proves (148).

11.3.4. Kazhdan-Lusztig conjecture. — From Section 11.1.3 we find the multiplicity matrix:

$$m_{\text{rep}} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and from Section 11.2.2 we find the normalised geometric multiplicity matrix

$$m'_{\text{geo}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $m_{\text{rep}}^t = m'_{\text{geo}}$, this proves the Kazhdan-Lusztig conjecture in this case.

Notice that

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and compare with Section 10.3.4.

11.3.5. Aubert duality and Fourier transform. — To verify (150), use Vogan’s bijection from Section 11.3.1 to compare Aubert duality from Section 11.1.5 with the Fourier transform from Section 11.2.3

To verify (152), observe that the twisting characters χ_ψ of A_ψ from Section 11.1.5 are trivial, as are the local systems \mathcal{T}_ψ from Section 11.2.7.

11.4. Endoscopy and equivariant restriction of perverse sheaves. — The material of Section 8.4 is trivial in this example, since $Z_{\widehat{G}}(\psi) = Z(\widehat{G})$.

12. SO(5) unipotent representations, regular parameter

In this example, of the four Langlands parameters with infinitesimal parameter λ below, only two are of Arthur type. Accordingly, we find two ABV-packet that are not Arthur packets.

Let $G = \mathrm{SO}(5)$, so $\widehat{G} = \mathrm{Sp}(4)$ and ${}^L G = \widehat{G} \times W_F$. As in the cases above,

$$H^1(F, G) = H^1(F, G_{\mathrm{ad}}) = H^1(F, \mathrm{Aut}(G)) \cong \mathbb{Z}/2\mathbb{Z},$$

so there are two isomorphism classes of rational forms of G , each pure. We will use the notation $G = G$ and G_1 for the non-quasisplit form of $\mathrm{SO}(5)$ given by the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -\varepsilon\varpi & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & \varpi & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\lambda : W_F \rightarrow \widehat{G}$ be the unramified homomorphism

$$\lambda(\mathrm{Fr}) = \begin{pmatrix} |w|^{3/2} & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 \\ 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & |w|^{-3/2} \end{pmatrix}.$$

Here and below we use the symplectic form $\langle x, y \rangle = {}^t x J y$ with matrix J given by $J_{ij} = (-1)^j \delta_{5-i, j}$ to determine a representation of $\widehat{G} = \mathrm{Sp}(4)$.

Although this example exhibits some interesting geometric phenomena, there is still no interesting endoscopy here. Nevertheless, this example will be important later when we consider other groups for which $\mathrm{SO}(5)$ is an endoscopic group.

12.1. Arthur packets. —

12.1.1. Parameters. — Up to $Z_{\widehat{G}}(\lambda)$ -conjugation, there are four Langlands parameters with infinitesimal parameter λ :

$$\begin{aligned}\phi_0(w, x) &= \nu_4(d_w) = \lambda(w), \\ \phi_1(w, x) &= \nu_2^2(d_w) \otimes \nu_2(x) = \left(\begin{array}{cc|cc} |w|x_{11} & |w|x_{11} & 0 & 0 \\ |w|x_{21} & |w|x_{22} & 0 & 0 \\ \hline 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12} \\ 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12} \end{array} \right), \\ \phi_2(w, x) &= \nu_2^3(d_w) \oplus \nu_2(x) = \left(\begin{array}{c|cc|c} |w|^{3/2} & 0 & 0 & 0 \\ \hline 0 & x_{11} & x_{12} & 0 \\ 0 & x_{21} & x_{22} & 0 \\ \hline 0 & 0 & 0 & |w|^{-3/2} \end{array} \right), \\ \phi_3(w, x) &= \nu_4(x),\end{aligned}$$

where $\nu_4 : \mathrm{SL}(2) \rightarrow \mathrm{Sp}(4)$ is the irreducible 4-dimensional representation of $\mathrm{SL}(2)$. Of the four Langlands parameters ϕ_0, ϕ_1, ϕ_2 and ϕ_3 , only ϕ_0 and ϕ_3 are of Arthur type; define

$$\psi_0(w, x, y) := \nu_4(y), \quad \text{and} \quad \psi_3(w, x, y) := \nu_4(x).$$

12.1.2. L -packets. — The component groups A_{ϕ_0} and A_{ϕ_1} are trivial, while the component groups A_{ϕ_2} and A_{ϕ_3} each have order two, being canonically isomorphic to $Z(\widehat{G})$. Therefore, the representations in play in this example are:

$$\begin{aligned}\Pi_{\phi_0}(G(F)) &= \{\pi(\phi_0)\}, & \Pi_{\phi_0}(G_1(F)) &= \emptyset, \\ \Pi_{\phi_1}(G(F)) &= \{\pi(\phi_1)\}, & \Pi_{\phi_1}(G_1(F)) &= \emptyset, \\ \Pi_{\phi_2}(G(F)) &= \{\pi(\phi_2, +)\}, & \Pi_{\phi_2}(G_1(F)) &= \{\pi(\phi_2, -)\}, \\ \Pi_{\phi_3}(G(F)) &= \{\pi(\phi_3, +)\}, & \Pi_{\phi_3}(G_1(F)) &= \{\pi(\phi_3, -)\}.\end{aligned}$$

Of the four admissible representations of $G(F)$ with infinitesimal parameter λ , only $\pi(\phi_3, +)$ is tempered – this is the Steinberg representation. The representation $\pi(\phi_1)$ (resp. $\pi(\phi_2, +)$) is denoted by $L(\nu\zeta \mathrm{St}_{\mathrm{GL}(2)})$ (resp. $L(\nu^{3/2}\zeta, \zeta \mathrm{St}_{\mathrm{SO}(3)})$) with $\zeta = 1$ in [Mat10]. When arranged into pure packets, we get

$$\begin{aligned}\Pi_{\mathrm{pure}, \phi_0}(G/F) &= \{[\pi(\phi_0), 0]\} \\ \Pi_{\mathrm{pure}, \phi_1}(G/F) &= \{[\pi(\phi_1), 0]\} \\ \Pi_{\mathrm{pure}, \phi_2}(G/F) &= \{[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]\} \\ \Pi_{\mathrm{pure}, \phi_3}(G/F) &= \{[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 1]\}.\end{aligned}$$

12.1.3. Multiplicities in standard modules. — The standard module $M(\phi_1)$ (resp. $M(\phi_2, +)$) is denoted by $\nu\zeta \mathrm{St}_{\mathrm{GL}(2)} \rtimes 1$ (resp. $\nu^{3/2}\zeta \rtimes \mathrm{St}_{\mathrm{SO}(3)}$) with $\zeta = 1$ in [Mat10]. The following table may be deduced from [Mat10, Proposition 3.3].

	$\pi(\phi_0)$	$\pi(\phi_1)$	$\pi(\phi_2, +)$	$\pi(\phi_3, +)$	$\pi(\phi_2, -)$	$\pi(\phi_3, -)$
$M(\phi_0)$	1	1	1	1	0	0
$M(\phi_1)$	0	1	0	1	0	0
$M(\phi_2, +)$	0	0	1	1	0	0
$M(\phi_3, +)$	0	0	0	1	0	0
$M(\phi_2, -)$	0	0	0	0	1	1
$M(\phi_3, -)$	0	0	0	0	0	1

12.1.4. Arthur packets. — The Arthur packets for these representations are

$$\begin{aligned}\Pi_{\psi_0}(G(F)) &= \{\pi(\phi_0)\}, & \Pi_{\psi_0}(G_1(F)) &= \{\pi(\phi_2, -)\}, \\ \Pi_{\psi_3}(G(F)) &= \{\pi(\phi_3, +)\}, & \Pi_{\psi_3}(G_1(F)) &= \{\pi(\phi_3, -)\}.\end{aligned}$$

When arranged into pure packets, we get

$$\begin{aligned}\Pi_{\text{pure}, \psi_0}(G/F) &= \{[\pi(\phi_0), 0], [\pi(\phi_2, -), 1]\}, \\ \Pi_{\text{pure}, \psi_3}(G/F) &= \{[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 1]\}.\end{aligned}$$

12.1.5. Aubert duality. — Aubert duality for $G(F)$ and $G_1(F)$ are given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_3, +)$
$\pi(\phi_1)$	$\pi(\phi_2, +)$
$\pi(\phi_2, +)$	$\pi(\phi_1)$
$\pi(\phi_3, +)$	$\pi(\phi_0)$
$\pi(\phi_2, -)$	$\pi(\phi_3, -)$
$\pi(\phi_3, -)$	$\pi(\phi_2, -)$

The twisting characters χ_{ψ_0} and χ_{ψ_1} are trivial.

12.1.6. Stable distributions and endoscopic transfer. — For $s \in Z(\hat{G}) \cong \mu_2$, the virtual representations $\eta_{\psi_0, s}$ and $\eta_{\psi_3, s}$ are given by

$$\begin{aligned}\eta_{\psi_0, 1} &= [\pi(\phi_0), 0] + [\pi(\phi_2, -), 1] \\ \eta_{\psi_0, -1} &= [\pi(\phi_0), 0] - [\pi(\phi_2, -), 1]\end{aligned}$$

and

$$\begin{aligned}\eta_{\psi_3, 1} &= [\pi(\phi_3, +), 0] - [\pi(\phi_3, -), 1] \\ \eta_{\psi_3, -1} &= [\pi(\phi_3, +), 0] + [\pi(\phi_3, -), 1].\end{aligned}$$

There are no endoscopic groups relevant to ψ_0 or ψ_3 other than G .

12.2. Vanishing cycles of perverse sheaves. —

12.2.1. *Vogan variety and orbit duality.* — Now

$$H := Z_{\widehat{G}}(\lambda) = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \mid \begin{matrix} t_1 \neq 0 \\ t_2 \neq 0 \end{matrix} \right\}.$$

The Vogan varieties V and V^* are given by

$$V = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid u, x \right\}, \quad V^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid u', x' \right\}.$$

The action of H on $T^*(V)$ is given by

$$\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} : \begin{pmatrix} 0 & u & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & u' & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & t_1 t_2^{-1} u & 0 & 0 \\ t_1^{-1} t_2 u' & 0 & t_2^2 x & 0 \\ 0 & t_2^{-2} x' & 0 & t_1 t_2^{-1} u \\ 0 & 0 & t_1^{-1} t_2 u' & 0 \end{pmatrix}.$$

The conormal bundle is

$$T_{H_\lambda}^*(V_\lambda) \cong \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{matrix} uu' = 0 \\ xx' = 0 \end{matrix} \right\}.$$

Now V is stratified into the following H -orbits:

$$C_0 := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad C_3 := \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \begin{matrix} u \neq 0 \\ x \neq 0 \end{matrix} \right\},$$

and

$$C_u := \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid u \neq 0 \right\}, \quad C_x := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x \neq 0 \right\}.$$

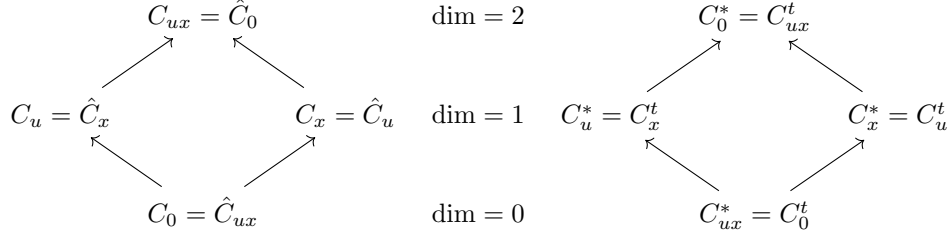
The dual orbits in V^* are

$$C_0^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{matrix} u' \neq 0 \\ x' \neq 0 \end{matrix} \right\}, \quad C_{ux}^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

and

$$C_u^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x' \neq 0 \right\}, \quad C_x^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid u' \neq 0 \right\}.$$

The following diagram gives the closure relations for these orbits.



12.2.2. Equivariant perverse sheaves. — The equivariant fundamental groups for C_0 and C_u are trivial, so they each carry only one equivariant local system, denoted by $\mathbb{1}_{C_0}$ and $\mathbb{1}_{C_u}$, respectively. The equivariant fundamental groups for C_x and C_{ux} have order two, so they each carry two equivariant local systems, denoted by $\mathbb{1}_{C_x}$, \mathcal{L}_{C_x} , $\mathbb{1}_{C_{ux}}$ and $\mathcal{L}_{C_{ux}}$. Thus,

$$\text{Per}_H(V)_{/\text{iso}}^{\text{simple}} = \{\mathcal{IC}(\mathbb{1}_{C_0}), \mathcal{IC}(\mathbb{1}_{C_u}), \mathcal{IC}(\mathbb{1}_{C_x}), \mathcal{IC}(\mathbb{1}_{C_{ux}}), \mathcal{IC}(\mathcal{L}_{C_x}), \mathcal{IC}(\mathcal{L}_{C_{ux}})\}.$$

The following table describes these perverse sheaves on H -orbits in V .

\mathcal{P}	$\mathcal{P} _{C_0}$	$\mathcal{P} _{C_u}$	$\mathcal{P} _{C_x}$	$\mathcal{P} _{C_{ux}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_u})$	$\mathbb{1}_{C_0}[1]$	$\mathbb{1}_{C_u}[1]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_x})$	$\mathbb{1}_{C_0}[1]$	0	$\mathbb{1}_{C_x}[1]$	0
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_u}[2]$	$\mathbb{1}_{C_x}[2]$	$\mathbb{1}_{C_{ux}}[2]$
$\mathcal{IC}(\mathcal{L}_{C_x})$	0	0	$\mathcal{L}_{C_x}[1]$	0
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	0	0	$\mathcal{L}_{C_x}[2]$	$\mathcal{L}_{C_{ux}}[2]$

We now explain how to make these calculations.

- (a) For the first four rows in the table above, those that deal with $\mathcal{IC}(\mathbb{1}_C)$, it is sufficient to observe that the closure \overline{C} of each strata C is smooth, hence the sheaf $\mathbb{1}_{\overline{C}}[\dim(C)]$ is perverse.
- (b) For the remaining two rows, those that deal with $\mathcal{IC}(\mathcal{L}_C)$, we observe that the closure \overline{C} of the strata C admits a finite equivariant double cover $\pi : \tilde{C} \rightarrow \overline{C}$ by taking \sqrt{x} . Because \tilde{C} is smooth, the sheaf $\mathbb{1}_{\tilde{C}}[\dim(C)]$ is perverse. The decomposition theorem for finite maps of perverse sheaves now yields that $\pi_!(\mathbb{1}_{\tilde{C}}[\dim(C)]) = \mathcal{IC}(\mathbb{1}_{\overline{C}}) \oplus \mathcal{IC}(\mathcal{L}_{\overline{C}})$. Proper base change, the decomposition theorem for finite étale maps, and our earlier computations for $\mathcal{IC}(\mathbb{1}_{\overline{C}})$ then allows us to readily compute the stalks of $\mathcal{IC}(\mathcal{L}_{\overline{C}})$.

From this, we easily find the normalised geometric multiplicity matrix is as follows.

	$\mathbb{1}_{C_0}$	$\mathbb{1}_{C_u}$	$\mathbb{1}_{C_x}$	$\mathbb{1}_{C_{ux}}$	\mathcal{L}_{C_x}	$\mathcal{L}_{C_{ux}}$
$\mathbb{1}_{C_0}^\sharp$	1	0	0	0	0	0
$\mathbb{1}_{C_u}^\sharp$	1	1	0	0	0	0
$\mathbb{1}_{C_x}^\sharp$	1	0	1	0	0	0
$\mathbb{1}_{C_{ux}}^\sharp$	1	1	1	1	0	0
$\mathcal{L}_{C_x}^\sharp$	0	0	0	0	1	0
$\mathcal{L}_{C_{ux}}^\sharp$	0	0	0	0	1	1

12.2.3. Cuspidal support decomposition and Fourier transform. — Up to conjugation, $\widehat{G} = \mathrm{Sp}(4)$ admits exactly two cuspidal Levi subgroups: $\widehat{M} = \mathrm{Sp}(2) \times \mathrm{GL}(1)$ and $\widehat{T} = \mathrm{GL}(1) \times \mathrm{GL}(1)$.

$$\mathrm{Per}_{H_\lambda}(V_\lambda) = \mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \mathrm{Per}_{H_\lambda}(V_\lambda)_{\widehat{M}}.$$

Simple objects in these two subcategories are listed below.

$\mathrm{Per}_H(V)_{\widehat{T}}$	$\mathrm{Per}_H(V)_{\widehat{M}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	
$\mathcal{IC}(\mathbb{1}_{C_u})$	
$\mathcal{IC}(\mathbb{1}_{C_x})$	$\mathcal{IC}(\mathcal{L}_{C_x})$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$\mathcal{IC}(\mathcal{L}_{C_{ux}})$

The Fourier transform is given as follows.

$$\begin{aligned}
 \mathrm{Ft} : \mathrm{Per}_H(V) &\longrightarrow \mathrm{Per}_H(V^*) \\
 \mathcal{IC}(\mathbb{1}_{C_0}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_0^*}) = \mathcal{IC}(\mathbb{1}_{C_{ux}^t}) \\
 \mathcal{IC}(\mathbb{1}_{C_u}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_u^*}) = \mathcal{IC}(\mathbb{1}_{C_x^t}) \\
 \mathcal{IC}(\mathbb{1}_{C_x}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_x^*}) = \mathcal{IC}(\mathbb{1}_{C_u^t}) \\
 \mathcal{IC}(\mathbb{1}_{C_{ux}}) &\mapsto \mathcal{IC}(\mathbb{1}_{C_{ux}^*}) = \mathcal{IC}(\mathbb{1}_{C_0^t}) \\
 \mathcal{IC}(\mathcal{L}_{C_x}) &\mapsto \mathcal{IC}(\mathcal{L}_{C_0^*}) = \mathcal{IC}(\mathcal{L}_{C_{ux}^t}) \\
 \mathcal{IC}(\mathcal{L}_{C_{ux}}) &\mapsto \mathcal{IC}(\mathcal{L}_{C_u^*}) = \mathcal{IC}(\mathcal{L}_{C_x^t})
 \end{aligned}$$

12.2.4. Equivariant local systems on the regular conormal bundle. — The regular conormal bundle to the H -action on V decomposes into H -orbits

$$T_H^*(V)_{\mathrm{reg}} = T_{C_0}^*(V)_{\mathrm{reg}} \sqcup T_{C_u}^*(V)_{\mathrm{reg}} \sqcup T_{C_x}^*(V)_{\mathrm{reg}} \sqcup T_{C_{ux}}^*(V)_{\mathrm{reg}},$$

where each $T_C^*(V)_{\mathrm{reg}}$ is given below. In each case, the microlocal fundamental group A_C^{mic} is canonically identified with $Z(\widehat{G}) \cong \{\pm 1\}$.

C_0 : Regular conormal bundle:

$$T_{C_0}^*(V)_{\mathrm{reg}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{matrix} u' \neq 0 \\ x' \neq 0 \end{matrix} \right\} = C_0 \times C_0^*$$

Base point:

$$(x_0, \xi_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in T_{C_0}^*(V)_{\text{reg}}$$

Fundamental groups:

$$1 = A_{x_0} \longleftarrow A_{(x_0, \xi_0)} \xrightarrow{\text{id}} A_{\xi_0} = \{\pm 1\}$$

Local systems:

$$\boxed{\begin{array}{ccc} \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) : & \mathbb{1}_{\mathcal{O}_0} & \mathcal{L}_{\mathcal{O}_0} \\ \text{Rep}(A_{(x_0, \xi_0)}) : & + & - \end{array}}$$

Pullback along the bundle map $T_{C_0}^*(V)_{\text{sreg}} \rightarrow C_0$:

$$\begin{array}{ccc} \text{Loc}_H(C_0) & \rightarrow & \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_0} & \mapsto & \mathbb{1}_{\mathcal{O}_0} \\ & & \mathcal{L}_{\mathcal{O}_0} \end{array}$$

C_u : Regular conormal bundle:

$$T_{C_u}^*(V)_{\text{reg}} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x' & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \begin{array}{l} u \neq 0 \\ x' \neq 0 \end{array} \right\} = C_u \times C_u^*$$

Base point:

$$(x_1, \xi_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_{C_u}^*(V)_{\text{reg}}$$

Fundamental groups:

$$1 = A_{x_1} \longleftarrow A_{(x_1, \xi_1)} \xrightarrow{\text{id}} A_{\xi_1} = \{\pm 1\}$$

Local systems:

$$\boxed{\begin{array}{ccc} \text{Loc}_H(T_{C_u}^*(V)_{\text{sreg}}) : & \mathbb{1}_{\mathcal{O}_u} & \mathcal{L}_{\mathcal{O}_u} \\ \text{Rep}(A_{(x_1, \xi_1)}) : & + & - \end{array}}$$

Pullback along the bundle map $T_{C_u}^*(V)_{\text{sreg}} \rightarrow C_u$:

$$\begin{array}{ccc} \text{Loc}_H(C_u) & \rightarrow & \text{Loc}_H(T_{C_u}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_u} & \mapsto & \mathbb{1}_{\mathcal{O}_u} \\ & & \mathcal{L}_{\mathcal{O}_u} \end{array}$$

C_x : Regular conormal bundle:

$$T_{C_x}^*(V)_{\text{reg}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ u' & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u' & 0 \end{pmatrix} \mid \begin{array}{l} u' \neq 0 \\ x \neq 0 \end{array} \right\} = C_x \times C_x^*$$

Base point:

$$(x_2, \xi_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in T_{C_x}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\{\pm 1\} = A_{x_2} \xleftarrow{\text{id}} A_{(x_2, \xi_2)} \longrightarrow A_{\xi_2} = 1$$

Local systems:

$$\begin{array}{ccc} \text{Loc}_H(T_{C_x}^*(V)_{\text{sreg}}) : & \mathbb{1}_{\mathcal{O}_x} & \mathcal{L}_{\mathcal{O}_x} \\ \text{Rep}(A_{(x_2, \xi_2)}) : & + & - \end{array}$$

Pullback along the bundle map $T_{C_x}^*(V)_{\text{sreg}} \rightarrow C_x$:

$$\begin{array}{ccc} \text{Loc}_H(C_x) & \rightarrow & \text{Loc}_H(T_{C_x}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_x} & \mapsto & \mathbb{1}_{\mathcal{O}_x} \\ \mathcal{L}_{C_x} & \mapsto & \mathcal{L}_{\mathcal{O}_x} \end{array}$$

C_{ux} : Regular conormal bundle:

$$T_{C_{ux}}^*(V)_{\text{reg}} = \left\{ \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \begin{array}{l} u \neq 0 \\ x \neq 0 \end{array} \right\} = C_{ux} \times C_{ux}^*$$

Base point:

$$(x_3, \xi_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_{C_{ux}}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\{\pm 1\} = A_{x_3} \xleftarrow{\text{id}} A_{(x_3, \xi_3)} \longrightarrow A_{\xi_3} = 1$$

Local systems:

$$\begin{array}{ccc} \text{Loc}_H(T_{C_{ux}}^*(V)_{\text{sreg}}) : & \mathbb{1}_{\mathcal{O}_{ux}} & \mathcal{L}_{\mathcal{O}_{ux}} \\ \text{Rep}(A_{(x_3, \xi_3)}) : & + & - \end{array}$$

Pullback along the bundle map $T_{C_{ux}}^*(V)_{\text{sreg}} \rightarrow C_{ux}$:

$$\begin{array}{ccc} \text{Loc}_H(C_{ux}) & \rightarrow & \text{Loc}_H(T_{C_{ux}}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_{ux}} & \mapsto & \mathbb{1}_{\mathcal{O}_{ux}} \\ \mathcal{L}_{C_{ux}} & \mapsto & \mathcal{L}_{\mathcal{O}_{ux}} \end{array}$$

12.2.5. Vanishing cycles of perverse sheaves. — We summarize the values of the functor $\text{Ev} : \text{Per}_H(V) \rightarrow \text{Per}_H(T_H^*(V)_{\text{reg}})$ on simple objects in Table 12.2.1 We now explain how to make these calculations.

(a) To compute $\text{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_x})$ we look at the vanishing cycles

$$\text{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_x}) = \text{R}\Phi_{xx'}(\mathbb{1}_{\overline{C_x} \times C_0^*})|_{T_{C_0}^*(V)_{\text{reg}}}[1].$$

TABLE 12.2.1. ${}^p\mathrm{Ev} : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 12.

$\mathrm{Per}_H(V)$	$\xrightarrow{{}^p\mathrm{Ev}}$	$\mathrm{Per}_H(T_H^*(V)_{\mathrm{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_u})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_u})$
$\mathcal{IC}(\mathbb{1}_{C_x})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_x})$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_{ux}})$
$\mathcal{IC}(\mathcal{L}_{C_x})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u})$

TABLE 12.2.2. $\mathrm{Evs} : \mathrm{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathrm{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\mathrm{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 12.

\mathcal{P}	$\mathrm{Evs}_{C_0} \mathcal{P}$	$\mathrm{Evs}_{C_u} \mathcal{P}$	$\mathrm{Evs}_{C_x} \mathcal{P}$	$\mathrm{Evs}_{C_{ux}} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	+	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_u})$	0	+	0	0
$\mathcal{IC}(\mathbb{1}_{C_x})$	0	0	+	0
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	0	0	0	+
$\mathcal{IC}(\mathcal{L}_{C_x})$	−	0	−	0
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	0	−	0	−

The singular locus of xx' is $x = x' = 0$ but this is not part of $T_{C_0}^*(V)_{\mathrm{reg}}$, so $\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_x}) = 0$. All the non-diagonal entries in the first four rows work similarly.

- (b) To compute the last two rows of the tables above consider the map $\pi : \tilde{C}' \rightarrow C'$ which comes from taking a square root of x . Rather than directly applying Ev to $\mathcal{IC}(\mathcal{L}_{C'})$ we apply it to $\pi_!(\mathbb{1}_{\tilde{C}'})$ and exploit the fact that we have already computed Ev for the \mathcal{IC} sheaves of constant local systems. For example, in the case of $\mathrm{Ev}_{C_0}(\pi_!(\mathbb{1}_{\tilde{C}_x}))$ we will compute:

$$(\pi_! \mathrm{R}\Phi_{x^2x'}(\mathbb{1}_{\tilde{C}'} \boxtimes \mathbb{1}_{C_0^*}))_{T_{C_0}^*(V)_{\mathrm{reg}}}.$$

The singular locus is precisely $x = 0$ (noting that x' is not actually zero on the variety under consideration). The local structure of the singularity is that it is a smooth family (in the variable u') over the singularity of x^2x' over $\mathbb{A} \times \mathbb{G}_m$. It follows from Lemma 6.2.4 that the vanishing cycles on such a singularity is the sheaf supported on $x = 0$ associated to the non-trivial double cover $\sqrt{x'}$. Finally, by observing that the map π is an isomorphism on the support of $\mathrm{R}\Phi$, we conclude that:

$${}^p\mathrm{Ev}_{C_0}(\pi_!(\mathbb{1}_{\tilde{C}_x})) = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}).$$

The other entries are computed similarly.

12.2.6. Normalization of Ev and the twisting local system. — From Table 12.2.1 we see that the twisting local system \mathcal{T} is trivial in this case, so ${}^pN\text{Ev} = {}^p\text{Ev}$.

12.2.7. Fourier transform and vanishing cycles. — Compare the table below with the Fourier transform from Section 12.2.3 to confirm (141) in this example.

$\text{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{{}^p\text{Ev}}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$	$\xrightarrow{a^*}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda^*)_{\text{reg}})$	$\xleftarrow{\text{Ev}^*}$	$\text{Per}_{H_\lambda}(V_\lambda^*)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*})$	\mapleftarrow	$\mathcal{IC}(\mathbb{1}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_u})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_u})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_u^*})$	\mapleftarrow	$\mathcal{IC}(\mathbb{1}_{C_u^*})$
$\mathcal{IC}(\mathbb{1}_{C_x})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_x})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_x^*})$	\mapleftarrow	$\mathcal{IC}(\mathbb{1}_{C_x^*})$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_{ux}})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_{ux}^*})$	\mapleftarrow	$\mathcal{IC}(\mathbb{1}_{C_{ux}^*})$
$\mathcal{IC}(\mathcal{L}_{C_x})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0^*})$	\mapleftarrow	$\mathcal{IC}(\mathcal{L}_{C_0^*})$
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_u^*})$	\mapleftarrow	$\mathcal{IC}(\mathcal{L}_{C_u^*})$

12.2.8. Arthur sheaves. —

Arthur sheaf	pure L-packet sheaves	coronal perverse sheaves
\mathcal{A}_{C_0}	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_x})$
\mathcal{A}_{C_u}	$\mathcal{IC}(\mathbb{1}_{C_u}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_{ux}})$
\mathcal{A}_{C_x}	$\mathcal{IC}(\mathbb{1}_{C_x}) \oplus \mathcal{IC}(\mathcal{L}_{C_x})$	
$\mathcal{A}_{C_{ux}}$	$\mathcal{IC}(\mathbb{1}_{C_{ux}}) \oplus \mathcal{IC}(\mathcal{L}_{C_{ux}})$	

12.3. ABV-packets. —

12.3.1. Admissible representations versus equivariant perverse sheaves. —

$\text{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$	$\Pi_{\text{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$(\pi(\phi_0), 0)$
$\mathcal{IC}(\mathbb{1}_{C_u})$	$(\pi(\phi_1), 0)$
$\mathcal{IC}(\mathbb{1}_{C_x})$	$(\pi(\phi_2, +), 0)$
$\mathcal{IC}(\mathbb{1}_{C_{ux}})$	$(\pi(\phi_3, +), 0)$
$\mathcal{IC}(\mathcal{L}_{C_x})$	$(\pi(\phi_2, -), 1)$
$\mathcal{IC}(\mathcal{L}_{C_{ux}})$	$(\pi(\phi_3, -), 1)$

The Arthur parameters ψ_0 and ψ_3 correspond uniquely to the base points (x_0, ξ_0) and (x_3, ξ_3) from Section 12.2.4 under the map $Q_\lambda({}^L G) \rightarrow T_H^*(V)_{\text{reg}}$ given by Proposition 5.1.1.

12.3.2. ABV-packets. — Using Section 12.2.5 and the bijection of Section 12.3.1, we simply read off the ABV-packets:

$$\begin{aligned}
 \Pi_{\text{pure},\phi_0}^{\text{ABV}}(G/F) &= \{[\pi(\phi_0), 0], [\pi(\phi_2, -), 1]\} \\
 \Pi_{\text{pure},\phi_1}^{\text{ABV}}(G/F) &= \{[\pi(\phi_1), 0], [\pi(\phi_3, -), 1]\} \\
 \Pi_{\text{pure},\phi_2}^{\text{ABV}}(G/F) &= \{[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]\} \\
 \Pi_{\text{pure},\phi_3}^{\text{ABV}}(G/F) &= \{[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 1]\}.
 \end{aligned}$$

Using Section 12.1.4, we see

$$\begin{aligned}\Pi_{\text{pure},\psi_0}(G/F) &= \Pi_{\text{pure},\phi_0}^{\text{ABV}}(G/F) \\ \Pi_{\text{pure},\psi_3}(G/F) &= \Pi_{\text{pure},\phi_3}^{\text{ABV}}(G/F),\end{aligned}$$

thus verifying that Arthur packets are ABV-packets for admissible representations with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 12.

12.3.3. Stable distributions and endoscopy. — For $s \in Z(\widehat{G}) \cong \mu_2$, the virtual representations $\eta_{\phi,s}^{\text{NEV}}$ of (149) are given by

$$\begin{aligned}\eta_{\phi_0,1}^{\text{NEV}} &= [\pi(\phi_0), 0] + [\pi(\phi_2, -), 1] \\ \eta_{\phi_0,-1}^{\text{NEV}} &= [\pi(\phi_0), 0] - [\pi(\phi_2, -), 1] \\ \eta_{\phi_1,1}^{\text{NEV}} &= [\pi(\phi_1), 0] - [\pi(\phi_3, -), 1] \\ \eta_{\phi_1,-1}^{\text{NEV}} &= [\pi(\phi_1), 0] + [\pi(\phi_3, -), 1] \\ \eta_{\phi_2,1}^{\text{NEV}} &= [\pi(\phi_2, +), 0] + [\pi(\phi_2, -), 1] \\ \eta_{\phi_2,-1}^{\text{NEV}} &= [\pi(\phi_2, +), 0] - [\pi(\phi_2, -), 1] \\ \eta_{\phi_3,1}^{\text{NEV}} &= [\pi(\phi_3, +), 0] - [\pi(\phi_3, -), 1] \\ \eta_{\phi_3,-1}^{\text{NEV}} &= [\pi(\phi_3, +), 0] + [\pi(\phi_3, -), 1].\end{aligned}$$

Comparing with Section 12.1.6, this proves (148) in this example.

12.3.4. Kazhdan-Lusztig conjecture. — Using the bijection of Section 12.3.1 we compare the normalised geometric multiplicity matrix from Section 12.2.2 with the multiplicity matrix from Section 12.1.3:

$$m_{\text{rep}} = \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad m'_{\text{geo}} = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

Since ${}^t m_{\text{rep}} = m'_{\text{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 12.

12.3.5. Aubert duality and Fourier transform. — To verify (150), use Vogan's bijection from Section 12.3.1 to compare Aubert duality from Section 12.1.5 with the Fourier transform from Section 12.2.3.

12.3.6. Normalisation. — To verify (152), observe that the twisting characters χ_ψ of A_ψ from Section 12.1.5 are trivial, as are the local systems \mathcal{T}_ψ from Section 12.2.7.

12.3.7. ABV-packets that are not pure Arthur packets. — The closed stratum C_0 and the open stratum C_3 are of Arthur type, while C_1 and C_2 are not of Arthur type.

Thus, there are two ABV-packets that are not Arthur packets in this example:

$$\begin{aligned}\Pi_{\text{pure}, \phi_1}^{\text{ABV}}(G/F) &= \{[\pi(\phi_1), 0], [\pi(\phi_3, -), 1]\} \\ \Pi_{\text{pure}, \phi_2}^{\text{ABV}}(G/F) &= \{[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]\}.\end{aligned}$$

From these we extract four stable distributions,

$$\begin{aligned}\Theta_{\phi_1}^G &:= \text{trace } \pi(\phi_1) & \Theta_{\phi_1}^{G_1} &:= \text{trace } \pi(\phi_3, -) \\ \Theta_{\phi_2}^G &:= \text{trace } \pi(\phi_2, +) & \Theta_{\phi_2}^{G_1} &:= -\text{trace } \pi(\phi_2, -).\end{aligned}$$

We will see more interesting examples of ABV-packets that are not pure Arthur packets in Section 14.3.6.

12.4. Endoscopy and equivariant restriction of perverse sheaves. — The material from Section 8.4 is trivial in this case.

13. $\text{SO}(5)$ unipotent representations, singular parameter

In this example we encounter an L-packet of representations of $\text{SO}(5, F)$ that is lifted from an L-packet of representations of $\text{SO}(3, F) \times \text{SO}(3, F)$. In Section 13.4 will see how this lifting may be understood through equivariant restriction of perverse sheaves on Vogan varieties, and their vanishing cycles.

Let $G = \text{SO}(5)$. Then $H^1(F, G) \cong \mathbb{Z}/2\mathbb{Z}$. Let G_1 be the non-split form of G , as in Section 12. We consider admissible representations of $G(F)$ and $G_1(F)$ with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ given by

$$\lambda(w) = \begin{pmatrix} |w|^{1/2} & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 \\ 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & |w|^{-1/2} \end{pmatrix}.$$

13.1. Arthur packets. —

13.1.1. Parameters. — There are three Langlands parameters with infinitesimal parameter λ , up to $Z_{\widehat{G}}(\lambda)$ -conjugacy, each of Arthur type. Set

$$\begin{aligned}\psi_0(w, x, y) &:= \nu_2(y) \oplus \nu_2(y), \\ \psi_2(w, x, y) &:= \nu_2(x) \oplus \nu_2(y), \\ \psi_3(w, x, y) &:= \nu_2(x) \oplus \nu_2(x),\end{aligned}$$

and observe that ψ_0 and ψ_3 are Aubert dual while ψ_2 is self dual. Let ϕ_0, ϕ_2 and ϕ_3 be the associated Langlands parameters; thus,

$$\begin{aligned}\phi_0(w, x) &:= \nu_2(d_w) \oplus \nu_2(d_w), \\ \phi_2(w, x) &:= \nu_2(x) \oplus \nu_2(d_w), \\ \phi_3(w, x) &:= \nu_2(x) \oplus \nu_2(x).\end{aligned}$$

13.1.2. L-packets. — The pure component groups for these three Langlands parameters are

$$A_{\phi_0} = 1, \quad A_{\phi_2} \cong \{\pm 1\}, \quad A_{\phi_3} \cong \{\pm 1\}.$$

Thus, there are five admissible representations of two pure forms of $\mathrm{SO}(5)$ in play in this example. When arranged into L-packets, these representations are:

$$\begin{aligned} \Pi_{\phi_0}(G(F)) &= \{\pi(\phi_0)\}, & \Pi_{\phi_0}(G_1(F)) &= \emptyset, \\ \Pi_{\phi_2}(G(F)) &= \{\pi(\phi_2, +)\}, & \Pi_{\phi_2}(G_1(F)) &= \{\pi(\phi_2, -)\}, \\ \Pi_{\phi_3}(G(F)) &= \{\pi(\phi_3, +), \pi(\phi_3, -)\}, & \Pi_{\phi_3}(G_1(F)) &= \emptyset. \end{aligned}$$

Of these five admissible representations, only $\pi(\phi_3, +)$ and $\pi(\phi_3, -)$ are tempered; these two representations are denoted by τ_2 and τ_1 , respectively, in [Mat10]. The admissible representation $\pi(\phi_0)$ is denoted by $L(\nu^{1/2}\zeta, \nu^{1/2}\zeta, 1)$ with $\zeta = 1$ in [Mat10] and $\pi(\phi_2, +)$ is denoted by $L(\nu^{1/2}\zeta, \zeta \mathrm{St}_{\mathrm{SO}(3)})$ with $\zeta = 1$.

13.1.3. Multiplicities in standard modules. — The standard module $M(\phi_0)$ is induced from the Levi subgroup $\mathrm{GL}(1, F) \times \mathrm{GL}(1, F) \times \mathrm{SO}(1, F)$ of $\mathrm{SO}(5, F)$; it is denoted by $\nu^{1/2}\zeta \times \nu^{1/2}\zeta \times 1$ with $\zeta = 1$ in [Mat10]. The standard module $M(\phi_2, +)$ is induced from the Levi subgroup $\mathrm{GL}(1, F) \times \mathrm{SO}(3, F)$ of $\mathrm{SO}(5, F)$; it is denoted by $\nu^{1/2}\zeta \times \zeta \mathrm{St}_{\mathrm{SO}(3)}$ with $\zeta = 1$ in [Mat10]. The standard module $M(\phi_3, \pm)$ coincides with the tempered representation $\pi(\phi_3, \pm)$. The 4×4 block in the following table may be deduced from [Mat10, Proposition 3.3].

	$\pi(\phi_0)$	$\pi(\phi_2, +)$	$\pi(\phi_3, +)$	$\pi(\phi_3, -)$	$\pi(\phi_2, -)$
$M(\phi_0)$	1	1	1	1	0
$M(\phi_2, +)$	0	1	1	0	0
$M(\phi_3, +)$	0	0	1	0	0
$M(\phi_3, -)$	0	0	0	1	0
$M(\phi_2, -)$	0	0	0	0	1

13.1.4. Arthur packets. — The component groups for the Arthur parameters in this example are

$$A_{\psi_0} \cong \{\pm 1\}, \quad A_{\psi_2} \cong \{\pm 1\} \times \{\pm 1\}, \quad A_{\psi_3} \cong \{\pm 1\}.$$

We may represent elements of each A_{ψ} as cosets with representatives taken from $\widehat{T}[2]$. The map $\widehat{T}[2] \rightarrow A_{\psi_0}$ is $s \mapsto s_1 s_2$; the map $\widehat{T}[2] \rightarrow A_{\psi_2}$ is $s \mapsto (s_1, s_2)$; the map $\widehat{T}[2] \rightarrow A_{\psi_3}$ is $s \mapsto s_1 s_2$.

The Arthur packets for Arthur parameters with infinitesimal parameter λ are:

$$\begin{aligned} \Pi_{\psi_0}(G(F)) &= \{\pi(\phi_0), \pi(\phi_2, +)\}, & \Pi_{\psi_0}(G_1(F)) &= \emptyset, \\ \Pi_{\psi_2}(G(F)) &= \{\pi(\phi_2, +), \pi(\phi_3, -)\}, & \Pi_{\psi_2}(G_1(F)) &= \{\pi(\phi_2, -)\}, \\ \Pi_{\psi_3}(G(F)) &= \{\pi(\phi_3, +), \pi(\phi_3, -)\}, & \Pi_{\psi_3}(G_1(F)) &= \emptyset. \end{aligned}$$

We arrange these representations into pure Arthur packets in the table below.

pure Arthur packets	pure L-packet representations	coronal representations
$\Pi_{\mathrm{pure}, \psi_0}(G/F)$	$[\pi(\phi_0), 0]$	$[\pi(\phi_2, +), 0]$
$\Pi_{\mathrm{pure}, \psi_2}(G/F)$	$[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]$	$[\pi(\phi_3, -), 0]$
$\Pi_{\mathrm{pure}, \psi_3}(G/F)$	$[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 0]$	

13.1.5. Aubert duality. — Aubert duality for $G(F)$ and $G_1(F)$ are given by the following table.

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_3, +)$
$\pi(\phi_2, +)$	$\pi(\phi_3, -)$
$\pi(\phi_3, +)$	$\pi(\phi_0, +)$
$\pi(\phi_3, -)$	$\pi(\phi_2, +)$
$\pi(\phi_2, -)$	$\pi(\phi_2, -)$

The twisting characters χ_{ψ_0} and χ_{ψ_3} are trivial. The twisting character χ_{ψ_2} of A_{ψ_2} is $\chi_{\psi_2}(s) = s_1 s_2 = \det(s)$. This is the first non-trivial twisting character to appear in this article.

13.1.6. Stable distributions and endoscopic transfer. — The stable distributions

$$\Theta_{\psi}^G = \sum_{\pi \in \Pi_{\psi}(G(F))} \langle s_{\psi}, \pi \rangle_{\psi} \text{trace } \pi$$

attached the Arthur parameters are:

$$\begin{aligned} \Theta_{\psi_0}^G &= \text{trace } \pi(\phi_0) + \text{trace } \pi(\phi_2, +) \\ \Theta_{\psi_2}^G &= \text{trace } \pi(\phi_2, +) - \text{trace } \pi(\phi_3, -) \\ \Theta_{\psi_3}^G &= \text{trace } \pi(\phi_3, +) + \text{trace } \pi(\phi_3, +). \end{aligned}$$

The distributions

$$\Theta_{\psi, s}^G = \sum_{\pi \in \Pi_{\psi}(G_{\delta}(F))} \langle ss_{\psi}, \pi \rangle_{\psi} \text{trace } \pi,$$

where $s \in Z_{\hat{G}}(\psi)$, are obtained by transfer from endoscopic groups. The coefficients above are given by

$$\langle ss_{\psi}, \pi \rangle_{\psi} = \langle s_{\psi}, \pi \rangle_{\psi} \langle s, \pi \rangle_{\psi}$$

where $\langle s_{\psi}, \pi \rangle_{\psi}$ appear above while $\langle s, \pi \rangle_{\psi}$ is given by the tables below.

We now give $\langle \cdot, \pi \rangle_{\psi}$ as a character of A_{ψ} , using the isomorphisms from Section 13.1.4.

π	$\langle \cdot, \pi \rangle_{\psi_0}$	$\langle \cdot, \pi \rangle_{\psi_2}$	$\langle \cdot, \pi \rangle_{\psi_3}$
$\pi(\phi_0)$	+	0	0
$\pi(\phi_2, +)$	—	++	0
$\pi(\phi_3, +)$	0	0	+
$\pi(\phi_3, -)$	0	--	—

The values of this character on the image of $s = \text{diag}(s_1, s_2, s_2^{-1}, s_1^{-1}) \in \hat{T}[2]$ in A_{ψ} are given by.

π	$\langle s, \pi \rangle_{\psi_0}$	$\langle s, \pi \rangle_{\psi_2}$	$\langle s, \pi \rangle_{\psi_3}$
$\pi(\phi_0)$	1	0	0
$\pi(\phi_2, +)$	$s_1 s_2$	1	0
$\pi(\phi_3, +)$	0	0	1
$\pi(\phi_3, -)$	0	$s_1 s_2$	$s_1 s_2$

For instance, if we take $s = \text{diag}(1, -1, -1, 1) \in \widehat{T}[2]$ then

$$\begin{aligned}\Theta_{\psi_0, s}^G &= \text{trace } \pi(\phi_0) - \text{trace } \pi(\phi_2, +), \\ \Theta_{\psi_2, s}^G &= \text{trace } \pi(\phi_2, +) + \text{trace } \pi(\phi_3, -), \\ \Theta_{\psi_3, s}^G &= \text{trace } \pi(\phi_3, +) - \text{trace } \pi(\phi_3, -).\end{aligned}$$

In this case, the elliptic endoscopic group G' for G determined by s is the group $G' = \text{SO}(3) \times \text{SO}(3)$, split over F .

13.2. Vanishing cycles of perverse sheaves. — We now assemble the geometric tools needed to calculate the Arthur packets, stable distributions and endoscopic transfer described above.

13.2.1. Vogan variety and its conormal bundle. —

$$V = \left\{ \left(\begin{array}{c|cc} & z & x \\ & y & -z \\ \hline & & \end{array} \right) \mid x, y, z \right\} \quad V^* = \left\{ \left(\begin{array}{cc|c} & & \\ \hline z' & y' & \\ x' & -z' & \end{array} \right) \mid x', y', z' \right\}$$

so

$$T^*(V) = \left\{ \left(\begin{array}{c|cc} & z & x \\ & y & -z \\ \hline z' & y' & \\ x' & -z' & \end{array} \right) \mid \begin{array}{c} x, y, z \\ x', y', z' \end{array} \right\} \subset \text{sp}(4)$$

The cotangent bundle $T^*(V)$ comes equipped with an action of

$$H := Z_{\widehat{G}}(\lambda) = \left\{ \left(\begin{array}{cc|cc} a_1 & b_1 & & \\ c_1 & d_1 & & \\ \hline & & a_2 & b_2 \\ & & c_2 & d_2 \end{array} \right) \in \text{Sp}(4) \right\}.$$

We will write $h_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $h_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then $h_2 = h_1 \det h_1^{-1}$, by the choice of symplectic form J in Section 12. In particular, $H \cong \text{GL}(2)$. The action of H on V , V^* and $T^*(V)$ is given by

$$\begin{aligned}h \cdot \begin{pmatrix} z & x \\ y & -z \end{pmatrix} &= h_1 \begin{pmatrix} z & x \\ y & -z \end{pmatrix} h_2^{-1} \\ h \cdot \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} &= h_2 \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} h_1^{-1}.\end{aligned}$$

The H -invariant function $(\cdot | \cdot) : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ is the quadratic form

$$\left(\begin{array}{c|cc} & z & x \\ & y & -z \\ \hline z' & y' & \\ x' & -z' & \end{array} \right) \mapsto xx' + yy' + 2zz'.$$

$$\left(\begin{array}{c|cc} & z & x \\ \hline & y & -z \\ \hline z' & y' & \\ x' & -z' & \end{array} \right) \mapsto \left(\begin{array}{cc|cc} zz' + xx' & zy' - xz' & & \\ yz' - zx' & yy' + zz' & & \\ \hline & & zz' + yy' & xz' - zy' \\ & & zx' - yz' & xx' + zz' \end{array} \right)$$
$$T_H^*(V) = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \mid \begin{array}{l} zz' + xx' = 0 \\ zz' + yy' = 0 \\ zx' - yz' = 0 \\ xz' - zy' = 0 \end{array} \right\}$$
$$C_0 = \left\{ \left(\begin{array}{c|cc} & 0 & 0 \\ \hline & 0 & 0 \end{array} \right) \right\}$$

$$C_2 = \left\{ \left(\begin{array}{c|cc} & z & y \\ \hline & x & -z \end{array} \right) \mid \begin{array}{l} xy + z^2 = 0 \\ (x, y, z) \neq (0, 0, 0) \end{array} \right\}$$

$$C_3 = \left\{ \left(\begin{array}{c|cc} & z & y \\ \hline & x & -z \end{array} \right) \mid xy + z^2 \neq 0 \right\}$$

$$C_0^* = \left\{ \left(\frac{\begin{array}{c|c} & \\ \hline z' & y' \\ x' & -z' \end{array}}{\quad} \mid x'y' + z'^2 \neq 0 \right) \right\}$$

$$C_2^* = \left\{ \left(\begin{array}{c|c} & \\ \hline z' & y' \\ x' & -z' \end{array} \right) \mid \begin{array}{l} x'y' + z'^2 = 0 \\ (x', y', z') \neq (0, 0, 0) \end{array} \right\}$$

$$C_3^* = \left\{ \left(\begin{array}{cc|c} & & \\ \hline 0 & 0 & \\ 0 & 0 & \end{array} \right) \right\}$$

The following diagram gives the dimensions of the H -orbits C and the dual orbits C^* , we well as the eccentricities $e_C = \dim C + \dim C^* - \dim V$ and closure relations

for the H -orbits C in V :

$$\begin{array}{ccccccc}
 C_3 & \dim C_3 = 3 & e_{C_3} = 0 & C_0^* \\
 \uparrow & & & \uparrow \\
 C_2 & \dim C_2 = 2 & e_{C_2} = 1 & C_2^* \\
 \uparrow & & & \uparrow \\
 C_0 & \dim C_0 = 0 & e_{C_0} = 0 & C_3^*
 \end{array}$$

Orbit C_2 is the first stratum in this article with non-zero eccentricity.

13.2.2. Equivariant local systems. —

C_0 : Regular conormal bundle above the closed H -orbit $C_0 \subset V$:

$$T_{C_0}^*(V)_{\text{reg}} = \left\{ \left(\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \mid x'y' - z'^2 \neq 0 \right) \right\}$$

Base point:

$$(x_0, \xi_0) = \left(\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 1 & 0 & & \end{array} \right) \right) \in T_{C_0}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c}
 \widehat{T}[2] \\
 \downarrow s \mapsto s_1 s_2 \\
 1 = A_{x_0} \longleftarrow A_{(x_0, \xi_0)} \xrightarrow{\text{id}} A_{\xi_0} \cong \{\pm 1\}
 \end{array}$$

Local systems:

$$\boxed{
 \begin{array}{ccc}
 \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) : & \mathbb{1}_{\mathcal{O}_0} & \mathcal{L}_{\mathcal{O}_0} \\
 \text{Rep}(A_{(x_0, \xi_0)}) : & + & -
 \end{array}
 }$$

Pullback along the bundle map $T_{C_0}^*(V)_{\text{sreg}} \rightarrow C_0$:

$$\begin{array}{ccc}
 \text{Loc}_H(C_0) & \rightarrow & \text{Loc}_H(T_{C_0}^*(V)_{\text{sreg}}) \\
 \mathbb{1}_{C_0} & \mapsto & \mathbb{1}_{\mathcal{O}_0} \\
 & & \mathcal{L}_{\mathcal{O}_0}
 \end{array}$$

C_2 : Regular conormal bundle above $C_2 \subset V$:

$$T_{C_2}^*(V)_{\text{reg}} = \left\{ \left(\left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \mid \begin{array}{l} xy + z^2 = 0 \\ [x : y : z] = [y' : x' : z'] \end{array} \right) \right\}$$

Base point:

$$(x_2, \xi_2) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array} \right) \in T_{C_2}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ \cong \downarrow s \mapsto (s_1, s_2) \\ \{\pm 1\} = A_{x_2} \xleftarrow{s_1 \mapsto (s_1, s_2)} A_{(x_2, \xi_2)} \xrightarrow{(s_1, s_2) \mapsto s_2} A_{\xi_2} = \{\pm 1\} \end{array}$$

Local systems:

$\text{Loc}_H(T_{C_2}^*(V)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}_2}$	$\mathcal{L}_{\mathcal{O}_2}$	$\mathcal{F}_{\mathcal{O}_2}$	$\mathcal{E}_{\mathcal{O}_2}$
$\text{Rep}(A_{(x_2, \xi_2)}) :$	$++$	$--$	$-+$	$+-$

Pullback along the bundle map $T_{C_2}^*(V)_{\text{sreg}} \rightarrow C_2$:

$\text{Loc}_H(C_2)$	\rightarrow	$\text{Loc}_H(T_{C_2}^*(V)_{\text{sreg}})$
$\mathbb{1}_{C_2}$	\mapsto	$\mathbb{1}_{\mathcal{O}_2}$
		$\mathcal{L}_{\mathcal{O}_2}$
\mathcal{F}_{C_2}	\mapsto	$\mathcal{F}_{\mathcal{O}_2}$
		$\mathcal{E}_{\mathcal{O}_2}$

C_3 : Regular conormal bundle above $C_3 \subset V$:

$$T_{C_3}^*(V)_{\text{reg}} = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \mid xy + z^2 \neq 0 \right\}$$

Base point:

$$(x_3, \xi_3) = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \in T_{C_3}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ s_1 s_2 \mapsto s \downarrow \\ \{\pm 1\} \cong A_{x_3} \xleftarrow{\text{id}} A_{(x_3, \xi_3)} \longrightarrow A_{\xi_3} = 1 \end{array}$$

Local systems:

$\text{Loc}_H(T_{C_3}^*(V)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}_3}$	$\mathcal{L}_{\mathcal{O}_3}$
$\text{Rep}(A_{(x_3, \xi_3)}) :$	$+$	$-$

Pullback along the bundle map $T_{C_3}^*(V)_{\text{sreg}} \rightarrow C_3$:

$$\begin{array}{ccc} \text{Loc}_H(C_3) & \rightarrow & \text{Loc}_H(T_{C_3}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_3} & \mapsto & \mathbb{1}_{\mathcal{O}_3} \\ \mathcal{L}_{C_3} & \mapsto & \mathcal{L}_{\mathcal{O}_3} \end{array}$$

13.2.3. Equivariant perverse sheaves. — The following table is helpful to understand the simple objects in $\text{Per}_H(V)$.

\mathcal{P}	$\mathcal{P} _{C_0}$	$\mathcal{P} _{C_2}$	$\mathcal{P} _{C_3}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_2}[2]$	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_2}[3]$	$\mathbb{1}_{C_3}[3]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mathbb{1}_{C_0}[1]$	0	$\mathcal{L}_{C_3}[3]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	$\mathcal{F}_{C_2}[2]$	0

We now explain how we made these calculations:

- (a) The first and third row of these tables are computed using the observation that when \overline{C} is smooth, the sheaf $\mathbb{1}_{\overline{C}}[\dim(C)]$ is perverse.
- (b) For the second row, the relevant cover $\tilde{C}_2^{(1)}$ is the blowup of the nilcone at the origin. We readily find using the decomposition theorem for semi-small maps that

$$\pi_2^{(1)*}(\mathbb{1}_{\tilde{C}_2^{(1)}}[2]) = \mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathbb{1}_{C_0}).$$

Proper base change and exactness allows us to deduce the fibres of $\mathcal{IC}(\mathbb{1}_{C_2})$ using what we already know about $\mathcal{IC}(\mathbb{1}_{C_0})$.

- (c) For the fourth row, we consider the double cover which arises from taking the square root of the determinant. Although this is singular at the origin, blowing up resolves this singularity. An alternate model for this blowup is the cover:

$$\tilde{C}_3 = \{([a : b], (x, y, z)) \in \mathbb{P}^1 \times V \mid a^2x + 2abz - b^2y = 0\}$$

with the obvious map π_3 to $V = \overline{C}_3$. The decomposition theorem for semi-small maps yields

$$\pi_{3!}(\mathbb{1}_{\tilde{C}_3}[3]) = \mathcal{IC}(\mathcal{L}_{C_3}) \oplus \mathcal{IC}(\mathbb{1}_{C_3}).$$

Proper base change and exactness again allows us to deduce the entries for $\mathcal{IC}(\mathcal{L}_{C_3})$, the key observation being that the map is 2 : 1 over C_3 , an isomorphism over C_2 and the fibre over C_0 is \mathbb{P}^1 .

- (d) Finally the fifth row is computed by considering the “symmetric squares” cover of the nilcone given by $\pi_2^{(2)} : (a, b) \mapsto (a^2, -b^2, ab)$. This map is 2 : 1 over C_2 and an isomorphism over C_0 ; we readily confirm using the decomposition theorem for finite maps that

$$\pi_2^{(2)*}(\mathbb{1}_{\tilde{C}}[2]) = \mathcal{IC}(\mathcal{F}_{C_2}) \oplus \mathcal{IC}(\mathbb{1}_{C_2}).$$

Computing the entries in the table is now immediate using our understanding of the fibres and what we already know about $\mathcal{IC}(\mathbb{1}_{C_2})$.

TABLE 13.2.1. ${}^p\text{Ev} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 13.

$\text{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{{}^p\text{Ev}}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_2})$

From this, we easily find the normalised geometric multiplicity matrix.

	$\mathbb{1}_{C_0}^\sharp$	$\mathbb{1}_{C_2}^\sharp$	$\mathbb{1}_{C_3}^\sharp$	$\mathcal{L}_{C_3}^\sharp$	$\mathcal{F}_{C_2}^\sharp$
$\mathbb{1}_{C_0}^\sharp$	1	0	0	0	0
$\mathbb{1}_{C_2}^\sharp$	1	1	0	0	0
$\mathbb{1}_{C_3}^\sharp$	1	1	1	0	0
$\mathcal{L}_{C_3}^\sharp$	1	0	0	1	0
$\mathcal{F}_{C_2}^\sharp$	0	0	0	0	1

13.2.4. Cuspidal support decomposition and Fourier transform. — Cuspidal Levi subgroups for \widehat{G} were given in Section 12.2.3, so the cuspidal support decomposition of $\text{Per}_{H_\lambda}(V_\lambda)$ takes the same form here:

$$\text{Per}_{H_\lambda}(V_\lambda) = \text{Per}_{H_\lambda}(V_\lambda)_{\widehat{T}} \oplus \text{Per}_{H_\lambda}(V_\lambda)_{\widehat{M}}.$$

However, simple objects in these two subcategories are quite different in this case:

$\text{Per}_H(V)_{\widehat{T}}$	$\text{Per}_H(V)_{\widehat{M}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathcal{IC}(\mathcal{F}_{C_2})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	
$\mathcal{IC}(\mathbb{1}_{C_3})$	
$\mathcal{IC}(\mathcal{L}_{C_3})$	

Here we record the functor $\text{Ft} : \text{Per}_H(V) \rightarrow \text{Per}_H(V^*)$ on simple objects, and the composition of that functor with the equivalence $\text{Per}_H(V^*) \rightarrow \text{Per}_H(V)$ described in Section 8.3.4; the composition is the functor ${}^\wedge : \text{Per}_H(V) \rightarrow \text{Per}_H(V)$ also discussed in Section 8.3.4.

$\text{Per}_H(V)$	$\xrightarrow{\text{Ft}}$	$\text{Per}_H(V^*)$	\longrightarrow	$\text{Per}_H(V)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_0^*})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_3})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{C_0^*})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{C_3})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_3^*})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_0})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_2^*})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_2})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_2^*})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_2})$

Note that the Fourier transform respects the cuspidal support decomposition.

TABLE 13.2.2. $\mathbf{Evs} : \mathbf{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 13.

\mathcal{P}	$\mathbf{Evs}_{C_0} \mathcal{P}$	$\mathbf{Evs}_{C_2} \mathcal{P}$	$\mathbf{Evs}_{C_3} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	+	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	—	—	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	0	0	+
$\mathcal{IC}(\mathcal{L}_{C_3})$	0	++	—
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	+—	0

13.2.5. Vanishing cycles. — Table 13.2.1 presents the calculation of \mathbf{Ev} on simple objects. We explain all the calculation here.

All the entries in row 1 and column 3 are a direct consequence of Lemma 6.2.1. We show how to compute column 2.

(a) We compute ${}^p\mathbf{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_2})$. By Lemma 6.6.1,

$${}^p\mathbf{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_2}) = \mathbf{R}\Phi_{xx'+yy'+2zz'}(\mathbb{1}_{C_2 \times C_2^*})|_{T_{C_2}^*(V)_{\text{reg}}}[3].$$

Consider the affine open subvariety $U_{xy'}$ of $C_2 \times C_2^*$ given by the equations $xy' \neq 0$; then

$$U = \left\{ \left(\begin{array}{cc|cc} & & z & x \\ & & y & -z \\ \hline z' & y' & & \\ x' & -z' & & \end{array} \right) \mid \begin{array}{l} xy + z^2 = 0 \\ x'y' + z'^2 = 0 \\ xy' \neq 0 \end{array} \right\}$$

Then $U_{xy'}$ has coordinate ring

$$\mathbb{C}[x, y, z, x', y', z']_{xy'} / (xy + z^2, x'y' + z'^2) \cong \mathbb{C}[x, z, y', z']_{xy'}.$$

Write $f_{xy'} : U_{xy'} \rightarrow \mathbb{A}^1$ for the restriction of f to $U_{xy'}$. Then $f_{xy'}$ is given on coordinate rings by

$$t \mapsto -x \frac{z'^2}{y'} - y' \frac{z^2}{x} + 2zz' = -\frac{1}{xy'}(zy' - xz')^2.$$

Using Section 6.2, especially Lemma 6.2.4, it follows that $\mathbf{R}\Phi_{f_{xy'}}(\mathbb{1}_{U_{xy'}})$ is the sheaf on

$$f_{xy'}^{-1}(0) = \text{Spec}(\mathbb{C}[x, z, y', z']_{xy'} / (zy' - xz'))$$

associated to the double cover $\text{Spec}(\mathbb{C}[x, z, y', z', s]_{xy'} / (zy' - xz', s^2 + xy'))$.

With reference to Section 13.2.1, the restriction of $[\cdot, \cdot] : T^*(V_\lambda) \rightarrow \mathfrak{h}_\lambda$ to $U_{xy'}$ is given by

$$\left(\begin{array}{cc|cc} & & z & x \\ & & * & -z \\ \hline z' & y' & & \\ * & -z' & & \end{array} \right) \mapsto \frac{zy' - xz'}{xy'} \left(\begin{array}{cc|cc} xz' & 1 & & \\ -zz' & -zy' & & \\ \hline & & -zy' & -1 \\ & & zz' & xz' \end{array} \right)$$

and so the equation $zy' - xz' = 0$ implies this product is zero. It follows that the support of $R\Phi_{f_{xy'}}(\mathbb{1}_{U_{xy'}})$ is contained in $T_{C_2}^*(V)_{\text{reg}} \cap U_{xy'}$. We may find the entire sheaf $R\Phi_{f|_{C_2 \times C_2^*}}(\mathbb{1}_{C_2 \times C_2^*})$ by also considering the affine open $U_{x'y}$ cut out by the condition $x'y \neq 0$. In this case we symmetrically obtain the local system on $zx' - yz' = 0$ associated to the double cover given by $s^2 + x'y = 0$. We may then glue these local systems together to see that ${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_2})[-3]$ is the non-trivial local system on $T_{C_2}^*(V)_{\text{reg}}$ trivialized by the double cover

$$\text{Spec}(\mathbb{C}[a^2, b^2, ab, a'^2, b'^2, a'b', ab', aa', ba', bb']) \rightarrow T_{C_2}^*(V)_{\text{reg}}$$

(subvariety of \mathbb{A}^{10} with all of the implied relations) given on the coordinate rings by

$$x \mapsto a^2, y \mapsto -b^2, z \mapsto ab, x' \mapsto a'^2, y' \mapsto -b'^2, z' \mapsto a'b'.$$

This is the diagonal quotient of the product of the symmetric squares.

It remains to see how to describe this local system in the language of Section 13.2.2. To do this, observe that $(x_2, \xi_2) \in U_{xy'}$ and return to the description of the local system $R\Phi_{f_{xy'}}(\mathbb{1}_{U_{xy'}})$ given above and observe that the covering group of $\text{Spec}(\mathbb{C}[x, z, y', z', s]_{xy'}/(zy' - xz', s^2 + xy'))$ over $f_{xy'}^{-1}(0)$ is $s \mapsto \pm s$, or equivalently, $(x, y') \mapsto (\pm x, \pm y')$. It follows that ${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_2}) = \mathcal{L}_{O_2}[3]$ where \mathcal{L}_{O_2} is the local system which corresponds to the character $(-)$ of $A_{(x_2, \xi_2)}$. Note that ${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_2}) \neq \mathbb{1}_{O_2}[3]$.

- (b) Here we compute ${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathcal{F}_{C_2})$ using the affine covering $U_{xy'} \cup U_{x'y} = C_2 \times C_2^*$ from Section 13.2.5, (a). Observe that

$${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathcal{F}_{C_2}) = R\Phi_{f|_{C_2 \times C_2^*}}(\mathcal{F}_{C_2} \boxtimes \mathbb{1}_{C_2^*})[3]$$

since $\mathcal{IC}(\mathcal{F}_{C_2}) = \mathcal{F}_{C_2}^\natural[2]$. Consider the cover $\tilde{U}_{xy'} = \text{Spec}(\mathbb{C}[a, b, y', z']_{ay'})$ of $U_{xy'} = \text{Spec}(\mathbb{C}[x, z, y', z']_{xy'})$ given on coordinate rings by $x \mapsto a^2, z \mapsto ab$ and the cover $\tilde{U}_{x'y} = \text{Spec}(\mathbb{C}[a, b, x', z']_{bx'})$ of $U_{x'y} = \text{Spec}(\mathbb{C}[y, z, x', z']_{x'y})$ given on coordinate rings by $y \mapsto -b^2, z \mapsto ab$. Together, this defines a double cover $\pi : \tilde{C}_2 \times C_2^* \rightarrow C_2 \times C_2^*$; the local system $\mathcal{F}_{C_2} \boxtimes \mathbb{1}_{C_2^*}$ is associated to the quadratic character of the covering group $(a, b) \mapsto (\pm a, \pm b)$. Then

$$(f \circ \pi)|_{U_{xy'}} = -\frac{1}{a^2 y'}(aby' - a^2 z')^2 = -\frac{1}{y'}(by' - az')^2.$$

By Lemma 6.2.4,

$$R\Phi_{-\frac{1}{y'}(by' - az')^2}(\mathbb{1}_{\tilde{U}_{xy'}})$$

is the local system associated to the cover $s^2 + y' = 0$ over the zero locus of $by' - az'$ and the quadratic character of the covering group $s \mapsto \pm s$, or equivalently, $(a, b) \mapsto (\pm a, \pm b)$. By proper base change,

$$(\pi|_{\tilde{U}_{xy'}})_* R\Phi_{-\frac{1}{y'}(by' - az')^2}(\mathbb{1}_{\tilde{U}_{xy'}}) = R\Phi_{f|_{U_{xy'}}}(\pi|_{\tilde{U}_{xy'}})_* \mathbb{1}_{\tilde{U}_{xy'}}$$

This shows that $\text{Ev}_{C_2} \mathcal{IC}(\mathcal{F}_2)[-3]$ is not the pullback \mathcal{F}_{O_2} of \mathcal{F}_2 to $T_{C_2}^*(V)_{\text{reg}}$, but rather the twist of that by the local system above, which is \mathcal{E}_{O_2} .

- (c) Recall the cover $\pi_3 : \widetilde{C}_3 \rightarrow \overline{C}_3$ from Section 13.2.3 (c). To compute $\mathrm{Ev}_{C_2} \mathcal{IC}(\mathcal{L}_{C_3})$ we consider

$$\mathrm{R}\Phi_{f \circ (\pi_3 \times \mathrm{id})}(\mathbb{I}_{\widetilde{C}_3 \times C_2^*}).$$

Localize $\overline{C}_3 = V$ at $xy' \neq 0$ to define $V_{xy'}$. Localize the fibre $(\pi_3 \times \mathrm{id})^{-1}(V_{xy'})$ away from the exceptional divisor to define $\tilde{V}_{xy'}$ with coordinate ring

$$\mathbb{C}[d, x, y, z, z', y', z']_{xy'} / (xy + z^2 - d^2, x'y' + z'^2) \cong \mathbb{C}[d, x, z, y', z']_{xy'}$$

and the cover $\pi_{xy'}^3 : \tilde{V}_{xy'} \rightarrow V_{xy'}$. Then

$$f \circ \pi_{xy'}^3 = -\frac{1}{xy'}(xz' - zy' - dy')(xz' - zy' + dy')$$

The functions $(xz' - zy' - dy')$ and $(xz' - zy' + dy')$ being smooth on $\tilde{V}_{xy'}$, and xy' being non-zero, it follows from Corollary 6.2.6 that

$$\mathrm{R}\Phi_{f \circ \pi_{xy'}^3}(\mathbb{I}_{\tilde{V}_{xy'}})$$

is the constant sheaf on the intersection of their zero loci, which is precisely $\mathcal{O}_2 \cap \tilde{V}_{xy'}$. Using proper base change, and noting that proper pushforward is an isomorphism on the regular conormal vectors, it follows that

$$\mathrm{Ev}_{C_2} \mathcal{IC}(\mathcal{L}_{C_3}) = \mathbb{I}_{\mathcal{O}_2}[3].$$

This concludes the computation of column 2 in Table 13.2.1.

We have now explained every entry in Table 13.2.1 except for the following case.

- (d) To compute $\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{I}_{C_2})$, consider

$$\mathrm{R}\Phi_{xx' + yy' + 2zz'}(\mathbb{I}_{\widetilde{C}_2^{(1)} \times C_0^*})$$

with

$$\widetilde{C}_2^{(1)} = \{([a : b], (x, y, z)) \in \mathbb{P}^1 \times \overline{C}_2 \mid -ax + bz = 0, az + by = 0\}.$$

The Jacobian condition for smoothness tells us that this is singular precisely when $x = y = z = 0$ and

$$-a^2x' + 2abz' + b^2y' = 0.$$

The restriction of $\widetilde{C}_2^{(1)} \times C_0^* \rightarrow C_2 \times C_0^*$ to the singular locus gives the non-trivial double cover of C_0^* . From this we conclude

$$\mathrm{Ev}_{C_0} \pi_2^{(1)!} \mathbb{I}_{\widetilde{C}_2^{(1)}}[2] = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathbb{I}_{\mathcal{O}_0}).$$

As we already know that $\mathcal{IC}(\mathbb{I}_{C_0})$ is the source of the second term, we conclude that

$$\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{I}_{C_2}) = \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}).$$

13.2.6. Normalization of Ev and the twisting local system. — From Table 13.2.1 we find the first interesting case of the local system \mathcal{T} on $T_H^*(V)_{\mathrm{sreg}}$ described in general in Section 6.9, and defined on $T_C^*(V)_{\mathrm{sreg}}$ by (97):

$$\mathcal{T}_C := \mathrm{Ev}_C \mathcal{IC}(C).$$

TABLE 13.2.3. ${}^p\text{NEv} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 13.

$\text{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{{}^p\text{NEv}}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})$

 TABLE 13.2.4. $\text{NEvs} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 13.

\mathcal{P}	\parallel	$\text{NEvs}_{\psi_0} \mathcal{P}$	$\text{NEvs}_{\psi_2} \mathcal{P}$	$\text{NEvs}_{\psi_3} \mathcal{P}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\parallel	+	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	\parallel	−	++	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	\parallel	0	0	+
$\mathcal{IC}(\mathcal{L}_{C_3})$	\parallel	0	−−	−
$\mathcal{IC}(\mathcal{F}_{C_2})$	\parallel	0	−+	0

From Table 13.2.1 we see that

$$\mathcal{T} = \mathbb{1}_{\mathcal{O}_0}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_2}^{\natural} \oplus \mathbb{1}_{\mathcal{O}_3}^{\natural},$$

or, in other notation,

$$\begin{array}{c} \parallel \\ \psi_0 \quad \psi_2 \quad \psi_3 \\ \hline \mathcal{T}_\psi \parallel \quad + \quad - - \quad + \end{array}$$

We use \mathcal{T} in Table 13.2.3 to calculate ${}^p\text{NEv} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ in two forms; compare with Table 13.2.1.

13.2.7. Fourier transform and vanishing cycles. — We may now verify (141) by comparing the functors below with the Fourier transform appearing in Section 13.2.4.

$\text{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{{}^p\text{NEv}}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$	$\xrightarrow{a_*}$	$\text{Per}_H(T_{H_\lambda}^*(V_\lambda^*)_{\text{reg}})$	$\xleftarrow{{}^p\text{Ev}^*}$	$\text{Per}_{H_\lambda}(V_\lambda^*)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0^*})$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0^*})$	\leftarrow	$\mathcal{IC}(\mathcal{L}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3^*})$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_3^*})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3^*}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2^*})$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_2^*})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2^*})$	\leftarrow	$\mathcal{IC}(\mathcal{F}_{C_2^*})$

13.2.8. Arthur sheaves. —

Arthur sheaf	packet sheaves	coronal sheaves
\mathcal{A}_{C_0}	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_2})$
\mathcal{A}_{C_2}	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_2}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_3})$
\mathcal{A}_{C_3}	$\mathcal{IC}(\mathbb{1}_{C_3}) \oplus \mathcal{IC}(\mathcal{F}_{C_3})$	

13.3. ABV-packets. —**13.3.1. Admissible representations versus equivariant perverse sheaves.** —

$\text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}$	$\Pi_{\text{pure}, \lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$[\pi(\phi_0), 0]$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$[\pi(\phi_2, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$[\pi(\phi_3, +), 0]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$[\pi(\phi_3, -), 0]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	$[\pi(\phi_2, -), 1]$

13.3.2. ABV-packets. —

ABV-packets	packet representations	coronal representations
$\Pi_{\text{pure}, \phi_0}^{\text{ABV}}(G/F) :$	$[\pi(\phi_0, +), 0]$	$[\pi(\phi_2, +), 0]$
$\Pi_{\text{pure}, \phi_2}^{\text{ABV}}(G/F) :$	$[\pi(\phi_2, +), 0], [\pi(\phi_2, -), 1]$	$[\pi(\phi_3, -), 0]$
$\Pi_{\text{pure}, \phi_3}^{\text{ABV}}(G/F) :$	$[\pi(\phi_3, +), 0], [\pi(\phi_3, -), 0]$	

13.3.3. Stable distributions and endoscopic transfer. — We now calculate the virtual representations $\eta_{\phi, s}^{\text{NEv}}$; see (149). In the list below, we use the notation $s = (s_1, s_2)$ for elements of $\widehat{T}[2]$.

$\phi_0:$

$$\eta_{\phi_0, s}^{\text{NEv}} = [\pi(\phi_0, +), 0] + (-)(s_1 s_2)[\pi(\phi_2, +), 0]$$

so

$$\begin{aligned} \eta_{\phi_0, (1,1)}^{\text{NEv}} &= [\pi(\phi_0), 0] + [\pi(\phi_2, +), 0] \\ \eta_{\phi_0, (1,-1)}^{\text{NEv}} &= [\pi(\phi_0), 0] - [\pi(\phi_2, +), 0] \\ \eta_{\phi_0, (-1,1)}^{\text{NEv}} &= [\pi(\phi_0), 0] - [\pi(\phi_2, +), 0] \\ \eta_{\phi_0, (-1,-1)}^{\text{NEv}} &= [\pi(\phi_0), 0] + [\pi(\phi_2, +), 0]. \end{aligned}$$

$\phi_2:$

$$\eta_{\phi_2, s}^{\text{NEv}} = [\pi(\phi_2, +), 0] - (+-)(s)[\pi(\phi_2, -), 1] - (--)(s)[\pi(\phi_3, -), 0]$$

so

$$\begin{aligned}
\eta_{\phi_2, (1,1)}^{\text{NEv}} &= [\pi(\phi_2, +), 0] - [\pi(\phi_2, -), 1] - [\pi(\phi_3, -), 0] \\
\eta_{\phi_2, (1,-1)}^{\text{NEv}} &= [\pi(\phi_2, +), 0] + [\pi(\phi_2, -), 1] + [\pi(\phi_3, -), 0] \\
\eta_{\phi_2, (-1,1)}^{\text{NEv}} &= [\pi(\phi_2, +), 0] - [\pi(\phi_2, -), 1] + [\pi(\phi_3, -), 0] \\
\eta_{\phi_2, (-1,-1)}^{\text{NEv}} &= [\pi(\phi_2, +), 0] + [\pi(\phi_2, -), 1] - [\pi(\phi_3, -), 0]
\end{aligned}$$

ϕ_3 :

$$\eta_{\phi_3, s}^{\text{NEv}} = [\pi(\phi_3, +), 0] + (-)(s_1 s_2)[\pi(\phi_3, -), 0]$$

so

$$\begin{aligned}
\eta_{\phi_3, (1,1)}^{\text{NEv}} &= [\pi(\phi_3, +), 0] + [\pi(\phi_3, -), 0] \\
\eta_{\phi_3, (1,-1)}^{\text{NEv}} &= [\pi(\phi_3, +), 0] - [\pi(\phi_3, -), 0] \\
\eta_{\phi_3, (-1,1)}^{\text{NEv}} &= [\pi(\phi_3, +), 0] - [\pi(\phi_3, -), 0] \\
\eta_{\phi_3, (-1,-1)}^{\text{NEv}} &= [\pi(\phi_3, +), 0] + [\pi(\phi_3, -), 0].
\end{aligned}$$

After comparing with Section 13.1.6, we see

$$\begin{aligned}
\eta_{\psi_0, s} &= \eta_{\psi_0, s}^{\text{NEv}} \\
\eta_{\psi_3, s} &= \eta_{\psi_3, s}^{\text{NEv}}.
\end{aligned}$$

This proves (148) for admissible representations with infinitesimal parameter λ given at the beginning of Section 13.

13.3.4. Kazhdan-Lusztig conjecture. — From Section 13.1.3 we find the multiplicity matrix m_{rep} and from Section 13.2.3 we find the normalised geometric multiplicity matrix m'_{geo} :

$$m_{\text{rep}} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad m'_{\text{geo}} = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Since ${}^t m_{\text{rep}} = m'_{\text{geo}}$, this confirms the Kazhdan-Lusztig conjecture as it applies to representations with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 13.

13.3.5. Aubert duality and Fourier transform. — To verify (150), use Vogan's bijection from Section 13.3.1 to compare Aubert duality from Section 13.1.5 with the Fourier transform from Section 13.2.4.

13.3.6. Normalisation. — To verify (152), observe that the twisting characters χ_ψ of A_ψ from Section 13.1.5 are trivial except for the Arthur parameter ψ_2 , as are the local systems \mathcal{T}_ψ from Section 13.2.7 and in both cases they are given by the

character $(-)$ of A_{ψ_2} determined by the isomorphism $A_{\psi_2} \cong \{\pm 1\} \times \{\pm 1\}$, fixed in Section 13.1.4.

13.4. Endoscopy and equivariant restriction of perverse sheaves. — As in Section 13.1.6, we now consider the split endoscopic group $G' = \mathrm{SO}(3) \times \mathrm{SO}(3)$ for G determined by $s = \mathrm{diag}(1, -1, -1, 1) \in \widehat{G}$. Then $\lambda : W_F \rightarrow {}^L G$ factors through $\epsilon : {}^L G' \rightarrow {}^L G$ to define $\lambda' : W_F \rightarrow {}^L G'$ by

$$\lambda'(w) = \left(\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right).$$

In this section we will calculate both sides of (154). This will illustrate how the Langlands-Shelstad lift of $\Theta_{\psi'}$ on $G'(F)$ to $\Theta_{\psi,s}$ on $G(F)$ is related to equivariant restriction of perverse sheaves from V to the Vogan variety V' for G' . Note that each component of λ' is the infinitesimal parameter $W_F \rightarrow {}^L \mathrm{SO}(3)$ that appeared in Section 10; here we will use that Section extensively.

13.4.1. Parameters. — There are four Arthur parameters with infinitesimal parameter $\lambda' : W_F \rightarrow {}^L G'$, up to H' -conjugacy. Using notation from Section 10, they are

$$\begin{aligned} \psi'_{00} &:= \psi_0 \times \psi_0, & \psi'_{11} &:= \psi_1 \times \psi_1, \\ \psi'_{10} &:= \psi_1 \times \psi_0, & \psi'_{01} &:= \psi_0 \times \psi_1, \end{aligned}$$

so

$$\begin{aligned} \psi'_{00}(w, x, y) &= (\nu_2(y), \nu_2(y)), & \psi'_{11}(w, x, y) &= (\nu_2(x), \nu_2(x)), \\ \psi'_{10}(w, x, y) &= (\nu_2(x), \nu_2(y)), & \psi'_{01}(w, x, y) &= (\nu_2(y), \nu_2(x)). \end{aligned}$$

Although $\psi_2 = \epsilon \circ \psi'_{10}$ is H -conjugate to $\epsilon \circ \psi'_{01}$, the Arthur parameters ψ'_{10} and ψ'_{01} for G' are not H' -conjugate.

13.4.2. Endoscopic Vogan variety. — The Vogan variety V' for λ' is simply two copies of the Vogan variety appearing in Section 10. As a subvariety of the conormal bundle to V , the conormal to the Vogan variety V' for $\lambda' : W_F \rightarrow {}^L G'$ is

$$T_{H'}^*(V') = \left\{ \left(\begin{array}{c|cc} & 0 & x \\ \hline & y & 0 \\ \hline 0 & y' & \\ x' & 0 & \end{array} \right) \mid \begin{array}{l} xy' = 0 \\ yx' = 0 \end{array} \right\}$$

(C'_0). Set $C'_0 = C_0 \times C_0$. Then the regular conormal above the closed H' -orbit $C'_0 \subset V'$ is

$$T_{C'_0}^*(V')_{\mathrm{reg}} = \left\{ \left(\begin{array}{c|cc} & 0 & 0 \\ \hline & 0 & 0 \\ \hline 0 & y' & \\ x' & 0 & \end{array} \right) \mid \begin{array}{l} x' \neq 0 \\ y' \neq 0 \end{array} \right\}$$

Base point:

$$(x'_0, \xi'_0) = \left(\begin{array}{c|cc} & 0 & 0 \\ \hline & 0 & 0 \\ \hline 0 & 1 & \\ 1 & 0 & \end{array} \right) \in T_{C'_0}^*(V')_{\mathrm{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ \downarrow s \mapsto (s_1, s_2) \\ 1 = A_{x'_0} \longleftarrow A_{(x'_0, \xi'_0)} \xrightarrow{\text{id}} A_{\xi'_0} = \{\pm 1\} \times \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\text{Loc}_H(T_{C'_0}^*(V)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}'_0}$	$\mathcal{L}_{\mathcal{O}'_0}$	$\mathcal{F}_{\mathcal{O}'_0}$	$\mathcal{E}_{\mathcal{O}'_0}$
$\text{Rep}(A_{(x'_0, \xi'_0)}) :$	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \text{Loc}_{H'}(C'_0) & \rightarrow & \text{Loc}_{H'}(T_{C'_0}^*(V')_{\text{sreg}}) \\ \mathbb{1}_{C'_0} & \mapsto & \begin{array}{c} \mathbb{1}_{\mathcal{O}'_0} \\ \mathcal{L}_{\mathcal{O}'_0} \\ \mathcal{F}_{\mathcal{O}'_0} \\ \mathcal{E}_{\mathcal{O}'_0} \end{array} \end{array}$$

(C'_x) . Set $C'_x = C_x \times C_0 \subset V'$. Then the regular conormal above C'_x is

$$T_{C'_x}^*(V)_{\text{reg}} = \left\{ \left(\left(\begin{array}{cc|cc} & & 0 & x \\ & & 0 & 0 \\ \hline 0 & y' & & \\ 0 & 0 & & \end{array} \right) \mid \begin{array}{l} x \neq 0 \\ y' \neq 0 \end{array} \right) \right\}$$

Base point:

$$(x'_{10}, \xi'_{10}) = \left(\left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array} \right) \right) \in T_{C'_x}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ \downarrow s \mapsto (s_1, s_2) \\ \{\pm 1\} = A_{x'_{10}} \xleftarrow{s_1 \mapsto (s_1, s_2)} A_{(x'_{10}, \xi'_{10})} \xrightarrow{(s_1, s_2) \mapsto s_2} A_{\xi'_{10}} = \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\text{Loc}_H(T_{C'_x}^*(V)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}'_x}$	$\mathcal{L}_{\mathcal{O}'_x}$	$\mathcal{F}_{\mathcal{O}'_x}$	$\mathcal{E}_{\mathcal{O}'_x}$
$\text{Rep}(A_{(x'_{01}, \xi'_{01})}) :$	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \text{Loc}_{H'}(C'_x) & \rightarrow & \text{Loc}_{H'}(T_{C'_x}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C'_x} & \mapsto & \begin{array}{c} \mathbb{1}_{\mathcal{O}'_x} \\ \mathcal{L}_{\mathcal{O}'_x} \\ \mathcal{F}_{\mathcal{O}'_x} \\ \mathcal{E}_{\mathcal{O}'_x} \end{array} \\ \mathcal{L}_{C'_x} & \mapsto & \mathcal{F}_{\mathcal{O}'_x} \end{array}$$

(C'_y) . Set $C'_y = C_0 \times C_x \subset V'$. Then the regular conormal above C'_y is

$$T_{C'_y}^*(V)_{\text{reg}} = \left\{ \left(\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & y & 0 \\ \hline 0 & 0 & & \\ x' & 0 & & \end{array} \right) \mid \begin{array}{l} x' \neq 0 \\ y \neq 0 \end{array} \right\}$$

Base point:

$$(x'_{01}, \xi'_{01}) = \left(\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 1 & 0 & & \end{array} \right) \right) \in T_{C'_y}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ \downarrow s \mapsto (s_1, s_2) \\ \{\pm 1\} = A_{x'_{01}} \xleftarrow{s_2 \leftarrow (s_1, s_2)} A_{(x'_{01}, \xi'_{01})} \xrightarrow{(s_1, s_2) \mapsto s_1} A_{\xi'_{01}} = \{\pm 1\} \end{array}$$

Local systems on strongly regular conormal:

$\text{Loc}_H(T_{C'_y}^*(V)_{\text{sreg}}):$	$\mathbb{1}_{\mathcal{O}'_y}$	$\mathcal{L}_{\mathcal{O}'_y}$	$\mathcal{F}_{\mathcal{O}'_y}$	$\mathcal{E}_{\mathcal{O}'_y}$
$\text{Rep}(A_{(x'_{10}, \xi'_{10})}):$	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C'_y) & \rightarrow & \text{Loc}_H(T_{C'_y}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C'_y} & \mapsto & \mathbb{1}_{\mathcal{O}'_y} \\ & & \mathcal{L}_{\mathcal{O}'_y} \\ & & \mathcal{F}_{\mathcal{O}'_y} \\ \mathcal{L}_{C'_y} & \mapsto & \mathcal{E}_{\mathcal{O}'_y} \end{array}$$

(C'_{xy}) . Set $C'_{xy} = C_x \times C_y \subset V'$. Then

$$T_{C'_{xy}}^*(V)_{\text{reg}} = \left\{ \left(\left(\begin{array}{cc|cc} & & 0 & x \\ & & y & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \mid xy \neq 0 \right\}$$

Base point:

$$(x'_{11}, \xi'_{11}) = \left(\left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array} \right) \right) \in T_{C'_{xy}}^*(V)_{\text{reg}}$$

Fundamental groups:

$$\begin{array}{c} \widehat{T}[2] \\ \downarrow s \mapsto (s_1, s_2) \\ \{\pm 1\} \times \{\pm 1\} = A_{x'_{11}} \xleftarrow{\text{id}} A_{(x'_{11}, \xi'_{11})} \longrightarrow A_{\xi'_{11}} = 1 \end{array}$$

Local systems on strongly regular conormal:

$\text{Loc}_H(T_{C'_{xy}}^*(V)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}'_{xy}}$	$\mathcal{L}_{\mathcal{O}'_{xy}}$	$\mathcal{F}_{\mathcal{O}'_{xy}}$	$\mathcal{E}_{\mathcal{O}'_{xy}}$
$\text{Rep}(A_{(x'_{11}, \xi'_{11})}) :$	$++$	$--$	$-+$	$+-$

Pullback along the bundle map:

$$\begin{array}{ccc} \text{Loc}_{H'}(C'_{xy}) & \rightarrow & \text{Loc}_H(T_{C'_{xy}}^*(V')_{\text{sreg}}) \\ \mathbb{1}_{C'_{xy}} & \mapsto & \mathbb{1}_{\mathcal{O}'_{xy}} \\ \mathcal{L}_{C'_{xy}} & \mapsto & \mathcal{L}_{\mathcal{O}'_{xy}} \\ \mathcal{F}_{C'_{xy}} & \mapsto & \mathcal{F}_{\mathcal{O}'_{xy}} \\ \mathcal{E}_{C'_{xy}} & \mapsto & \mathcal{E}_{\mathcal{O}'_{xy}} \end{array}$$

13.4.3. Vanishing cycles. — The functor

$${}^p\text{NEv}' : \text{Per}_{H'}(V') \rightarrow \text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$$

may be deduced from Section 10.2.6 using the Sebastiani-Thom isomorphism [III17] and [Mas01]; see Table 13.4.1. Here we show the calculation of the last three rows, to illustrate the method.

$$\begin{aligned} {}^p\text{NEv}' \mathcal{IC}(\mathcal{L}_{C'_x}) &= {}^p\text{NEv}' (\mathcal{IC}(\mathcal{E}_{C_x}) \boxtimes \mathcal{IC}(\mathbb{1}_{C_0})) \\ &= ({}^p\text{NEv} \mathcal{IC}(\mathcal{E}_{C_x})) \boxtimes ({}^p\text{NEv} \mathcal{IC}(\mathbb{1}_{C_0})) \\ &= (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ &= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ &= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \\ &= \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0}) \end{aligned}$$

Similarly,

$$\begin{aligned} {}^p\text{NEv}' \mathcal{IC}(\mathcal{L}_{C'_y}) &= {}^p\text{NEv}' (\mathcal{IC}(\mathbb{1}_{C_0}) \boxtimes \mathcal{IC}(\mathcal{E}_{C_y})) \\ &= ({}^p\text{NEv} \mathcal{IC}(\mathbb{1}_{C_0})) \boxtimes ({}^p\text{NEv} \mathcal{IC}(\mathcal{E}_{C_y})) \\ &= \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})) \\ &= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ &= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_y} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \\ &= \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y}) \end{aligned}$$

TABLE 13.4.1. ${}^p\text{NEv}' : \text{Per}_{H_{\lambda'}}(V_{\lambda'}) \rightarrow \text{Per}_{H_{\lambda'}}(T_{H_{\lambda'}}^*(V_{\lambda'})_{\text{reg}})$ on simple objects, for $\lambda' : W_F \rightarrow {}^L G'$ given at the beginning of Section 13.

$\text{Per}_{H'}(V')$	$\xrightarrow{{}^p\text{NEv}'}$	$\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$
$\mathcal{IC}(\mathbb{1}_{C'_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})$
$\mathcal{IC}(\mathbb{1}_{C'_x})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x})$
$\mathcal{IC}(\mathbb{1}_{C'_y})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_y})$
$\mathcal{IC}(\mathbb{1}_{C'_{xy}})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_{xy}})$
$\mathcal{IC}(\mathcal{L}_{C'_x})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})$
$\mathcal{IC}(\mathcal{L}_{C'_y})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})$
$\mathcal{IC}(\mathcal{L}_{C'_{xy}})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})$

TABLE 13.4.2. $\text{NEvs}' : \text{Per}_{H_{\lambda'}}(V_{\lambda'}) \rightarrow \text{Loc}_{H_{\lambda'}}(T_{H_{\lambda'}}^*(V_{\lambda'})_{\text{sreg}})$ on simple objects, for $\lambda' : W_F \rightarrow {}^L G'$ given at the beginning of Section 13.

\mathcal{P}'	\parallel	$\text{NEvs}_{\psi'_{10}} \mathcal{P}'$	$\text{NEvs}_{\psi'_{10}} \mathcal{P}'$	$\text{NEvs}_{\psi'_{01}} \mathcal{P}'$	$\text{NEvs}_{\psi'_{11}} \mathcal{P}'$
$\mathcal{IC}(\mathbb{1}_{C'_0})$	\parallel	++	0	0	0
$\mathcal{IC}(\mathbb{1}_{C'_x})$	\parallel	0	++	0	0
$\mathcal{IC}(\mathbb{1}_{C'_y})$	\parallel	0	0	++	0
$\mathcal{IC}(\mathbb{1}_{C'_{xy}})$	\parallel	0	0	0	++
$\mathcal{IC}(\mathcal{L}_{C'_x})$	\parallel	-+	-+	0	0
$\mathcal{IC}(\mathcal{L}_{C'_y})$	\parallel	+-	0	+-	0
$\mathcal{IC}(\mathcal{L}_{C'_{xy}})$	\parallel	--	--	--	--

and

$$\begin{aligned}
{}^p\text{NEv}' \mathcal{IC}(\mathcal{L}_{C'_{xy}}) &= {}^p\text{NEv}' (\mathcal{IC}(\mathcal{E}_{C_x}) \boxtimes \mathcal{IC}(\mathcal{E}_{C_y})) \\
&= ({}^p\text{NEv} \mathcal{IC}(\mathcal{E}_{C_x})) \boxtimes ({}^p\text{NEv} \mathcal{IC}(\mathcal{E}_{C_y})) \\
&= (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})) \\
&= \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\
&\quad \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\
&= \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0}).
\end{aligned}$$

13.4.4. Restriction. —

$$\begin{aligned}
 \text{res} : \text{Per}_H(V) &\longrightarrow \text{KPer}_{H'}(V') \\
 \mathcal{IC}(\mathbb{1}_{C_0}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_0}) \\
 \mathcal{IC}(\mathbb{1}_{C_2}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1] \\
 \mathcal{IC}(\mathbb{1}_{C_3}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_{xy}})[1] \\
 \mathcal{IC}(\mathcal{L}_{C_3}) &\mapsto \mathcal{IC}(\mathbb{1}_{C'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_{xy}})[1] \\
 \mathcal{IC}(\mathcal{F}_{C_2}) &\mapsto \mathcal{IC}(\mathcal{L}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1]
 \end{aligned}$$

13.4.5. Restriction and vanishing cycles. — In this example the inclusion $V' \hookrightarrow V$ induces a map of conormal bundles $\epsilon : T_{H'}^*(V') \hookrightarrow T_H^*(V)$; this is not true in general, as Section 14.4.3 shows. Here we have

$$\begin{aligned}
 T_{C_0}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C'_0}^*(V')_{\text{reg}} \\
 T_{C_2}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C'_x}^*(V')_{\text{reg}} \cup T_{C'_y}^*(V')_{\text{reg}} \\
 T_{C_3}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C'_{xy}}^*(V')_{\text{reg}}.
 \end{aligned}$$

We now calculate both sides of (153) in three cases: when $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$, $\mathcal{IC}(\mathcal{L}_{C_3})$ and $\mathcal{IC}(\mathcal{F}_{C_2})$.

The case $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$. — We now calculate both sides of (153) when $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$. By Section 13.4,

$$\mathcal{IC}(\mathbb{1}_{C_2})|_{V'} \equiv \mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1],$$

after passing to the Grothendieck group of $\text{Per}_{H'}(V')$. So, by Section 13.4.3,

$$\begin{aligned}
 &\text{NEV}'(\mathcal{IC}(\mathbb{1}_{C_2})|_{V'}) \\
 &\equiv \text{NEV}'\left(\mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C'_0})[1]\right) \\
 &= \mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_0})[1]
 \end{aligned}$$

in the Grothendieck group of $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$. Thus, for each $(x', \xi') \in T_{C'}^*(V')_{\text{reg}}$ with image $(x, \xi) \in T_C^*(V)_{\text{reg}}$, the left-hand side of (153) is

$$\begin{aligned}
 &(-1)^{\dim C'} \text{trace}_{a'_s}(\text{NEV}' \mathcal{IC}(\mathbb{1}_{C_2})|_{V'})_{(x', \xi')} \\
 &= (-1)^{\dim C'} \text{trace}_{a'_s}\left(\mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_0})[1]\right)_{(x', \xi')}
 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned}
 &(-1)^{\dim C} \text{trace}_{a_s}(\text{Ev } \mathcal{IC}(\mathbb{1}_{C_2}))_{(x, \xi)} \\
 &= (-1)^{\dim C} \text{trace}_{a_s}(\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{L}_{C_0}))_{(x, \xi)}.
 \end{aligned}$$

We now calculate both sides of (153) when $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$.

(C'_0) . If $(x', \xi') \in T_{C'_0}^*(V')_{\text{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (153) is

$$\begin{aligned}
 &(-1)^{\dim C'_0} \text{trace}_{(+1, -1)} \mathcal{IC}(\mathbb{1}_{C'_0})[1] \\
 &= -(-1)^0 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathbb{1}_{C'_0}) \\
 &= -(++) (+1, -1) \\
 &= -1,
 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_0} \operatorname{trace}_{(+1, -1)} (\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}))|_{T_{C_0}^*(V)_{\text{reg}}} \\ &= \operatorname{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\ &= (-)(+1, -1) \\ &= -1. \end{aligned}$$

This confirms (153) on $T_{C'_0}^*(V')_{\text{reg}}$.

(C'_x) . If $(x', \xi') \in T_{C'_x}^*(V')_{\text{reg}}$ then $C' = C'_x$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_x} \operatorname{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \\ &= -(-1)^1 \operatorname{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \\ &= (-)(+1, -1) \\ &= -1 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_2} \operatorname{trace}_{(-1)} (\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}))|_{T_{C_2}^*(V)_{\text{reg}}} \\ &= \operatorname{trace}_{(-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \\ &= (-)(-1) \\ &= -1. \end{aligned}$$

This confirms (153) on $T_{C'_x}^*(V')_{\text{reg}}$.

(C'_y) . If $(x', \xi') \in T_{C'_y}^*(V')_{\text{reg}}$ then $C' = C'_y$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_y} \operatorname{trace}_{(-1, +1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \\ &= -(-1)^1 \operatorname{trace}_{(-1, +1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \\ &= (-)(-1, +1) \\ &= -1, \end{aligned}$$

while the right-hand side of (153) is -1 , as in the case above. This confirms (153) on $T_{C'_y}^*(V')_{\text{reg}}$.

(C'_{xy}) . If $(x', \xi') \in T_{C'_{xy}}^*(V')_{\text{reg}}$ then $C' = C'_{xy}$ and $C = C_3$ and the left-hand side of (153) is

$$(-1)^{\dim C'_{xy}} \operatorname{trace}_{(+1, -1)} \left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1] \right)_{(x', \xi')}$$

while the right-hand side of (153) is

$$(-1)^{\dim C_3} \operatorname{trace}_{(-1)} (\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}))|_{T_{C_3}^*(V)_{\text{reg}}},$$

both of which are trivially 0. This confirms (153) on $T_{C'_{xy}}^*(V')_{\text{reg}}$.

This proves (154) when $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_2})$.

The case $\mathcal{P} = \mathcal{IC}(\mathcal{L}_{C_3})$. — We now calculate both sides of (153) when $\mathcal{P} = \mathcal{IC}(\mathcal{L}_{C_3})$. By Section 13.4,

$$\mathcal{IC}(\mathcal{L}_{C_3})|_{V'} \equiv \mathcal{IC}(\mathbb{1}_{C'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_{xy}})[1],$$

after passing to the Grothendieck group of $\text{Per}_{H'}(V')$. So, by Section 13.4.3,

$$\begin{aligned} & \text{NEv}'(\mathcal{IC}(\mathcal{L}_{C_3})|_{V'}) \\ & \equiv \text{NEv}'\left(\mathcal{IC}(\mathbb{1}_{C'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_{xy}})[1]\right) \\ & = \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1] \end{aligned}$$

in the Grothendieck group of $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$. Thus, for each $(x', \xi') \in T_{C'}^*(V')_{\text{reg}}$ with image $(x, \xi) \in T_C^*(V)_{\text{reg}}$, the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'} \text{trace}_{a'_s}(\text{NEv}' \mathcal{IC}(\mathcal{L}_{C_3})|_{V'})_{(x', \xi')} \\ & = (-1)^{\dim C'} \text{trace}_{a'_s} \left(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \right. \\ & \quad \left. \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1] \right)_{(x', \xi')} \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C} \text{trace}_{a_s}(\text{Ev} \mathcal{IC}(\mathcal{L}_{C_3}))_{(x, \xi)} \\ & = (-1)^{\dim C} \text{trace}_{a_s}(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}))_{(x, \xi)}. \end{aligned}$$

We now calculate both sides of (153) in every case.

(C'_0) . If $(x', \xi') \in T_{C'_0}^*(V')_{\text{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_0} \text{trace}_{(+1, -1)}(\mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0})[1] \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})[1]) \\ & = (-1)^0 (-\text{trace}_{(+1, -1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}'_0}) - \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_0})) \\ & = -(++)(+1, -1) - (--)(+1, -1) \\ & = +1 - 1 \\ & = 0, \end{aligned}$$

while the right-hand side of (153) is

$$(-1)^{\dim C_0} \text{trace}_{(-1)}(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}))|_{T_{C_0}^*(V)_{\text{reg}}} = 0.$$

This confirms (153) on $T_{C'_0}^*(V')_{\text{reg}}$.

(C'_x) . If $(x', \xi') \in T_{C'_x}^*(V')_{\text{reg}}$ then $C' = C'_x$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_x} \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x})[1] \\ & = -(-1)^1 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_x}) \\ & = (--)(+1, -1) \\ & = -1 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_2} \text{trace}_{(-1)}(\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}))|_{T_{C_2}^*(V)_{\text{reg}}} \\ & = \text{trace}_{(-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \\ & = (-)(-1) \\ & = -1. \end{aligned}$$

This confirms (153) on $T_{C'_x}^*(V')_{\text{reg}}$.

(C'_y) . If $(x', \xi') \in T_{C'_y}^*(V')_{\text{reg}}$ then $C' = C'_y$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_y} \text{trace}_{(-1, +1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y})[1] \\ &= -(-1)^1 \text{trace}_{(-1, +1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_y}) \\ &= (-)(-1, +1) \\ &= -1, \end{aligned}$$

while the right-hand side of (153) is -1 , as in the case above. This confirms (153) on $T_{C'_y}^*(V')_{\text{reg}}$.

(C'_{xy}) . If $(x', \xi') \in T_{C'_{xy}}^*(V')_{\text{reg}}$ then $C' = C'_{xy}$ and $C = C_3$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_{xy}} \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}})[1] \\ &= -(-1)^2 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}'_{xy}}) \\ &= -(-)(+1, -1) \\ &= -(-1) \\ &= +1 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_3} \text{trace}_{(-1)} (\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}))|_{T_{C_3}^*(V)_{\text{reg}}} \\ &= -\text{trace}_{(-1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \\ &= -(-)(-1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C'_{xy}}^*(V')_{\text{reg}}$.

This proves (154) when $\mathcal{P} = \mathcal{IC}(\mathcal{L}_{C_3})$.

The case $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_2})$. — We now calculate both sides of (153) when $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_2})$. By Section 13.4,

$$\mathcal{IC}(\mathcal{F}_{C_2})|_{V'} \equiv \mathcal{IC}(\mathbb{1}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1],$$

after passing to the Grothendieck group of $\text{Per}_{H'}(V')$. So, by Section 13.4.3,

$$\begin{aligned} & \text{NEV}'(\mathcal{IC}(\mathcal{F}_{C_2})|_{V'}) \\ &\equiv \text{NEV}'\left(\mathcal{IC}(\mathcal{L}_{C'_x})[1] \oplus \mathcal{IC}(\mathcal{L}_{C'_y})[1]\right) \\ &= \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1] \end{aligned}$$

in the Grothendieck group of $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$. Thus, for each $(x', \xi') \in T_{C'}^*(V')_{\text{reg}}$ with image $(x, \xi) \in T_C^*(V)_{\text{reg}}$, the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'} \text{trace}_{a'_s} \left(\text{NEV}' \mathcal{IC}(\mathcal{F}_{C_2})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^{\dim C'} \text{trace}_{a'_s} \left(\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_y})[1] \right. \\ &\quad \left. \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1] \right)_{(x', \xi')} \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C} \operatorname{trace}_{a_s} (\operatorname{Ev}_{(x,\xi)} \mathcal{IC}(\mathcal{F}_{C_2})) \\ &= (-1)^{\dim C} \operatorname{trace}_{a_s} (\operatorname{Ev} \mathcal{IC}(\mathcal{F}_{C_2}))|_{T_C^*(V)_{\text{reg}}} \\ &= (-1)^{\dim C} \operatorname{trace}_{a_s} (\mathcal{IC}(\mathcal{F}_{C_2}))|_{T_C^*(V)_{\text{reg}}}. \end{aligned}$$

We now calculate both sides of (153) in every case.

(C'_0) . If $(x', \xi') \in T_{C'_0}^*(V')_{\text{reg}}$ then $C' = C'_0$ and $C = C_0$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_0} \operatorname{trace}_{(+1,-1)} (\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1] \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})[1]) \\ &= (-1)^0 (-\operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0}) - \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_0})) \\ &= -(-+)(+1, -1) - (+-)(+1, -1) \\ &= +1 - 1 \\ &= 0, \end{aligned}$$

while the right-hand side of (153) is

$$(-1)^{\dim C_0} \operatorname{trace}_{(-1)} (\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}))|_{T_{C_0}^*(V)_{\text{reg}}} = 0.$$

This confirms (153) on $T_{C'_0}^*(V')_{\text{reg}}$.

(C'_x) . If $(x', \xi') \in T_{C'_x}^*(V')_{\text{reg}}$ then $C' = C'_x$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_x} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \\ &= -(-1)^1 \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x}) \\ &= (-+)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_2} \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})|_{T_{C_2}^*(V)_{\text{reg}}} \\ &= \operatorname{trace}_{(+1,-1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \\ &= (-+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C'_x}^*(V')_{\text{reg}}$.

(C'_y) . If $(x', \xi') \in T_{C'_y}^*(V')_{\text{reg}}$ then $C' = C'_y$ and $C = C_2$ and the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C'_y} \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y})[1] \\ &= -(-1)^1 \operatorname{trace}_{(-1,+1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}'_y}) \\ &= (+-)(-1, +1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is $+1$, as in the case above. This confirms (153) on $T_{C'_y}^*(V')_{\text{reg}}$.

(C'_{xy}) . If $(x', \xi') \in T_{C'_{xy}}^*(V')_{\text{reg}}$ then $C' = C'_{xy}$ and $C = C_3$ and the left-hand side of (153) is

$$(-1)^{\dim C'_{xy}} \operatorname{trace}_{(+1,-1)} (\mathcal{IC}(\mathcal{F}_{\mathcal{O}'_x})[1] \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}'_0})[1])_{(x', \xi')}$$

while the right-hand side of (153) is

$$(-1)^{\dim C} \operatorname{trace}_{a_s} (\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}))|_{T_{\mathcal{C}_3}^*(V)_{\text{reg}}},$$

both of which are trivially 0. This confirms (153) on $T_{C'_{xy}}^*(V')_{\text{reg}}$.

This proves (154) when $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_2})$.

14. $\text{SO}(7)$ unipotent representations, singular parameter

Let $G = \text{SO}(7)$. The calculation of pure inner twists and inner twists and their forms for G is the same as in Section 12. Let G_1 be the non-quasisplit form of G , given by the quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon\varpi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varpi & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One readily verifies that the Hasse invariant of this form is $(\varpi, \varepsilon) = -1$ so that the form is not split. Note that the choice $\varepsilon = 1$ would give a split form.

Consider the infinitesimal parameter $\lambda : W_F \rightarrow \hat{G}$ given by

$$\lambda(w) := \begin{pmatrix} |w|^{3/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & |w|^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & |w|^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & |w|^{-3/2} \end{pmatrix}.$$

Here, and below, we use the symplectic form $\langle x, y \rangle = {}^t x J y$ with matrix J given by $J_{ij} = (-1)^j \delta_{7-i,j}$ to determine a representation of $\text{Sp}(6)$. Note that, in contrast to the unramified infinitesimal parameters in Sections 10 and 12, in this case the image of Frobenius is singular semisimple.

14.1. Arthur packets. —

14.1.1. Parameters. — Up to H_λ -conjugation, there are eight Langlands parameters with infinitesimal parameter λ , of which six are of Arthur type. The six Langlands parameters of Arthur type are most easily described through their Arthur parameters:

$$\begin{aligned} \psi_0(w, x, y) &= \nu_4(y) \oplus \nu_2(y), & \psi_7(w, x, y) &= \nu_4(x) \oplus \nu_2(x), \\ \psi_2(w, x, y) &= \nu_4(y) \oplus \nu_2(x), & \psi_6(w, x, y) &= \nu_4(x) \oplus \nu_2(y), \\ \psi_4(w, x, y) &= \nu_2(x) \otimes \nu_3(y), & \psi_5(w, x, y) &= \nu_3(x) \otimes \nu_2(y). \end{aligned}$$

where $\nu_4 : \text{SL}(2) \rightarrow \text{Sp}(4)$ is a 4-dimensional *symplectic* irreducible representation of $\text{SL}(2)$, $\nu_3 : \text{SL}(2) \rightarrow \text{SO}(3)$ is a 3-dimensional *orthogonal* irreducible representation of

$\mathrm{SL}(2)$ and, as above, $\nu_2 : \mathrm{SL}(2) \rightarrow \mathrm{SL}(2)$ is the identity representation. Note that ψ_0 is the Aubert dual of ψ_7 , ψ_2 is the Aubert dual of ψ_6 , and ψ_4 is the Aubert dual of ψ_5 .

These Arthur parameters define the following six Langlands parameters:

$$\begin{aligned} \phi_0(w, x) &= \nu_4(d_w) \oplus \nu_2(d_w), & \phi_7(w, x) &= \nu_4(x) \oplus \nu_2(x), \\ \phi_2(w, x) &= \nu_4(d_w) \oplus \nu_2(x), & \phi_6(w, x) &= \nu_4(x) \oplus \nu_2(d_w), \\ \phi_4(w, x) &= \nu_2(x) \otimes \nu_3(d_w), & \phi_5(w, x) &= \nu_3(x) \otimes \nu_2(d_w). \end{aligned}$$

The remaining two Langlands parameters in $P_\lambda({}^L G)/Z_{\widehat{G}}(\lambda)$ that are not of Arthur type are given here:

$$\begin{aligned} \phi_1(w, x) &= \left(\begin{array}{cc|cc|cc} |w|x_{11} & |w|x_{12} & 0 & 0 & 0 & 0 \\ |w|x_{21} & |w|x_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & |w|^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & |w|^{-1/2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & |w|^{-1}x_{11} & |w|^{-1}x_{12} \\ 0 & 0 & 0 & 0 & |w|^{-1}x_{21} & |w|^{-1}x_{22} \end{array} \right), \\ \phi_3(w, x) &= \left(\begin{array}{cc|cc|cc} |w|^{3/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & x_{12} & 0 \\ \hline 0 & 0 & x_{11} & x_{12} & 0 & 0 \\ 0 & 0 & x_{21} & x_{22} & 0 & 0 \\ \hline 0 & -x_{21} & 0 & 0 & -x_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & |w|^{-3/2} \end{array} \right). \end{aligned}$$

14.1.2. L -packets. — In total, there are 15 admissible representations with infinitesimal parameter λ , of which 10 are representations of $G(F)$ while 5 are representations of $G_1(F)$. In order to list them, we must enumerate the irreducible representations A_ϕ , for each $\phi \in P_\lambda({}^L G)$. In every case but one, the group A_ϕ is trivial or has order 2; in the latter case, the irreducible representations of these groups are unambiguously labeled with $+$ or $-$; in the former case, we simply elide the trivial representation, such as in the list below.

$$\begin{aligned} \Pi_{\phi_0}(G(F)) &= \{\pi(\phi_0)\} & \Pi_{\phi_0}(G_1(F)) &= \emptyset \\ \Pi_{\phi_1}(G(F)) &= \{\pi(\phi_1)\} & \Pi_{\phi_1}(G_1(F)) &= \emptyset \\ \Pi_{\phi_2}(G(F)) &= \{\pi(\phi_2, +)\} & \Pi_{\phi_2}(G_1(F)) &= \{\pi(\phi_2, -)\} \\ \Pi_{\phi_3}(G(F)) &= \{\pi(\phi_3, +), \pi(\phi_3, -)\} & \Pi_{\phi_3}(G_1(F)) &= \emptyset \\ \Pi_{\phi_4}(G(F)) &= \{\pi(\phi_4, +)\} & \Pi_{\phi_4}(G_1(F)) &= \{\pi(\phi_4, -)\} \\ \Pi_{\phi_5}(G(F)) &= \{\pi(\phi_5)\} & \Pi_{\phi_5}(G_1(F)) &= \emptyset \\ \Pi_{\phi_6}(G(F)) &= \{\pi(\phi_6, +)\} & \Pi_{\phi_6}(G_1(F)) &= \{\pi(\phi_6, -)\} \\ \Pi_{\phi_7}(G(F)) &= \{\pi(\phi_7, ++), \pi(\phi_7, --)\} & \Pi_{\phi_7}(G_1(F)) &= \{\pi(\phi_7, +-), \pi(\phi_7, -+)\} \end{aligned}$$

The centraliser of ϕ_7 is the following subgroup of 2-torsion elements $\widehat{T}[2]$ in the diagonal dual torus \widehat{T} :

$$Z_{\widehat{G}}(\phi_7) = \left\{ \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1 \end{pmatrix} \in \widehat{T}[2] \mid s_1 = s_2 \right\}.$$

We fix the isomorphism $Z_{\widehat{G}}(\phi_7) \cong \{\pm 1\} \times \{\pm 1\}$ so that the image of $Z(\widehat{G})$ in $Z_{\widehat{G}}(\phi_7)$ is $\{(+1, +1), (-1, -1)\}$; using this isomorphism, we label irreducible representations of $A_{\phi_7} \cong Z_{\widehat{G}}(\phi_7)$ by the symbols $++$, $+-$, $-+$ and $--$. Note that the restriction of these representations to $Z(\widehat{G})$ is trivial for $++$ and $--$ only.

Of these 15 admissible representations, only the representation $\pi(\phi_7, +-)$ of $G_1(F)$ is supercuspidal. In fact, $\pi(\phi_7, +-)$ is a unipotent supercuspidal depth-zero representation. In Lusztig's classification of unipotent representations, $\pi(\phi_7, +-)$ is the case $n = 3$, $a = 1$, $b = 1$ of [Lus95a, 7.55]; it corresponds to the unique cuspidal unipotent local system for \widehat{G} , see $\tilde{C}_3/(C_3 \times C_0)$ in [Lus95a, 7.55]. Lusztig's classification also shows how $\pi(\phi_7, +-)$ may be constructed by compact induction, as follows; see $\tilde{B}_3/(D_1 \times B_2)$ in [Lus95a, 7.55]. Let \underline{G}_1 be the parahoric \mathcal{O}_F -group scheme associated to an almost self-dual lattice chain and the quadratic form at the beginning of Section 14. The generic fibre of \underline{G}_1 is the inner form G_1 of G^* , and $\underline{G}_1(\mathcal{O}_F)$ is a maximal parahoric subgroup of the F -points on the generic fibre of \underline{G}_1 . The reductive quotient $\underline{G}_{1, \mathbb{F}_q}^{\text{red}}$ of the special fibre of \underline{G}_1 is $\text{SO}(5) \times \text{SO}(2)$ over \mathbb{F}_q , where $\text{SO}(5)$ and $\text{SO}(2)$ are determined, respectively, by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\epsilon & 0 \\ 0 & 1 \end{pmatrix},$$

with $\epsilon = \varepsilon \bmod \mathcal{O}_F$. Note that the parahoric $\underline{G}_1(\mathcal{O}_K)$ is not hyperspecial. The finite group $\text{SO}(5, \mathbb{F}_q) \times \text{SO}(2, \mathbb{F}_q)$ admits a unique cuspidal unipotent irreducible representation, ${}^o\sigma$. Let $\text{inf}({}^o\sigma)$ be the representation of $\underline{G}_1(\mathcal{O}_F)$ obtained by inflation of ${}^o\sigma$ along $\underline{G}_1(\mathcal{O}_F) \rightarrow (\underline{G}_1)_{\mathbb{F}_q}^{\text{red}}(\mathbb{F}_q)$. Now extend $\text{inf}({}^o\sigma)$ to the representation $\text{inf}({}^o\sigma)^+$ of $N_{G_1(F)}(\underline{G}_1(\mathcal{O}_F))$ by tensoring with an unramified character which has order 2 on $N_{G_1(F)}(\underline{G}_1(\mathcal{O}_F))/\underline{G}_1(\mathcal{O}_F)$. Then

$$\pi(\phi_7, +-)=\text{cInd}_{N_{G_1(F)}(\underline{G}_1(\mathcal{O}_F))}^{\underline{G}_1(F)}(\text{inf}(\sigma)^+).$$

We remark that $N_{G_1(F)}(\underline{G}_1(\mathcal{O}_F))$ also admits a smooth model over \mathcal{O}_F , for which the reductive quotient of the special fibre is $\text{S}(\text{O}(5) \times \text{O}(2)) \cong \text{SO}(5) \times \text{O}(2)$.

14.1.3. Multiplicities in standard modules. — In order to describe the other admissible representations appearing in this example, we give the multiplicity of $\pi(\phi, \rho)$ in the

standard modules $M(\phi', \rho')$ for representations of the pure form $G(F)$ in Table 14.5. To save space there we write π_i for $\pi(\phi_i)$ and π_i^ϵ for $\pi(\phi_i, \epsilon)$; a similar convention applies to the notation for the standard modules. Let us see how to calculate row 8 in Table 14.5. Consider the standard module

$$M_6^+ := \text{Ind}(|^{1/2} \otimes \pi(\nu_4, +))$$

for $G(F)$. It is clear that this will contain π_6^+ . Moreover, it has an irreducible submodule π_7^{++} . To show there is nothing else, we can compute the Jacquet module of M_6^+ with respect to the standard parabolic subgroup P , whose Levi component is $GL(1) \times SO(5)$. By the geometric lemma, we get

$$\text{s.s. Jac}_P M_6^+ = ||^{3/2} \otimes \text{Ind}(|^{1/2} \otimes \pi(\nu_2, +)) \oplus ||^{1/2} \otimes \pi(\nu_4, +) \oplus ||^{-1/2} \otimes \pi(\nu_4, +)$$

and

$$\text{s.s. Ind}(|^{1/2} \otimes \pi(\nu_2, +)) = \pi(\nu_2 \oplus \nu_2, ++)\oplus \pi'$$

where π' is the unique irreducible quotient. Here, s.s. denotes the semi-simplification of the module. On the other hand,

$$\text{s.s. Jac}_P \pi_6^+ = ||^{-1/2} \otimes \pi(\nu_4, +) \oplus ||^{3/2} \otimes \pi'$$

and

$$\text{s.s. Jac}_P \pi_7^{++} = ||^{1/2} \otimes \pi(\nu_4, +) \oplus ||^{3/2} \otimes \pi(\nu_2 \oplus \nu_2, ++).$$

Therefore,

$$\text{s.s. } M_6^+ = \pi_6^+ \oplus \pi_7^{++}.$$

This explains row 8 in Table 14.5.

The multiplicity of $\pi(\phi, \rho)$ in the standard modules $M(\phi', \rho')$, for representations of the form $G_1(F)$ are also displayed in Table 14.5.

14.1.4. Arthur packets. — In order to describe the component groups A_ψ , consider the torus

$$S := \left\{ \left(\begin{array}{c|c|c|c} s_1 & & & \\ \hline & s_2 & 0 & \\ & 0 & s_3 & \\ \hline & & s_3^{-1} & 0 \\ & & 0 & s_2^{-1} \\ \hline & & & s_1^{-1} \end{array} \right) \mid s_1 = s_2 \right\} \subset \hat{T} \subset \hat{G}.$$

Let $S[2]$ be the 2-torsion subgroup of S ; Note that $Z(\hat{G}) \subset S[2]$. Let us the notation

$$s(s_2, s_3) := \left(\begin{array}{c|c|c|c} s_2 & & & \\ \hline & s_2 & 0 & \\ & 0 & s_3 & \\ \hline & & s_3 & 0 \\ & & 0 & s_2 \\ \hline & & & s_2 \end{array} \right) \in S[2]$$

and let $S[2] \cong \{\pm 1\} \times \{\pm 1\}$ be the isomorphism determined by this notation. Then $Z(\widehat{G}) \cong \{\pm 1\}$ is the diagonal subgroup, for which we will use the notation

$$s(s_1, s_1) := \begin{pmatrix} s_1 & & & \\ & s_1 & 0 & \\ & 0 & s_1 & \\ & & & s_1 \end{pmatrix} \in Z(\widehat{G}) \subset S[2].$$

We can now give the component groups A_ψ :

$$\begin{aligned} A_{\psi_0} &= S[2], & A_{\psi_7} &= S[2], \\ A_{\psi_2} &= S[2], & A_{\psi_6} &= S[2], \\ A_{\psi_4} &= Z(\widehat{G}), & A_{\psi_5} &= Z(\widehat{G}). \end{aligned}$$

The Arthur packets for admissible representations of $G(F)$ with infinitesimal parameter λ are

$$\begin{aligned} \Pi_{\psi_0}(G(F)) &= \{\pi(\phi_0), \pi(\phi_2, +)\}, \\ \Pi_{\psi_2}(G(F)) &= \{\pi(\phi_2, +), \pi(\phi_3, -)\}, \\ \Pi_{\psi_4}(G(F)) &= \{\pi(\phi_4, +)\}, \\ \Pi_{\psi_5}(G(F)) &= \{\pi(\phi_5)\}, \\ \Pi_{\psi_6}(G(F)) &= \{\pi(\phi_6, +), \pi(\phi_7, --)\}, \\ \Pi_{\psi_7}(G(F)) &= \{\pi(\phi_7, ++), \pi(\phi_7, --)\}, \end{aligned}$$

and the Arthur packets for admissible representations of $G_1(F)$ with infinitesimal parameter λ are

$$\begin{aligned} \Pi_{\psi_0}(G_1(F)) &= \{\pi(\phi_4, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_2}(G_1(F)) &= \{\pi(\phi_2, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_4}(G_1(F)) &= \{\pi(\phi_4, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_5}(G_1(F)) &= \{\pi(\phi_7, -+), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_6}(G_1(F)) &= \{\pi(\phi_6, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_7}(G_1(F)) &= \{\pi(\phi_7, -+), \pi(\phi_7, +-)\}. \end{aligned}$$

We arrange these representations into pure Arthur packets in Table 14.5; see also Table 14.5.

14.1.5. Aubert duality. — The following table gives Aubert duality for the admissible representations of $G(F)$ with infinitesimal parameter λ .

π	$\hat{\pi}$
$\pi(\phi_0)$	$\pi(\phi_7, ++)$
$\pi(\phi_1, +)$	$\pi(\phi_3, +)$
$\pi(\phi_2, +)$	$\pi(\phi_7, --)$
$\pi(\phi_3, +)$	$\pi(\phi_1, +)$
$\pi(\phi_3, -)$	$\pi(\phi_6, +)$
$\pi(\phi_4, +)$	$\pi(\phi_5)$
$\pi(\phi_5)$	$\pi(\phi_4, +)$
$\pi(\phi_6, +)$	$\pi(\phi_3, -)$
$\pi(\phi_7, ++)$	$\pi(\phi_0)$
$\pi(\phi_7, --)$	$\pi(\phi_2, +)$

Aubert duality for the admissible representations of $G_1(F)$ with infinitesimal parameter λ is given by the following table.

π	$\hat{\pi}$
$\pi(\phi_2, -)$	$\pi(\phi_6, -)$
$\pi(\phi_4, -)$	$\pi(\phi_7, -+)$
$\pi(\phi_6, -)$	$\pi(\phi_2, -)$
$\pi(\phi_7, +-)$	$\pi(\phi_7, +-)$

The twisting characters χ_{ψ_0} , χ_{ψ_4} , χ_{ψ_5} and χ_{ψ_7} are trivial. The twisting characters χ_{ψ_2} and χ_{ψ_6} are nontrivial, both given $(-)$, using the respective isomorphisms $A_{\psi_2} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$ and $A_{\psi_6} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$ fixed in Section 14.1.4.

14.1.6. Stable distributions and endoscopy. — The stable distributions on $G(F)$ attached to these Arthur packets are:

$$\begin{aligned}
 \Theta_{\psi_0}^G &= \text{trace } \pi(\phi_0) + \text{trace } \pi(\phi_2, +), & \Theta_{\psi_7}^G &= \text{trace } \pi(\phi_7, ++) + \text{trace } \pi(\phi_7, --), \\
 \Theta_{\psi_2}^G &= \text{trace } \pi(\phi_2, +) - \text{trace } \pi(\phi_3, -), & \Theta_{\psi_6}^G &= \text{trace } \pi(\phi_6, +) - \text{trace } \pi(\phi_7, --), \\
 \Theta_{\psi_4}^G &= \text{trace } \pi(\phi_4, +), & \Theta_{\psi_5}^G &= \text{trace } \pi(\phi_5).
 \end{aligned}$$

The characters $\langle \cdot, \pi \rangle_{\psi}$ of A_{ψ} are given in Table 14.5. With this, we easily find the coefficients $\langle ss_{\psi}, \pi \rangle_{\psi}$ in $\Theta_{\psi, s}^G$. First calculate $s_{\psi} := \psi(1, 1, -1)$:

$$\begin{aligned}
 s_{\psi_0} &= \nu_4(-1) \oplus \nu_2(-1) = s(-1, -1), & s_{\psi_7} &= \nu_4(1) \oplus \nu_2(1) = s(1, 1), \\
 s_{\psi_2} &= \nu_4(-1) \oplus \nu_2(1) = s(-1, 1), & s_{\psi_6} &= \nu_4(1) \oplus \nu_2(-1) = s(1, -1), \\
 s_{\psi_4} &= \nu_2(1) \otimes \nu_3(-1) = s(1, 1), & s_{\psi_5} &= \nu_3(1) \otimes \nu_2(-1) = s(-1, -1).
 \end{aligned}$$

Then, using the notation $s = s(s_2, s_3)$ from Section 14.1.4, we have:

$$\begin{aligned}
 \Theta_{\psi_0, s}^G &= \text{trace } \pi(\phi_0) + s_2 s_3 \text{ trace } \pi(\phi_2, +), \\
 \Theta_{\psi_2, s}^G &= \text{trace } \pi(\phi_2, +) - s_2 s_3 \text{ trace } \pi(\phi_3, -), \\
 \Theta_{\psi_4, s}^G &= \text{trace } \pi(\phi_4, +),
 \end{aligned}$$

and

$$\begin{aligned}
 \Theta_{\psi_7, s}^G &= \text{trace } \pi(\phi_7, ++) + s_2 s_3 \text{ trace } \pi(\phi_7, --), \\
 \Theta_{\psi_6, s}^G &= \text{trace } \pi(\phi_6, +) - s_2 s_3 \text{ trace } \pi(\phi_7, --), \\
 \Theta_{\psi_5, s}^G &= \text{trace } \pi(\phi_5).
 \end{aligned}$$

We now turn our attention to the distributions on $G_1(F)$ attached to these Arthur packets:

$$\begin{aligned}\Theta_{\psi_0}^{G_1} &= - \operatorname{trace} \pi(\phi_4, +) - \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_2}^{G_1} &= + \operatorname{trace} \pi(\phi_2, -) - \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_4}^{G_1} &= + \operatorname{trace} \pi(\phi_4, -) + \operatorname{trace} \pi(\phi_7, +-) \end{aligned}$$

and

$$\begin{aligned}\Theta_{\psi_7}^{G_1} &= + \operatorname{trace} \pi(\phi_7, -+) + \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_6}^{G_1} &= + \operatorname{trace} \pi(\phi_6, -) - \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_5}^{G_1} &= - \operatorname{trace} \pi(\phi_7, -+) - \operatorname{trace} \pi(\phi_7, +-) \end{aligned}$$

The characters $\langle \cdot, \pi \rangle_\psi$ of A_ψ for these representations are also given in Table 14.5. With this, we easily find the coefficients $\langle ss_\psi, \pi \rangle_\psi$ in $\Theta_{\psi, s}^{G_1}$, again using the notation $s = s(s_2, s_3)$ or $s = s(s_1, s_1)$ from Section 14.1.4 from which we deduce

$$\begin{aligned}\Theta_{\psi_0, s}^{G_1} &= -s_2 \operatorname{trace} \pi(\phi_4, +) - s_3 \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_2, s}^{G_1} &= +s_3 \operatorname{trace} \pi(\phi_2, -) - s_2 \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_4, s}^{G_1} &= +s_1 \operatorname{trace} \pi(\phi_4, -) + s_1 \operatorname{trace} \pi(\phi_7, +-) \end{aligned}$$

and

$$\begin{aligned}\Theta_{\psi_7, s}^{G_1} &= +s_2 \operatorname{trace} \pi(\phi_7, -+) + s_3 \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_6, s}^{G_1} &= +s_2 \operatorname{trace} \pi(\phi_6, -) - s_3 \operatorname{trace} \pi(\phi_7, +-) \\ \Theta_{\psi_5, s}^{G_1} &= -s_1 \operatorname{trace} \pi(\phi_7, -+) - s_1 \operatorname{trace} \pi(\phi_7, +-) \end{aligned}$$

The endoscopic group for G attached to $s = s(1, -1)$ or $s = s(-1, 1)$ is the group $G' = \mathrm{SO}(5) \times \mathrm{SO}(3)$, in which case $\Theta_{\psi, s}^G$ is the endoscopic transfer of a stable distribution $\Theta_{\psi'}^{G'}$. We write $\psi' = (\psi^{(2)}, \psi^{(1)})$ where $\psi^{(1)}$ is an Arthur parameter for $\mathrm{SO}(3)$ and $\psi^{(2)}$ is an Arthur parameter for $\mathrm{SO}(5)$. The following table gives $\psi^{(1)}$ from Section 10.1.1 and $\psi^{(2)}$ from Section 12.1.1, for each Arthur parameter ψ appearing in Section 14.1.1 that factors through ${}^L G'$.

$\S 14.1.1$ ψ	$\S 12.1.1$ $\psi^{(2)}$	$\S 10.1.1$ $\psi^{(1)}$
ψ_0	ψ_0	ψ_0
ψ_2	ψ_0	ψ_1
ψ_6	ψ_3	ψ_0
ψ_7	ψ_3	ψ_1

14.2. Vanishing cycles of perverse sheaves. —

14.2.1. Vogan variety and its conormal bundle. — The centralizer in \widehat{G} of the infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ is

$$H_\lambda := \left\{ \left(\begin{array}{c|c|c|c} h_1 & & & \\ \hline & a_2 & b_2 & \\ & c_2 & d_2 & \\ \hline & & & a_3 & b_3 \\ & & & c_3 & d_3 \\ \hline & & & & h_4 \end{array} \right) \in \widehat{G} \right\} \cong \mathrm{GL}(1) \times \mathrm{GL}(2)$$

We will write $h_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and $h_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$. Then $h_3 = h_2 \det h_2^{-1}$ and $h_4 = h_1^{-1}$, by the choice of symplectic form J at the beginning of Section 14. The Vogan varieties V_λ and V_λ^* are:

$$V_\lambda = \left\{ \left(\begin{array}{c|c|c|c} & u & v & \\ \hline & & & z & x \\ & & & y & -z \\ \hline & & & & -v \\ & & & & u \end{array} \right) \right\}, \quad V_\lambda^* = \left\{ \left(\begin{array}{c|c|c|c} u' & & & \\ \hline v' & & & \\ \hline & z' & y' & \\ & x' & -z' & \\ \hline & & & -v' & u' \end{array} \right) \right\}$$

so

$$T^*(V_\lambda) = \left\{ \left(\begin{array}{c|c|c|c} & u & v & \\ \hline u' & & & z & x \\ v' & & & y & -z \\ \hline & z' & y' & & -v \\ & x' & -z' & & u \\ \hline & & & -v' & u' \end{array} \right) \mid \begin{array}{l} u, v, x, y, z \\ u', v', x', y', z' \end{array} \right\} \subset \widehat{\mathfrak{g}}$$

The action of H_λ on V , V^* and $T^*(V_\lambda)$ is simply the restriction of the adjoint action of $H \subset \widehat{G}$ on $T^*(V) \subset \widehat{\mathfrak{g}}$. This action is given by

$$\begin{aligned} h \cdot \begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix} &= h_1 \begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix} h_2^{-1} \\ h \cdot \begin{pmatrix} z & x \\ y & -z \end{pmatrix} &= h_2 \begin{pmatrix} z & x \\ y & -z \end{pmatrix} h_3^{-1} \end{aligned}$$

and

$$\begin{aligned} h \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} &= h_2 \begin{pmatrix} u' \\ v' \end{pmatrix} h_1^{-1}, \\ h \cdot \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} &= h_3 \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix} h_2^{-1}. \end{aligned}$$

We remark that for $\mu \in \mathbb{C}$,

$$\begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix} = \mu \begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix}$$

if and only if

$$h \cdot \begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix} h = (\mu \det h_2) h \cdot \begin{pmatrix} u & v \\ z & x \\ y & -z \end{pmatrix}.$$

The H -invariant function $(\cdot | \cdot) : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ is the quadratic form

$$\left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array} \right) \mapsto 2uu' + 2vv' + xx' + yy' + 2zz'.$$

The H_λ -invariant function $[\cdot, \cdot] : T^*(V_\lambda) \rightarrow \mathfrak{h}_\lambda$ is given by

$$\left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array} \right) \mapsto \begin{aligned} & (uu' + vv')H_1 + (xx' + zz')H_2 \\ & + (yy' + zz')H_3 + (zy' - xz')E \\ & + (yz' - zx')F \end{aligned}$$

where, $\{H_1, H_2, H_3\}$ is the standard basis for the standard Cartan in $\widehat{\mathfrak{g}}$ and, with reference to $H_\lambda \subset \widehat{G}$ and $\mathfrak{h}_\lambda \subset \widehat{\mathfrak{g}}$, $\{H_1\}$, $\{H_2, H_3, E, F\}$ is the Chevalley basis for $gl(2)$ in $\mathfrak{sp}(6)$. Thus, the conormal bundle is

$$T_H^*(V_\lambda) = \left\{ \left(\begin{array}{c|cc|cc|c} & u & v & & & \\ \hline u' & & & z & x & \\ v' & & & y & -z & \\ \hline & z' & y' & & & -v \\ & x' & -z' & & & u \\ \hline & & & -v' & u' & \end{array} \right) \mid \begin{aligned} & uu' + vv' = 0 \\ & xx' + zz' = 0 \\ & yy' + zz' = 0 \\ & zy' - xz' = 0 \\ & yz' - zx' = 0 \end{aligned} \right\}$$

Note that the fibre of $(\cdot | \cdot) : V_\lambda \times V_\lambda^t \rightarrow \mathbb{A}^1$ above 0 properly contains the conormal bundle $T_H^*(V_\lambda)$ as a codimension-4 subvariety.

Although it is possible to continue to work with V and $T^*(V)$ as matrices in $\widehat{\mathfrak{g}}$ and make all the following calculations, we now switch to the perspective on Vogan varieties discussed in Section 8.2.1. This new perspective has several advantages: it is notationally less awkward, it generalises to all classical groups after unramification in the sense of Theorem 4.1.1 and it helps clarify the proper covers which play a crucial role in the calculations of the vanishing cycles that we make later in this section. Write $\widehat{\mathfrak{g}} = \mathfrak{sp}(E, J)$, so E is a six-dimensional vector space equipped with the symplectic form described in Section 14.1.1. Let E_1 be the eigenspace of $\lambda(\text{Fr})$ with eigenvalue $q^{3/2}$; let E_2 be the eigenspace of $\lambda(\text{Fr})$ with eigenvalue $q^{1/2}$; let E_3 be the eigenspace of $\lambda(\text{Fr})$ with eigenvalue $q^{-1/2}$; let E_4 be the eigenspace of $\lambda(\text{Fr})$ with eigenvalue $q^{-3/2}$. Then $\text{GL}(E_4) \times \text{GL}(E_3) \times \text{GL}(E_2) \times \text{GL}(E_1)$ acts naturally on the variety $\text{Hom}(E_3, E_4) \times \text{Hom}(E_2, E_3) \times \text{Hom}(E_1, E_2)$. If we identify E_3 with the dual space E_2^* and E_4 with E_1^* then V may be identified with the subvariety of (w_1, w_2, w_3) in $\text{Hom}(E_1, E_2) \times \text{Hom}(E_2, E_2^*) \times \text{Hom}(E_2^*, E_1^*)$ such that ${}^t w_3 = w_1$ and ${}^t w_2 = w_2$, so

$$\begin{aligned} V &\cong \{ (w, X) \in \text{Hom}(E_1, E_2) \times \text{Hom}(E_2, E_2^*) \mid {}^t X = X \} \\ &\cong \text{Hom}(E_1, E_2) \times \text{Sym}^2(E_2^*). \end{aligned}$$

The action of H on V now corresponds to the natural action of $\mathrm{GL}(E_1) \times \mathrm{GL}(E_2)$ on $\mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_2^*)$. After choosing bases for E_1 and E_2 , the conversion from the matrices in $\widehat{\mathfrak{g}}$ to pairs $(w, X) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Sym}^2(E_2^*)$ is given by

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad X := \begin{pmatrix} -x & z \\ z & y \end{pmatrix} = \begin{pmatrix} z & x \\ y & -z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will use coordinates (w, X) for V_λ when convenient. The same perspective gives coordinates (w', X') for V_λ^* where

$$w' = \begin{pmatrix} u' & v' \end{pmatrix} \quad \text{and} \quad X' := \begin{pmatrix} -x' & z' \\ z' & y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z' & y' \\ x' & -z' \end{pmatrix}.$$

In these coordinates, the action of H on V is given by

$$\begin{aligned} h \cdot w &= {}^t h_2^{-1} w {}^t h_1 & h \cdot w' &= h_1 w' h_2^{-1} \\ h \cdot X &= h_2 X {}^t h_2 & h \cdot X' &= h_2 X' {}^t h_2, \end{aligned}$$

the H -invariant function $(\cdot | \cdot) : T^*(V_\lambda) \rightarrow \mathbb{A}^1$ is given by

$$((w, X) | (w', X')) = w'w + \mathrm{trace} X'X,$$

and the H -invariant function $[\cdot, \cdot] : T^*(V_\lambda) \rightarrow \mathfrak{h}_\lambda$ is given by

$$[(w, X), (w', X')] = (w'w, X'X).$$

In particular, the conormal may be written as

$$T^*(V_\lambda) \cong \{((w, X), (w', X')) \in V \times V^* \mid w'w = 0, X'X = 0\}.$$

14.2.2. Equivariant local systems and orbit duality. — The variety V_λ is stratified into H -orbits according to the possible values of $\mathrm{rank} X$ (either 2, 1 or 0), $\mathrm{rank} {}^t w$ (either 1 or 0) and $\mathrm{rank} {}^t w X w$ (either 1 or 0). There are eight compatible values for these ranks. We now describe these eight locally closed subvarieties $C \subset V$, the singularities in the closure $\bar{C} \subset V$ and the equivariant local systems on C . For each H -orbit $C \subset V$ except the open orbit $C_7 \subset V$, the H -equivariant fundamental group of C is trivial or of order 2. So in each of these cases we use the notation $\mathbb{1}_C$ for the constant local system and \mathcal{L}_C or \mathcal{F}_C for the non-constant irreducible equivariant local system on C . (The choice of \mathcal{L}_C or \mathcal{F}_C will be explained in Section 14.2.4.)

C_0 : Closed orbit:

$$C_0 = \{0\}.$$

This corresponds to the minimal rank values

$$\mathrm{rank} X = 0, \quad \mathrm{rank} {}^t w = 0, \quad \mathrm{rank} {}^t w X w = 0.$$

This is the only closed orbit in V_λ .

C_1 : Punctured plane:

$$C_1 = \{(w, X) \in V_\lambda \mid X = 0, w \neq 0\}.$$

This corresponds to the rank values

$$\mathrm{rank} X = 0, \quad \mathrm{rank} {}^t w = 1, \quad \mathrm{rank} {}^t w X w = 0.$$

While C_1 is not affine, its closure $\bar{C}_1 = \{(w, X) \in V_\lambda \mid X = 0\}$ is \mathbb{A}^2 . This orbit is not of Arthur type. Since A_{C_1} is trivial, $\mathbb{1}_{C_1}$ is the only simple equivariant local system on C_1 .

C_2 : Smooth cone:

$$C_2 = \{(w, X) \in V_\lambda \mid \text{rank } X = 1, w = 0\}.$$

This corresponds to the rank values

$$\text{rank } X = 1, \quad \text{rank } {}^t w = 0, \quad \text{rank } {}^t w X w = 0.$$

Then C_2 is not an affine variety and the singular locus of its closure

$$\bar{C}_2 \cong \{(x, y, z) \mid xy + z^2 = 0\}$$

is precisely C_0 . We remark that $xy + z^2$ is a semi-invariant of V_λ with character $h \mapsto \det h_2^2$. Now $A_{C_2} \cong \{\pm 1\}$; let \mathcal{F}_{C_2} be the equivariant local system for the non-trivial character of A_{C_2} . Then \mathcal{F}_{C_2} coincides with the local system denoted by the same symbol in Section 13.2.3.

C_3 : The rank values

$$\text{rank } X = 2, \quad \text{rank } {}^t w = 0, \quad \text{rank } {}^t w X w = 0.$$

determine

$$C_3 = \{(w, X) \in V_\lambda \mid \text{rank } X = 2, w = 0\} \cong \{(x, y, z) \mid xy + z^2 \neq 0\}.$$

The closure of C_3 is smooth:

$$\bar{C}_3 = \{(w, X) \in V_\lambda \mid w = 0\} \cong \mathbb{A}^3.$$

This orbit is not of Arthur type. Since $A_{C_3} \cong \{\pm 1\}$, there are two simple equivariant local systems on C_3 , denoted by $\mathbb{1}_{C_3}$ and \mathcal{L}_{C_3} . Then \mathcal{L}_{C_3} coincides with the local system denoted by the same symbol in Section 13.2.3.

C_4 : The rank values

$$\text{rank } X = 1, \quad \text{rank } {}^t w = 1, \quad \text{rank } {}^t w X w = 0$$

determine

$$C_4 = \{(w, X) \in V_\lambda \mid \text{rank } X = 1, w \neq 0, Xw = 0\}.$$

The singular locus of the closure

$$\bar{C}_4 \cong \{(u, v, x, y, z) \mid xy + z^2 = 0, -xu + zv = 0 = zu + yv\}$$

is C_0 . Here, $A_{C_4} \cong \{\pm 1\}$. Let $\mathbb{1}_{C_4}$ and \mathcal{F}_{C_4} be the local systems for the trivial and non-trivial characters, respectively, of A_{C_4} .

C_5 : The rank values

$$\text{rank } X = 2, \quad \text{rank } {}^t w = 1, \quad \text{rank } {}^t w X w = 0$$

determine

$$C_5 = \{(w, X) \in V_\lambda \mid \text{rank } X = 2, w \neq 0, {}^t w X w = 0\}.$$

The closure of C_5 ,

$$\bar{C}_5 \cong \{(u, v, x, y, z) \mid -u^2 x + 2uvz + v^2 y = 0\},$$

has singular locus \bar{C}_3 . We remark that $-u^2x + 2uvz + v^2y$ is a semi-invariant of V_λ with character $h \mapsto h_1^2$. The group A_{C_5} is trivial.

C_6 : The rank values

$$\text{rank } X = 1, \quad \text{rank } {}^t w = 1, \quad \text{rank } {}^t w X w = 1$$

determine

$$C_6 = \{(w, X) \in V_\lambda \mid \text{rank } X = 1, w \neq 0, {}^t w X w \neq 0\}.$$

The singular locus of

$$\bar{C}_6 \cong \{(u, v, x, y, z) \mid xy + z^2 = 0\}$$

is \bar{C}_1 . Then $A_{C_6} \cong \{\pm 1\}$. Let $\mathbb{1}_{C_6}$ and \mathcal{F}_{C_6} be the local systems for the trivial and non-trivial characters, respectively, of A_{C_6} . The local system \mathcal{F}_{C_6} is associated to the double cover from adjoining $d^2 = -u^2x + 2uvz + v^2y$, which is isomorphic to the pullback of the double cover from \mathcal{F}_{C_2} .

C_7 : Open dense orbit:

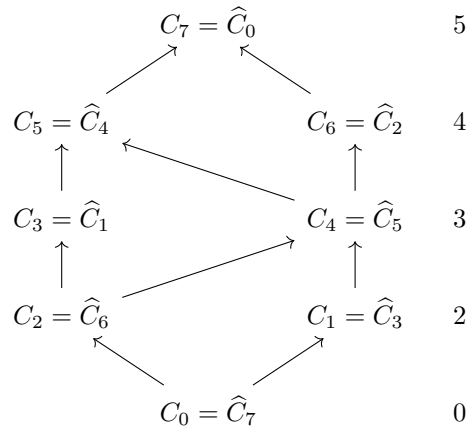
$$C_7 = \{(w, X) \in V_\lambda \mid \text{rank } X = 2, w \neq 0, {}^t w X w \neq 0\}.$$

This corresponds to the maximal rank values:

$$\text{rank } X = 2, \quad \text{rank } {}^t w = 1, \quad \text{rank } {}^t w X w = 1.$$

Now, $A_{C_7} = S[2] \cong \{\pm 1\} \times \{\pm 1\}$. Let $\mathbb{1}_{C_7}$ be the local system for the trivial character $(++)$ of A_{C_7} ; let \mathcal{L}_{C_7} be the local system for the character $(--)$ of A_{C_7} ; let \mathcal{F}_{C_7} be the local system for the character $(-+)$ of A_{C_7} ; let \mathcal{E}_{C_7} be the local system for the character $(+-)$ of A_{C_7} . Equivalently, \mathcal{L}_{C_7} is the local system on C_7 associated to the double cover $d^2 = xy + z^2$, \mathcal{F}_{C_7} is the local system associated to the double cover $d^2 = -u^2x + 2uvz + v^2y$, and \mathcal{E}_{C_7} is the local system associated to the double cover $d^2 = (xy + z^2)(-u^2x + 2uvz + v^2y)$.

Dimensions, closure relations for these eight orbits in V , and their dual orbits in V^* , are given as follows:



From this table one can find the eccentricities, as defined in Section 6.7, of these strata:

$$\begin{aligned}
e_{C_0} &= \dim C_0 + \dim C_7 - \dim V = 0 + 5 - 5 = 0 \\
e_{C_1} &= \dim C_2 + \dim C_3 - \dim V = 2 + 3 - 5 = 0 \\
e_{C_2} &= \dim C_2 + \dim C_6 - \dim V = 2 + 4 - 5 = 1 \\
e_{C_3} &= \dim C_3 + \dim C_2 - \dim V = 3 + 2 - 5 = 0 \\
e_{C_4} &= \dim C_4 + \dim C_5 - \dim V = 3 + 4 - 5 = 2 \\
e_{C_5} &= \dim C_5 + \dim C_4 - \dim V = 4 + 3 - 5 = 2 \\
e_{C_6} &= \dim C_6 + \dim C_2 - \dim V = 4 + 2 - 5 = 1 \\
e_{C_7} &= \dim C_7 + \dim C_0 - \dim V = 5 + 0 - 5 = 0
\end{aligned}$$

14.2.3. Equivariant perverse sheaves. — Table 14.2.1 shows the results of calculating $\mathcal{P}|_C$ for every simple equivariant perverse sheaf $\mathcal{IC}(C, \mathcal{L})$ and every stratum C in V . Using this, Table 14.5.6 gives the normalised geometric multiplicity matrix, m'_{geo} . Notice that m'_{geo} decomposes into block matrices of size 10×10 , 4×4 and 1×1 .

TABLE 14.2.1. Standard sheaves and perverse sheaves in $\text{Per}_{H_\lambda}(V_\lambda)$

\mathcal{P}	$\mathcal{P} _{C_0}$	$\mathcal{P} _{C_1}$	$\mathcal{P} _{C_2}$	$\mathcal{P} _{C_3}$	$\mathcal{P} _{C_4}$	$\mathcal{P} _{C_5}$	$\mathcal{P} _{C_6}$	$\mathcal{P} _{C_7}$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$\mathbb{1}_{C_0}[0]$	0	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_1})$	$\mathbb{1}_{C_0}[2]$	$\mathbb{1}_{C_1}[2]$	0	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_2})$	$\mathbb{1}_{C_0}[2]$	0	$\mathbb{1}_{C_2}[2]$	0	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathbb{1}_{C_0}[3]$	0	$\mathbb{1}_{C_2}[3]$	$\mathbb{1}_{C_3}[3]$	0	0	0	0
$\mathcal{IC}(\mathcal{L}_{C_3})$	$\mathbb{1}_{C_0}[1]$	0	0	$\mathcal{L}_{C_3}[3]$	0	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_4})$	$\mathbb{1}_{C_0}[1] \oplus \mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_1}[3]$	$\mathbb{1}_{C_2}[3]$	0	$\mathbb{1}_{C_4}[3]$	0	0	0
$\mathcal{IC}(\mathbb{1}_{C_5})$	$\mathbb{1}_{C_0}[2] \oplus \mathbb{1}_{C_0}[4]$	$\mathbb{1}_{C_1}[4]$	$\mathbb{1}_{C_2}[4]$	$\mathbb{1}_{C_3}[4] \oplus \mathcal{L}_{C_3}[4]$	$\mathbb{1}_{C_4}[4]$	$\mathbb{1}_{C_5}[4]$	0	0
$\mathcal{IC}(\mathbb{1}_{C_6})$	$\mathbb{1}_{C_0}[4]$	$\mathbb{1}_{C_1}[4]$	$\mathbb{1}_{C_2}[4]$	0	$\mathbb{1}_{C_4}[4]$	0	$\mathbb{1}_{C_6}[4]$	0
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mathbb{1}_{C_0}[5]$	$\mathbb{1}_{C_1}[5]$	$\mathbb{1}_{C_2}[5]$	$\mathbb{1}_{C_3}[5]$	$\mathbb{1}_{C_4}[5]$	$\mathbb{1}_{C_5}[5]$	$\mathbb{1}_{C_6}[5]$	$\mathbb{1}_{C_7}[5]$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$\mathbb{1}_{C_0}[3]$	$\mathbb{1}_{C_1}[3]$	0	$\mathcal{L}_{C_3}[5]$	0	$\mathbb{1}_{C_5}[5]$	0	$\mathcal{L}_{C_7}[5]$
$\mathcal{IC}(\mathcal{F}_{C_2})$	0	0	$\mathcal{F}_{C_2}[2]$	0	0	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_4})$	0	0	$\mathcal{F}_{C_2}[3]$	0	$\mathcal{F}_{C_4}[3]$	0	0	0
$\mathcal{IC}(\mathcal{F}_{C_6})$	0	0	$\mathcal{F}_{C_2}[4]$	0	$\mathcal{F}_{C_4}[4]$	0	$\mathcal{F}_{C_6}[4]$	0
$\mathcal{IC}(\mathcal{F}_{C_7})$	0	0	$\mathcal{F}_{C_2}[5]$	0	0	0	$\mathcal{F}_{C_6}[5]$	$\mathcal{F}_{C_7}[5]$
$\mathcal{IC}(\mathcal{E}_{C_7})$	0	0	0	0	0	0	0	$\mathcal{E}_{C_7}[5]$

We now give a few explicit examples of the technique, sketched in Section 8.2.3, which we used to find the local systems appearing in Table 14.2.1.

- (a) The calculations from Section 13.2.3 show how to find rows 1–5 and row 11 so here we begin with row 6.
- (b) To compute $\mathcal{IC}(\mathbb{1}_{C_4})|_C$ for every H -orbit $C \subset V$, observe that

$$\overline{C}_4 = \{(w, X) \in V_\lambda \mid {}^t w X = 0, \det(X) = 0\}.$$

Note that ${}^t w X = 0$ implies $\det(X) = 0$ provided $w \neq 0$. This variety is singular precisely when w and X are both zero; in other words, C_0 is the singular locus

of \overline{C}_4 , as we remarked in Section 14.2.1. The blowup of \overline{C}_4 at the origin is:

$$\tilde{C}_4^{(1)} := \left\{ ((w, X), [a : b]) \in V_\lambda \times \mathbb{P}^1 \mid \begin{array}{ll} (-b & a) w = 0, & (a & b) X = 0 \\ {}^t w X = 0, & \det X = 0 \end{array} \right\}.$$

Let $\pi^{(1)} : \tilde{C}_4^{(1)} \rightarrow \overline{C}_4$ be the obvious projection. In the definition of $\tilde{C}_4^{(1)}$, the first two equations imply the second two; this observation greatly simplifies checking the following claims. The cover $\pi^{(1)} : \tilde{C}_4^{(1)} \rightarrow \overline{C}_4$ is proper and the variety \tilde{C}_4 is smooth. Moreover, the fibres of $\pi^{(1)}$ have the following structure:

- above C_4 , C_2 and C_1 , $\pi^{(1)}$ is an isomorphism;
- the fibre of $\pi^{(1)}$ above C_0 is \mathbb{P}^1 .

It follows that $\pi^{(1)}$ is semi-small. By the decomposition theorem,

$$\pi_!^{(1)}(\mathbb{1}_{\tilde{C}_4^{(1)}}[3]) = \mathcal{IC}(\mathbb{1}_{C_4}).$$

By proper base change,

$$\begin{array}{llll} \mathcal{IC}(\mathbb{1}_{C_4})|_{C_4} & = & \mathbb{1}_{C_4}[3] & \mathcal{IC}(\mathbb{1}_{C_4})|_{C_2} & = & \mathbb{1}_{C_2}[3] \\ \mathcal{IC}(\mathbb{1}_{C_4})|_{C_1} & = & \mathbb{1}_{C_1}[3] & \mathcal{IC}(\mathbb{1}_{C_4})|_{C_0} & = & \mathbb{1}_{C_0}[1] \oplus \mathbb{1}_{C_0}[3], \end{array}$$

and $\mathcal{IC}(\mathbb{1}_{C_4})|_C = 0$ for all other strata C .

- (c) Next, we show how to compute $\mathcal{IC}(\mathcal{F}_{C_4})$. The singular variety \overline{C}_4 also admits a finite double cover:

$$\tilde{C}_4^{(2)} := \left\{ ((w, X), (\alpha, \beta)) \in V_\lambda \times \mathbb{A}^2 \mid \begin{array}{ll} X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}, & \begin{pmatrix} \alpha & \beta \end{pmatrix} w = 0 \\ {}^t w X = 0, & \det X = 0 \end{array} \right\}.$$

Again, the first two equations imply the second two. This variety is singular precisely when w , X , and (α, β) are all zero. Consider the pullback:

$$\begin{array}{ccccc} & & \tilde{C}_4^{(3)} & & \\ & \swarrow & \downarrow \pi^{(3)} & \searrow & \\ \tilde{C}_4^{(1)} & & & & \tilde{C}_4^{(2)} \\ & \searrow \pi^{(1)} & \downarrow & \swarrow \pi^{(2)} & \\ & & \overline{C}_4 & & \end{array}$$

Then $\tilde{C}_4^{(3)}$ is smooth and the projections onto $\tilde{C}_4^{(2)}$, $\tilde{C}_4^{(1)}$ and \overline{C}_4 are all proper.

The fibres of $\pi^{(3)} : \tilde{C}_4^{(3)} \rightarrow \overline{C}_4$ have the following structure:

- the fibre of $\pi^{(3)}$ over C_4 is the non-split double cover of C_4 ;
- the fibre of $\pi^{(3)}$ over C_2 is the non-split double cover of C_2 ;
- the fibre of $\pi^{(3)}$ over C_1 is isomorphic to C_1 ;
- the fibre of $\pi^{(3)}$ over C_0 is \mathbb{P}^1 .

It follows that $\pi^{(3)}$ is semi-small and, by the Decomposition Theorem, that:

$$\pi_!^{(3)}(\mathbb{1}_{\tilde{C}_4^{(3)}}[3]) = \mathcal{IC}(\mathbb{1}_{C_4}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}).$$

It now follows that:

$$\begin{array}{ll} \mathcal{IC}(\mathcal{F}_{C_4})|_{C_4} &= \mathcal{F}_{C_4}[3] & \mathcal{IC}(\mathcal{F}_{C_4})|_{C_2} &= \mathcal{F}_{C_2}[3] \\ \mathcal{IC}(\mathcal{F}_{C_4})|_{C_1} &= 0 & \mathcal{IC}(\mathcal{F}_{C_4})|_{C_0} &= 0. \end{array}$$

We simply list the other covers needed to calculate $\mathcal{P}|_C$ in all other cases except $\mathcal{P} = \mathcal{E}_7$, following the procedure illustrated above in the cases $\mathcal{P} = \mathcal{IC}(\mathbb{1}_{C_4})$ and $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$$\tilde{C}_5 = \left\{ ((w, X), [a : b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} a \\ b \end{pmatrix} = 0, (-b \ a) w = 0 \right\}$$

$$\tilde{C}_6^{(1)} = \left\{ ((w, X), [a : b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix} a & b \end{pmatrix} X = 0 \right\}$$

$$\tilde{C}_6^{(2)} = \left\{ ((w, X), (\alpha, \beta)) \in V \times \mathbb{A}^2 \mid X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} \right\}$$

$$\tilde{C}_7^{(1)} = \left\{ (w, X, [a : b]) \in V \times \mathbb{P}^1 \mid \begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} a \\ b \end{pmatrix} = 0 \right\}$$

$$\tilde{C}_7^{(2)} = \left\{ ((w, X), [a : b : r]) \in V \times \mathbb{P}^2 \mid \begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} a \\ b \end{pmatrix} = r^2, (-b \ a) w = 0 \right\}$$

Finally there is the most complex example: the smooth cover \tilde{V} of $\overline{C_7} = V$ needed to understand $\mathcal{IC}(\mathcal{E}_7)$. The construction of the smooth cover \tilde{V} of V proceeds by first adjoining a square root of

$$(-u^2x + 2uvz + v^2y)(xy + z^2).$$

This results in a variety which is singular on $\overline{C_4}$. After blowing up along $\overline{C_4}$ the result will still be singular along $\overline{C_3}$, so a further blow up along $\overline{C_3}$ is needed. The following steps construct \tilde{V} in detail.

- (i) Let $\tilde{V}^{(1)}$ be the blow up V along $\overline{C_4}$. This is equivalent to adding coordinates $[a : b] \in \mathbb{P}^1$ and the condition

$$\begin{pmatrix} a & b \end{pmatrix} X w = 0,$$

because the two equations $Xw = 0$ define $\overline{C_4}$.

- (ii) Let $\tilde{V}^{(2)}$ be the blow up of $\tilde{V}^{(1)}$ along $\overline{C_3}$. For this one must add coordinates $[c : d] \in \mathbb{P}^1$ with the condition

$$\begin{pmatrix} -d & c \end{pmatrix} w = 0,$$

because the equation $w = 0$ defines $\overline{C_3}$. The additional equation necessary to define the blow up is

$$\begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} c \\ d \end{pmatrix} = 0.$$

(iii) Next, we replace $[a : b]$ with $[a : b : r]$ and add the equation

$$\begin{pmatrix} a & b \end{pmatrix} X \begin{pmatrix} a \\ b \end{pmatrix} = r^2.$$

The resulting variety, $\tilde{V}^{(3)}$ has coordinates:

$$(w, X, [a : b : r], [c : d])$$

together with all the above equations. Then $\tilde{V}^{(3)}$ is a double cover of $\tilde{V}^{(2)}$ and is singular precisely when

$$X \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \text{and} \quad [a : b] = [c : d].$$

(iv) We now form the blowup \tilde{V} of $\tilde{V}^{(3)}$ along the singular locus. In order to have homogeneous equations we write our relations in the form

$$X \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = 0 \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = 0.$$

Then \tilde{V} is formed by introducing coordinates $[Y : y]$, where Y is a 2 by 2 matrix, and the conditions

$$X \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} y = Y \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix}$$

and

$$\begin{pmatrix} c \\ d \end{pmatrix} Y = 0 \quad \text{Trace}(Y) = 0.$$

Note that $[c : d]$ determines Y up to rescaling.

14.2.4. Cuspidal support decomposition and Fourier transform. — Up to conjugation, $\widehat{G} = \text{Sp}(6)$ admits three cuspidal Levi subgroups: $\widehat{G} = \text{Sp}(6)$ itself, the group $\widehat{M} = \text{Sp}(2) \times \text{GL}(1) \times \text{GL}(1)$ and the torus $\widehat{T} = \text{GL}(1) \times \text{GL}(1) \times \text{GL}(1)$. Simple objects in these three subcategories are listed below. This decomposition is responsible for the choice of symbols \mathcal{L} , \mathcal{F} and \mathcal{E} made in Section 14.2.3.

$\text{Per}_H(V)_{\widehat{T}}$		\parallel	$\text{Per}_H(V)_{\widehat{M}}$	\parallel	$\text{Per}_H(V)_{\widehat{G}}$
$\mathcal{IC}(\mathbb{1}_{C_0})$					
$\mathcal{IC}(\mathbb{1}_{C_1})$					
$\mathcal{IC}(\mathbb{1}_{C_2})$			$\mathcal{IC}(\mathcal{F}_{C_2})$		
$\mathcal{IC}(\mathbb{1}_{C_3})$	$\mathcal{IC}(\mathcal{L}_{C_3})$		$\mathcal{IC}(\mathcal{F}_{C_4})$		
$\mathcal{IC}(\mathbb{1}_{C_4})$			$\mathcal{IC}(\mathcal{F}_{C_6})$		
$\mathcal{IC}(\mathbb{1}_{C_5})$			$\mathcal{IC}(\mathcal{F}_{C_7})$		$\mathcal{IC}(\mathcal{E}_{C_7})$
$\mathcal{IC}(\mathbb{1}_{C_6})$					
$\mathcal{IC}(\mathbb{1}_{C_7})$	$\mathcal{IC}(\mathcal{L}_{C_7})$				

The Fourier transform respects the cuspidal support decomposition; see Table 14.2.2.

TABLE 14.2.2. Fourier transform

$\mathrm{Per}_{H_\lambda}(V_\lambda)$	$\xrightarrow{\mathcal{F}}$	$\mathrm{Per}_{H_\lambda}(V_\lambda^*)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_1^*})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_3^*})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_2^*})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_4^*})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_5^*})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{C_1^*})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_7^*})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{C_6^*})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_2^*})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_0^*})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_6^*})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{C_5^*})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{C_0^*})$

14.2.5. Equivariant perverse sheaves on the regular conormal bundle. — For each stratum C , we pick $(x, \xi) \in T_C^*(V)_{\mathrm{reg}}$ such that the H -orbit $T_C^*(V)_{\mathrm{sreg}}$ of (x, ξ) is open in $T_{C_i}^*(V)_{\mathrm{reg}}$. Then, we find all equivariant local systems on each $T_C^*(V)_{\mathrm{sreg}}$. The perverse extensions of these local systems to the regular conormal bundle $T_H^*(V)_{\mathrm{reg}}$ will be needed when we compute vanishing cycles of perverse sheave on V in Section 14.2.6. Here we revert to expressing V as a subvariety in $\widehat{\mathfrak{g}}$, largely for typographic reasons.

C_0 : Base point for $T_{C_0}^*(V_\lambda)_{\mathrm{sreg}}$:

$$(x_0, \xi_0) = \left(\begin{array}{c|cc|cc|c} & 0 & 0 & & & \\ \hline 1 & & & 0 & 0 & \\ 0 & & & 0 & 0 & \\ \hline & 0 & 1 & & & 0 \\ & 1 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array} \right)$$

The equivariant fundamental group is $A_{(x_0, \xi_0)} = Z_{H_\lambda}(x_0, \xi_0) = S[2]$. Thus, $T_{C_0}^*(V_\lambda)_{\mathrm{sreg}}$ carries four local systems. The following table displays how we label equivariant local systems on $T_{C_0}^*(V_\lambda)_{\mathrm{sreg}}$ by showing the matching representation of $A_{(x_0, \xi_0)}$:

$\mathrm{Loc}_{H_\lambda}(T_{C_0}^*(V_\lambda)_{\mathrm{sreg}})$	$\mathbb{1}_{\mathcal{O}_0}$	$\mathcal{L}_{\mathcal{O}_0}$	$\mathcal{F}_{\mathcal{O}_0}$	$\mathcal{E}_{\mathcal{O}_0}$
$\mathrm{Rep}(A_{(x_0, \xi_0)})$	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_0, \xi_0)} \rightarrow A_{x_0}$ induced from the projection $T_{C_0}^*(V)_{\text{sreg}} \rightarrow C_0$ is trivial; on the other hand, the map on equivariant fundamental groups $A_{(x_0, \xi_0)} \rightarrow A_{\xi_0}$ induced from the projection $T_{C_0}^*(V)_{\text{sreg}} \rightarrow C_0^* = C_7^t$ is the identity isomorphism.

$$\begin{array}{c} S[2] \\ \downarrow \text{id} \\ 1 = A_{x_0} \longleftarrow A_{(x_0, \xi_0)} \xrightarrow{\text{id}} A_{\xi_0} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Per}_H(C_0) & \rightarrow & \text{Per}_H(T_{C_0}^*(V)_{\text{reg}}) \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto & \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\ & & \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}) \end{array}$$

C_1 : Base point for $T_{C_1}^*(V_\lambda)_{\text{sreg}}$:

$$(x_1, \xi_1) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 0 & \\ 0 & & & 0 & 0 & \\ \hline & 0 & 1 & & & 0 \\ & 1 & 0 & & & 1 \\ \hline & & & 0 & 0 & \end{array} \right),$$

The equivariant fundamental group is $A_{(x_1, \xi_1)} = Z_{H_\lambda}(x_1, \xi_1) = S[2]$. Thus, $T_{C_1}^*(V_\lambda)_{\text{reg}}$ carries four local systems. The following table displays how we label equivariant local systems on $T_{C_1}^*(V_\lambda)_{\text{sreg}}$ by showing the matching representation of $A_{(x_1, \xi_1)}$:

$\text{Loc}_{H_\lambda}(T_{C_1}^*(V_\lambda)_{\text{sreg}}):$	$\mathbb{1}_{\mathcal{O}_1}$	$\mathcal{L}_{\mathcal{O}_1}$	$\mathcal{F}_{\mathcal{O}_1}$	$\mathcal{E}_{\mathcal{O}_1}$
$\text{Rep}(A_{(x_1, \xi_1)}):$	$++$	$--$	$-+$	$+-$

For use below, we remark that $\mathcal{L}_{\mathcal{O}_1}$ is the local system associated to the double cover arising from taking $\sqrt{\det X'}$.

The map on equivariant fundamental groups $A_{(x_1, \xi_1)} \rightarrow A_{x_1}$ induced from the projection $T_{C_1}^*(V)_{\text{sreg}} \rightarrow C_1$ is trivial; on the other hand, the map on equivariant fundamental groups $A_{(x_1, \xi_1)} \rightarrow A_{\xi_1}$ induced from the projection $T_{C_1}^*(V)_{\text{sreg}} \rightarrow C_1^* = C_3^t$ is $(s_2, s_3) \mapsto s_2 s_3$.

$$\begin{array}{c} S[2] \\ \downarrow \text{id} \\ 1 = A_{x_1} \longleftarrow A_{(x_1, \xi_1)} \xrightarrow{(s_2, s_3) \mapsto s_2 s_3} A_{\xi_1} = \{\pm 1\} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathrm{Loc}_H(C_1) & \rightarrow & \mathrm{Loc}_H(T_{C_1}^*(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_1} & \mapsto & \mathbb{1}_{\mathcal{O}_1} \\ & & \mathcal{L}_{\mathcal{O}_1} \\ & & \mathcal{F}_{\mathcal{O}_1} \\ & & \mathcal{E}_{\mathcal{O}_1} \end{array}$$

C_2 : Base point for $T_{C_2}^*(V)_{\mathrm{sreg}}$:

$$(x_2, \xi_2) = \left(\begin{array}{c|cc|cc|c} & 0 & 0 & & & \\ \hline 1 & & & 0 & 0 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 1 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array} \right)$$

The equivariant fundamental group is $A_{(x_2, \xi_2)} = Z_{H_\lambda}((x_2, \xi_2)) = S[2]$. Thus, $T_{C_2}^*(V_\lambda)_{\mathrm{reg}}$ carries four local systems.

$\mathrm{Loc}_{H_\lambda}(T_{C_2}^*(V_\lambda)_{\mathrm{sreg}}) :$	$\mathbb{1}_{\mathcal{O}_2}$	$\mathcal{L}_{\mathcal{O}_2}$	$\mathcal{F}_{\mathcal{O}_2}$	$\mathcal{E}_{\mathcal{O}_2}$
$\mathrm{Rep}(A_{(x_2, \xi_2)}) :$	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_2, \xi_2)} \rightarrow A_{x_2}$ induced from the projection $T_{C_2}^*(V)_{\mathrm{sreg}} \rightarrow C_2$ is given by projection to the second factor while the map on equivariant fundamental groups $A_{(x_2, \xi_2)} \rightarrow A_{\xi_2}$ induced from the projection $T_{C_2}^*(V)_{\mathrm{sreg}} \rightarrow C_2^* = C_6^t$ is projection to the first factor:

$$\begin{array}{c} S[2] \\ \downarrow \mathrm{id} \\ \{\pm 1\} = A_{x_2} \xleftarrow{s_3 \leftarrow (s_2, s_3)} A_{(x_2, \xi_2)} \xrightarrow{(s_2, s_3) \mapsto s_2} A_{\xi_2} = \{\pm 1\} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \mathrm{Loc}_H(C_2) & \rightarrow & \mathrm{Loc}_H(T_{C_2}^*(V)_{\mathrm{sreg}}) \\ \mathbb{1}_{C_2} & \mapsto & \mathbb{1}_{\mathcal{O}_2} \\ & & \mathcal{L}_{\mathcal{O}_2} \\ & & \mathcal{F}_{\mathcal{O}_2} \\ \mathcal{F}_{C_2} & \mapsto & \mathcal{E}_{\mathcal{O}_2} \end{array}$$

C_3 : Base point for $T_{C_3}^*(V)_{\mathrm{sreg}}$:

$$(x_3, \xi_3) = \left(\begin{array}{c|cc|cc|c} & 0 & 0 & & & \\ \hline 1 & & & 0 & 1 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 0 & 0 & & & 0 \\ \hline & & & 0 & 1 & \end{array} \right)$$

The equivariant fundamental group is $A_{(x_3, \xi_3)} = Z_{H_\lambda}((x_3, \xi_3)) = S[2]$. Thus, $T_{C_3}^*(V_\lambda)_{\text{reg}}$ carries four local systems.

$$\begin{array}{c} \text{Loc}_{H_\lambda}(T_{C_3}^*(V_\lambda)_{\text{reg}}) : \quad \mathbb{1}_{\mathcal{O}_3} \quad \mathcal{L}_{\mathcal{O}_3} \quad \mathcal{F}_{\mathcal{O}_3} \quad \mathcal{E}_{\mathcal{O}_3} \\ \text{Rep}(A_{(x_3, \xi_3)}) : \quad ++ \quad -- \quad -+ \quad +- \end{array}$$

The map on equivariant fundamental groups $A_{(x_3, \xi_3)} \rightarrow A_{x_3}$ induced from the projection $T_{C_3}^*(V)_{\text{reg}} \rightarrow C_2$ has kernel $Z(H)$, while $A_{(x_3, \xi_3)} \rightarrow A_{\xi_3}$ is trivial.

$$\begin{array}{c} S[2] \\ \downarrow \text{id} \\ \{\pm 1\} = A_{x_3} \xleftarrow{s_2 s_3 \leftarrow (s_2, s_3)} A_{(x_3, \xi_3)} \longrightarrow A_{\xi_3} = 1 \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C_3) & \rightarrow & \text{Loc}_H(T_{C_3}^*(V)_{\text{reg}}) \\ \mathbb{1}_{C_3} & \mapsto & \mathbb{1}_{\mathcal{O}_3} \\ \mathcal{L}_{C_3} & \mapsto & \mathcal{L}_{\mathcal{O}_0} \\ & & \mathcal{F}_{\mathcal{O}_0} \\ & & \mathcal{E}_{\mathcal{O}_0} \end{array}$$

C_4 : Base point for $T_{C_4}^*(V_\lambda)_{\text{reg}}$:

$$(x_4, \xi_4) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 0 & \\ 1 & & & 1 & 0 & \\ \hline & 1 & 0 & & & 0 \\ & 0 & -1 & & & 1 \\ \hline & & & -1 & 0 & \end{array} \right)$$

The equivariant fundamental group is $A_{(x_4, \xi_4)} = Z_{H_\lambda}((x_4, \xi_4)) = Z(\widehat{G})$. Thus, $T_{C_4}^*(V_\lambda)_{\text{reg}}$ carries two local systems.

$$\begin{array}{c} \text{Loc}_{H_\lambda}(T_{C_4}^*(V_\lambda)_{\text{reg}}) : \quad \mathbb{1}_{\mathcal{O}_4} \quad \mathcal{F}_{\mathcal{O}_4} \\ \text{Rep}(A_{(x_4, \xi_4)}) : \quad + \quad - \end{array}$$

The map on equivariant fundamental groups $A_{(x_4, \xi_4)} \rightarrow A_{x_4}$ induced from the projection $T_{C_4}^*(V)_{\text{reg}} \rightarrow C_4$ is the identity isomorphism, while $A_{(x_4, \xi_4)} \rightarrow A_{\xi_4}$ is trivial.

$$\begin{array}{c} Z(\widehat{G}) \\ \downarrow \text{id} \\ \{\pm 1\} = A_{x_4} \xleftarrow{\text{id}} A_{(x_4, \xi_4)} \longrightarrow A_{\xi_4} = 1 \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C_4) & \rightarrow & \text{Loc}_H(T_{C_4}^*(V)_{\text{reg}}) \\ \mathbb{1}_{C_4} & \mapsto & \mathbb{1}_{\mathcal{O}_4} \\ \mathcal{F}_{C_4} & \mapsto & \mathcal{F}_{\mathcal{O}_4} \end{array}$$

C_5 : Base point for $T_{C_5}^*(V_\lambda)_{\text{sreg}}$:

$$(x_5, \xi_5) = \left(\begin{array}{c|cc|c|c} & 0 & 1 & & \\ \hline 1 & & & 1 & 0 \\ 0 & & & 0 & -1 \\ \hline & 0 & 1 & & -1 \\ & 0 & 0 & & 0 \\ \hline & & & 0 & 1 \end{array} \right)$$

The equivariant fundamental group is $A_{(x_5, \xi_5)} = Z_{H_\lambda}((x_5, \xi_5)) = Z(\widehat{G})$. Thus, $T_{C_5}^*(V_\lambda)_{\text{reg}}$ carries two local systems.

$\text{Loc}_{H_\lambda}(T_{C_5}^*(V_\lambda)_{\text{sreg}})$	$\mathbb{1}_{\mathcal{O}_5}$	$\mathcal{F}_{\mathcal{O}_5}$
$\text{Rep}(A_{(x_5, \xi_5)})$	$+$	$-$

The map on equivariant fundamental groups $A_{(x_5, \xi_5)} \rightarrow A_{x_5}$ induced from the projection $T_{C_5}^*(V)_{\text{sreg}} \rightarrow C_5$ is trivial, while $A_{(x_5, \xi_5)} \rightarrow A_{\xi_5}$ is the identity isomorphism.

$$\begin{array}{c} Z(\widehat{G}) \\ \downarrow \text{id} \\ 1 = A_{x_5} \longleftarrow A_{(x_5, \xi_5)} \xrightarrow{\text{id}} A_{\xi_5} = \{\pm\} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C_5) & \rightarrow & \text{Loc}_H(T_{C_5}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_5} & \mapsto & \mathbb{1}_{\mathcal{O}_5} \\ & & \mathcal{F}_{\mathcal{O}_5} \end{array}$$

C_6 : Base point for $T_{C_6}^*(V_\lambda)_{\text{sreg}}$:

$$(x_6, \xi_6) = \left(\begin{array}{c|cc|c|c} & 1 & 0 & & \\ \hline 0 & & & 0 & 1 \\ 0 & & & 0 & 0 \\ \hline & 0 & 1 & & 0 \\ & 0 & 0 & & 1 \\ \hline & & & 0 & 0 \end{array} \right)$$

The equivariant fundamental group is $A_{(x_6, \xi_6)} = Z_{H_\lambda}((x_6, \xi_6)) = S[2]$. Thus, $T_{C_6}^*(V_\lambda)_{\text{reg}}$ carries four local systems.

$\text{Loc}_{H_\lambda}(T_{C_6}^*(V_\lambda)_{\text{sreg}})$	$\mathbb{1}_{\mathcal{O}_6}$	$\mathcal{L}_{\mathcal{O}_6}$	$\mathcal{F}_{\mathcal{O}_6}$	$\mathcal{E}_{\mathcal{O}_6}$
$\text{Rep}(A_{(x_6, \xi_6)})$	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_6, \xi_6)} \rightarrow A_{x_6}$ induced from the projection $T_{C_6}^*(V)_{\text{sreg}} \rightarrow C_6$ is given by projection to the first factor while the map on equivariant fundamental groups $A_{(x_6, \xi_6)} \rightarrow A_{\xi_6}$ induced from the

projection $T_{C_6}^*(V)_{\text{sreg}} \rightarrow C_6^* = C_2^t$ is projection to the second factor:

$$\begin{array}{c} S[2] \\ \downarrow \text{id} \\ \{\pm 1\} = A_{x_6} \xleftarrow{s_2 \leftarrow (s_2, s_3)} A_{(x_6, \xi_6)} \xrightarrow{(s_2, s_3) \mapsto s_3} A_{\xi_6} = \{\pm 1\} \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C_6) & \rightarrow & \text{Loc}_H(T_{C_6}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_6} & \mapsto & \mathbb{1}_{\mathcal{O}_6} \\ & & \mathcal{L}_{\mathcal{O}_6} \\ \mathcal{F}_{C_6} & \mapsto & \mathcal{F}_{\mathcal{O}_6} \\ & & \mathcal{E}_{\mathcal{O}_6} \end{array}$$

C_7 : Base point for $T_{C_7}^*(V_\lambda)_{\text{sreg}}$:

$$(x_7, \xi_7) = \left(\begin{array}{c|cc|cc|c} & 1 & 0 & & & \\ \hline 0 & & & 0 & 1 & \\ 0 & & & 1 & 0 & \\ \hline & 0 & 0 & & & 0 \\ & 0 & 0 & & & 1 \\ \hline & & & 0 & 0 & \end{array} \right)$$

The equivariant fundamental group is $A_{(x_7, \xi_7)} = Z_{H_\lambda}((x_7, \xi_7)) = S[2]$. Thus, $T_{C_7}^*(V_\lambda)_{\text{reg}}$ carries four local systems.

$\text{Loc}_{H_\lambda}(T_{C_7}^*(V_\lambda)_{\text{sreg}}) :$	$\mathbb{1}_{\mathcal{O}_7}$	$\mathcal{L}_{\mathcal{O}_7}$	$\mathcal{F}_{\mathcal{O}_7}$	$\mathcal{E}_{\mathcal{O}_7}$
$\text{Rep}(A_{(x_7, \xi_7)}) :$	$++$	$--$	$-+$	$+-$

The map on equivariant fundamental groups $A_{(x_7, \xi_7)} \rightarrow A_{x_7}$ induced from the projection $T_{C_7}^*(V)_{\text{sreg}} \rightarrow C_7$ is the identity, while the map on equivariant fundamental groups $A_{(x_7, \xi_7)} \rightarrow A_{\xi_7}$ induced from the projection $T_{C_7}^*(V)_{\text{sreg}} \rightarrow C_7^* = C_0^t$ is trivial.

$$\begin{array}{c} S[2] \\ \downarrow \text{id} \\ A_{x_7} \xleftarrow{\text{id}} A_{(x_7, \xi_7)} \longrightarrow A_{\xi_7} = 1 \end{array}$$

Pull-back along the bundle map:

$$\begin{array}{ccc} \text{Loc}_H(C_7) & \rightarrow & \text{Loc}_H(T_{C_7}^*(V)_{\text{sreg}}) \\ \mathbb{1}_{C_7} & \mapsto & \mathbb{1}_{\mathcal{O}_7} \\ \mathcal{L}_{C_7} & \mapsto & \mathcal{L}_{\mathcal{O}_7} \\ \mathcal{F}_{C_7} & \mapsto & \mathcal{F}_{\mathcal{O}_7} \\ \mathcal{E}_{C_7} & \mapsto & \mathcal{E}_{\mathcal{O}_7} \end{array}$$

14.2.6. Vanishing cycles of perverse sheaves. — Tables 14.2.3 and 14.2.4 record the functor ${}^p\text{Ev}$ on simple objects, from two perspectives. In this section we explain some of the calculations.

Rows 1–5 and row 11 of Table 14.2.4 follow from Section 13.2.5.

We show how to calculate row 6. First note that it follows from Proposition 6.5.1 that all of ${}^p\mathrm{Ev}_{C_3} \mathcal{IC}(\mathbb{1}_{C_4})$, ${}^p\mathrm{Ev}_{C_5} \mathcal{IC}(\mathbb{1}_{C_4})$, ${}^p\mathrm{Ev}_{C_6} \mathcal{IC}(\mathbb{1}_{C_4})$ and ${}^p\mathrm{Ev}_{C_7} \mathcal{IC}(\mathbb{1}_{C_4})$ vanish. We calculate ${}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_4})$, ${}^p\mathrm{Ev}_{C_1} \mathcal{IC}(\mathbb{1}_{C_4})$, ${}^p\mathrm{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_4})$ and ${}^p\mathrm{Ev}_{C_4} \mathcal{IC}(\mathbb{1}_{C_4})$, here.

- (a) To calculate ${}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_4})$, recall the cover $\pi_4^{(1)} : \tilde{C}_4^{(1)} \rightarrow \overline{C}_4$ from Section 14.2.3. For reasons explained in Section 8.2.6, we begin by finding the singularities of the composition $f \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\tilde{C}_4^{(1)} \times C_0^*$. The equations that define $\tilde{C}_4^{(1)} \times C_0^*$ as a subvariety of $V \times \mathbb{P}^1 \times V^*$ with coordinates $(w, X, [a : b], w', X')$ are

$$\begin{aligned} (-b \ a) w &= 0, & (a \ b) X &= 0, \\ {}^t w X &= 0, & \det(X) &= 0 \end{aligned}$$

together with the equations that define C_0^* in terms of w' and X' . The singularities of $f \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\tilde{C}_4^{(1)} \times C_0^*$, are found by examining the Jacobian for the functions taking $(w, X, [a : b], w', X')$ to

$$(-b \ a) w, \quad (a \ b) X, \quad w'w + \mathrm{trace}(X'X);$$

this Jacobian is given here:

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$

where the second and third rows correspond to the function with value $(a \ b) X$. This system of equations forms an H -bundle over \mathbb{P}^1 , so we can specialize the $[a : b]$ coordinates to $[1 : 0]$ without loss of generality. Now we can see that if the rank of this matrix is less than 4 on $\tilde{C}_4^{(1)} \times C_0^*$ then $u' = y' = 0$, which implies ${}^t w' X' w' = 0$, which contradicts $(w', X') \in C_0^*$. Therefore, $f \circ (\pi_4^{(1)} \times \mathrm{id})$ is smooth on $\tilde{C}_4^{(1)} \times C_0^*$. Now, by Lemma 6.2.2,

$$\mathrm{R}\Phi_{f \circ (\pi_4^{(1)} \times \mathrm{id})}(\mathbb{1}_{\tilde{C}_4^{(1)} \times C_0^*}) = 0.$$

By smooth base change, this implies

$${}^p\mathrm{Ev}_{C_0} \mathcal{IC}(\mathbb{1}_{C_4}) = 0.$$

- (b) The argument showing ${}^p\mathrm{Ev}_{C_1} \mathcal{IC}(\mathbb{1}_{C_4}) = 0$ is similar to (a) above. To find the singularities of $f \circ (\pi_4^{(1)} \times \mathrm{id})$ on $\tilde{C}_4^{(1)} \times C_1^*$ we simply add the equation that defines \overline{C}_1^* to the list of functions in the case above. The Jacobian for the functions

$$(-b \ a) w, \quad (a \ b) X, \quad w'w + \mathrm{trace}(X'X), \quad w' = 0,$$

is given here,

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$
0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

where, as above, rows two and three refer to $(a \ b) X$. Arguing as above, by setting $[a : b] = [1 : 0]$ we find $x = z = u = 0$. If the rank of this Jacobian were less than 6 then $u' = y' = 0$ so ${}^t w X' w = 0$, which would force the point to be non-regular in the conormal bundle. It follows that $f \circ (\pi_4^{(1)} \times \text{id})$ is smooth on the regular part of $\tilde{C}_4^{(1)} \times C_1^*$. Therefore,

$${}^p\text{Ev}_{C_1} \mathcal{IC}(\mathbb{1}_{C_4}) = 0.$$

- (c) The closed equation that cuts out $\overline{C_2^*}$ is $\text{rank } X' = 1$. Thus, to find the singular locus of $f \circ (\pi_4^{(1)} \times \text{id})$ on $\tilde{C}_4^{(1)} \times C_2^*$ we consider the functions

$$(-b \ a) w, \quad (a \ b) X, \quad w'w + \text{trace}(X'X), \quad \det X',$$

and the associated Jacobian, below.

du	dv	dx	dy	dz	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	0	v	$-u$	0	0	0	0	0
0	0	$-a$	0	b	$-x$	z	0	0	0	0	0
0	0	0	b	a	z	y	0	0	0	0	0
u'	v'	x'	y'	$2z'$	0	0	u	v	x	y	$2z$
0	0	0	0	0	0	0	0	0	y'	x'	$2z'$

If the rank of this Jacobian is not maximal, then $u' = y' = 0$, which implies ${}^t w' X' w' = 0$ which contradicts $(w', X') \in C_2^*$. Thus, $f \circ (\pi_4^{(1)} \times \text{id})$ is smooth on $\tilde{C}_4^{(1)} \times C_2^*$. It follows that

$${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathbb{1}_{C_4}) = 0.$$

- (d) The closure of $C_4 \times C_4^*$ is cut out by the equations

$$xu + zv = 0, \quad zu + yv = 0, \quad u'^2 x' + 2u'v'z' + v'^2 y'.$$

We wish to find the restriction of

$$f = xx' + 2zz' + yy' + uu' + vv'$$

to $C_4 \times C_4^*$ in local coordinates. Localize away from $u = 0$, and $v' = 0$ and note that this implies that $y \neq 0$ on $C_4 \times C_4^*$; note also that (x_4, ξ_4) lies in this open subvariety. Then

$$z = \frac{-v}{u}y, \quad x = \frac{-vz}{u} = \frac{v^2}{u^2}y, \quad y' = -\left(\frac{u'}{v'}\right)^2 x' - 2\frac{v'}{u'}z'$$

so we may rewrite

$$\begin{aligned}
f &= xx' + 2zz' + yy' + uu' + vv' \\
&= \left(\frac{v}{u}\right)^2 yx' - 2\frac{v}{u}yz' - y\left(\frac{u'}{v'}\right)^2 x' + 2\frac{v'}{u'}z + uu' + vv' \\
&= y\left(\left(\frac{v}{u}\right)^2 - \left(\frac{u'}{v'}\right)^2 x' - 2\left(\frac{v}{u} + \frac{u'}{v'}\right)z'\right) + uu' + vv' \\
&= \frac{1}{u^2v'^2}(uu' + vv')\left((vv' - uu')x'y - 2uv'z'y + u^2v'^2\right)
\end{aligned}$$

This gives us f expressed in the form cXY where c is non-vanishing and non-singular, X and Y are both non-singular on $C_4 \times C_4^*$ (for X this is because the differential of u' is non-zero for Y it is because the differential of z' is non-zero). It follows from Corollary 6.2.6 that the vanishing cycles functor evaluates to a constant sheaf, so

$${}^p\mathrm{Ev}_{C_4} \mathcal{IC}(\mathbb{1}_{C_4}) = \mathcal{IC}(\mathbb{1}_{C_4}).$$

This completes the calculations needed for row 6 of Table 14.2.4.

We show how to compute row 12. As recalled in Section 8.2.6, ${}^p\mathrm{Ev}_C \mathcal{IC}(\mathcal{F}_{C_4}) = 0$ unless $C \subset \overline{C}_4$, and ${}^p\mathrm{Ev}_{C_4} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{C_4})$; see Section 14.2.5. So here we determine ${}^p\mathrm{Ev}_{C_i} \mathcal{IC}(\mathcal{F}_{C_4})$ for $i = 0, 1, 2$. Recall the cover $\pi_4^{(3)} : \tilde{C}_4^{(3)} \rightarrow \overline{C}_4$ from Section 14.2.3. As above, we begin by finding the singularities of the composition $f \circ (\pi_4^{(3)} \times \mathrm{id})$ on $\tilde{C}_4^{(3)} \times C_i^*$. The equations that define $\tilde{C}_4^{(3)} \times C_i^*$ as a subvariety of $V \times \mathbb{A}^2 \times \mathbb{P}^1 \times V^*$ with coordinates $(w, X, A, B, [a : b], w', X')$ are

$$\begin{aligned}
(a \ b) \begin{pmatrix} A \\ B \end{pmatrix} &= 0, & (a \ b) X &= 0, \\
(-b \ a) w &= 0, & X &= (A \ B) \begin{pmatrix} A \\ B \end{pmatrix}, \\
(-B \ A) w &= 0, \quad {}^t w X = 0, & \det(X) &= 0,
\end{aligned}$$

together with the equations that define C_i^* in terms of w' and X' . The conormal bundle to this variety is generated by the differentials of the functions

$$(a \ b) \begin{pmatrix} A \\ B \end{pmatrix}, \quad (-b \ a) w,$$

together with the equations that define \overline{C}_i^* . We find the singular locus of $f \circ (\pi_4^{(3)} \times \mathrm{id})$ on $\tilde{C}_4^{(3)} \times C_i^*$ by checking the rank of the Jacobian of these functions. This will determine the support of the sheaf

$$(158) \quad \mathrm{R}\Phi_{f \circ (\pi_4^{(3)} \times \mathrm{id})}(\mathbb{1}_{\tilde{C}_4^{(3)} \times C_i^*}).$$

If $f \circ (\pi_4^{(3)} \times \mathrm{id})$ is smooth on $\tilde{C}_4^{(3)} \times C_i^*$ or if the restriction of this sheaf to the preimage of $(\overline{C}_4 \times C_i^*)_{\mathrm{reg}}$ under $\pi_4^{(3)} \times \mathrm{id}$ is 0, then $\mathrm{Ev}_{C_i} \mathcal{F}_{C_4} = 0$. However, if the restriction of (158) to the preimage of $(\overline{C}_4 \times C_i^*)_{\mathrm{reg}}$ under $\pi_4^{(3)} \times \mathrm{id}$ is not 0, then to determine ${}^p\mathrm{Ev}_{C_i} \mathcal{IC}(\mathcal{F}_{C_4})$ we must calculate the pushforward of this restriction

along the proper morphism $\pi_4^{(3)} \times \text{id}$ (and in principle eliminate any contribution from ${}^p\text{Ev}_{C_i}(\mathbb{1}_{C_4})$, however in each of the following three cases there is none). We now show the remaining calculations for row 12.

- (e) To find the support of (158) when $C_i = C_2$, we consider the differentials of the following functions.

$$(-b \ a)w, \quad (a \ b) \begin{pmatrix} A \\ B \end{pmatrix}, \quad w'w + \text{Tr}(X'X), \quad \det X'.$$

This gives the following Jacobian, in which we hide x, y and z since we have $x = -A^2$, $z = AB$ and $y = B^2$. In this table the rows are the differentials of the above functions, in that order, and to save space, we set $A' := -Ax' + Bz'$ and $B' := Az' + By'$:

du	dv	dA	dB	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0	0	0
0	0	a	b	A	B	0	0	0	0	0
u'	v'	$2A'$	$2B'$	0	0	u	v	$-A^2$	B^2	$2AB$
0	0	0	0	0	0	0	0	y'	x'	$2z'$

Again we observe that this system of equations is an H -bundle over \mathbb{P}^1 and therefore we can set $[a : b] = [1 : 0]$ without loss of generality. If we do this we find $v = x = z = A = 0$. Moreover, if we suppose that the rank is not maximal, then $u' = 0$ by inspecting the first four columns and $y' = 0$ by inspecting the fourth column. This implies ${}^t w' X' w' = 0$ with contradicts $(w', X') \in C_2^*$. Thus, the singular locus of $f \circ (\pi_4^{(3)} \times \text{id})$ on $\tilde{C}_4^{(3)} \times C_2^*$ is empty. It follows that

$${}^p\text{Ev}_{C_2} \mathcal{IC}(\mathcal{F}_4) = 0.$$

- (f) To find the support of (158) when $C_i = C_1$, we consider the differentials of the following functions.

$$(-b \ a)w, \quad (a \ b) \begin{pmatrix} A \\ B \end{pmatrix}, \quad \text{trace}(X'X).$$

In this case we have $u' = v' = 0$, so they may be omitted, and so the relevant Jacobian is:

du	dv	dA	dB	da	db	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0
0	0	a	b	A	B	0	0	0
0	0	$2A'$	$2B'$	0	0	$-A^2$	B^2	$2AB$

where, as above, we set $A' := -Ax' + Bz'$ and $B' := Az' + By'$. On $\tilde{C}_4^{(3)} \times C_1^*$ we find that the singular locus of $f \circ (\pi_4^{(3)} \times \text{id})$ is cut out by

$$A = B = 0, \quad (-b \ a)w = 0.$$

This is already sufficient to conclude that ${}^p\text{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) \neq 0$. Since we only need to compute the vanishing cycles (158) over the regular part of the conormal bundle, we may assume $w \neq 0$. We claim that local coordinates for the regular

part of the conormal bundle are given by (X', w) . Indeed, the coordinate $[a : b]$ is determined by w and all other coordinates are zero on the singular locus. It follows from this that the map from the singular locus to $T_C^*(V_\lambda)_{\text{reg}}$ is one-to-one. Moreover, we are free to localize away from the exceptional divisor of the blowup and thus essentially ignore $[a : b]$ while computing the vanishing cycles. Doing this, we can give new coordinates for our variety by setting

$$\begin{pmatrix} A \\ B \end{pmatrix} = cw$$

for some new coordinate c . That is, on this open we have local coordinates u, v, c, x', y', z' , with no relations, and we wish to compute

$$R\Phi_{c^2(-u^2x'+2uvz'+v^2z')}(\mathbb{1}).$$

The function $-u^2x' + 2uvz' + v^2z'$ is smooth and non-vanishing on the regular part of the conormal bundle, so by setting $h = -u^2x' + 2uvz' + v^2z'$, we may consider the smooth map on our open subvariety induced from the map $\mathbb{A}^6 \rightarrow \mathbb{A}^2$ given on coordinates by $(u, v, c, x', y', z') \mapsto (c, h)$. By smooth base change $R\Phi_{c^2(-u^2x'+2uvz'+v^2z')}(\mathbb{1})$ is the pullback of $R\Phi_{c^2h}(\mathbb{1})$. It follows from Lemma 6.2.4 that $R\Phi_{c^2h}(\mathbb{1})$ is the skyscraper sheaf on $c = 0$ associated to the cover arising from taking the square root of h . Pulling this back, we have the same. This is the cover associated to the sheaf $\mathcal{F}_{\mathcal{O}_1}$ in Section 14.2.5, so

$${}^p\text{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}).$$

- (g) To find the support of (158) when $C_i = C_0$, we consider the differentials of the following functions.

$$\begin{pmatrix} -b & a \end{pmatrix} w, \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad w'w + \text{Tr}(X'X).$$

This determines the following Jacobian, in which we again use the notation $A' := -Ax' + Bz'$ and $B' := Az' + By'$:

du	dv	dA	dB	da	db	du'	dv'	dx'	dy'	dz'
$-b$	a	0	0	v	$-u$	0	0	0	0	0
0	0	a	b	A	B	0	0	0	0	0
u'	v'	$2A'$	$2B'$	0	0	u	v	$-A^2$	B^2	$2AB$

The singular locus of $f \circ (\pi_4^{(3)} \times \text{id})$ on $\tilde{C}_4^{(3)} \times C_0^*$ is

$$u = v = A = B = 0, \quad \begin{pmatrix} a & b \end{pmatrix} w' = 0.$$

Note, this is already sufficient to conclude that ${}^p\text{Ev}_{C_1} \mathcal{IC}(\mathcal{F}_{C_4}) \neq 0$. We may assume $w' \neq 0$, since we only need to compute (158) the vanishing cycles over the regular part of the conormal bundle. Local coordinates for the conormal bundle are now given by (w', X') . Since $[a : b]$ is determined by w' , and all other coordinates are zero, it follows that the map from the singular locus to $T_C^*(V)_{\text{reg}}$ is one-to-one. In the following, wherever we write (a, b) you should interpret this as either $(1, b)$ or $(a, 1)$ as though we were working in one of the two charts for \mathbb{P}^1 .

We pick new local coordinates in a neighbourhood of the singular locus: these will be $[a : b], c, d, X', w'$ with the change of coordinates given by $(A, B) = c(-b, a)$ and $(u, v) = d(a, b)$. The function $ww' + \text{trace}(XX')$ may now be re-written in the form

$$d \begin{pmatrix} a & b \end{pmatrix} w' + c^2 \begin{pmatrix} -b & a \end{pmatrix} X' \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The functions $h = \begin{pmatrix} a & b \end{pmatrix} w'$ and $g = \begin{pmatrix} -b & a \end{pmatrix} X' \begin{pmatrix} -b \\ a \end{pmatrix}$ are smooth (on the regular part of the conormal bundle). We may thus consider the map to \mathbb{A}^4 induced by:

$$([a : b], c, d, X', w') \mapsto (c, d, h, g)$$

The map $ww' + \text{trace}(XX')$ is simply the pullback of $dh + c^2g$. Thus, if we can compute $R\Phi_{dh+c^2g}(\mathbb{1})$ on \mathbb{A}^4 , by smooth base change this will give us $R\Phi_{w'w+\text{trace}(X'X)}(\mathbb{1})$ over the regular part of the conormal bundle. By Proposition 6.2.5, we see that $R\Phi_{df+c^2g}(\mathbb{1})$ is the skyscraper sheaf over $d = h = c = 0$ associated to the cover coming from adjoining the square root of g . Pulling this back to $\widetilde{C}_4^{(3)} \times C_0^*$ and identifying the singular locus with the regular part of the conormal, we conclude that

$${}^p\text{Ev}_{C_0} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$$

by comparing the covers associated to the local systems in Section 14.2.5.

- (h) The computations for ${}^p\text{Ev}_{C_4} \mathcal{IC}(\mathcal{F}_{C_4})$ are essentially the same as those for ${}^p\text{Ev}_{C_4} \mathcal{IC}(\mathbb{1}_{C_4})$. While working on the cover one has $x = A^2, y = B^2, z = AB$, the result is that one finds

$$\begin{aligned} f &= xx' + 2zz' + yy' + uu' + vv' \\ &= \frac{1}{u^2v'^2} (uu' + vv') \left((vv' - uu') x' B^2 - 2uv' z' B^2 + u^2 v'^2 \right). \end{aligned}$$

Taking the proper pushforward we obtain a direct sum of two sheaves, however as ${}^p\text{Ev}_{C_4} \mathcal{IC}(\mathbb{1}_{C_4})$ was the constant sheaf, we realize ${}^p\text{Ev}_{C_4} \mathcal{IC}(\mathcal{F}_{C_4})$ will be the non-trivial factor.

$${}^p\text{Ev}_{C_4} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}).$$

14.2.7. Normalization of Ev and the twisting local system. — Using Table 14.2.4 we find our second case when the equivariant local system \mathcal{T} is non-trivial:

$$\mathcal{T} = \mathbb{1}_{\mathcal{O}_0}^\sharp \oplus \mathbb{1}_{\mathcal{O}_1}^\sharp \oplus \mathcal{L}_{\mathcal{O}_2}^\sharp \oplus \mathbb{1}_{\mathcal{O}_3}^\sharp \oplus \mathbb{1}_{\mathcal{O}_4}^\sharp \oplus \mathbb{1}_{\mathcal{O}_5}^\sharp \oplus \mathcal{L}_{\mathcal{O}_6}^\sharp \oplus \mathbb{1}_{\mathcal{O}_7}^\sharp.$$

We use \mathcal{T} in Table 14.5.7 to calculate ${}^p\text{NEv} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$; compare with Table 14.2.4

14.2.8. Vanishing cycles and the Fourier transform. — Compare Table 14.2.6 with the Fourier transform from Section 14.2.4 to confirm (141) in this example.

14.2.9. Arthur sheaves. — Arthur perverse sheaves in $\text{Per}_{H_\lambda}(V_\lambda)$, decomposed into pure packet sheaves and conormal perverse sheaves, are displayed in Table 14.2.7.

TABLE 14.2.3. ${}^p\text{Ev} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 14. See also Table 14.2.4.

$\text{Per}_H(V)$	$\xrightarrow{{}^p\text{Ev}}$	$\text{Per}_H(T_H^*(V)_{\text{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_4})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_5})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_7})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4})$ $\oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})$

14.3. ABV-packets. —

14.3.1. Admissible representations versus perverse sheaves. — Using Vogan’s bijection between $\text{Per}_{H_\lambda}(V_\lambda)_{\text{iso}}^{\text{simple}}$ and $\Pi_{\text{pure},\lambda}(G/F)$ as discussed in Section 8.3.1, we now match the 8 Langlands parameters from Section 14.1.1 with the 8 strata from Section 14.2.1 and the 15 admissible representations from Section 14.1.2 with the 15 perverse sheaves from Section 14.2.3; see Table 14.3.1.

14.3.2. ABV-packets. — Using the bijection from Section 14.3.1 and the calculation of the functor Ev from Section 14.2.6, we now easily find the ABV-packets $\Pi_{\text{pure},\phi}^{\text{ABV}}(G/F)$ for Langlands parameters ϕ with infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$, using Section 8.3.2. We record the stable distributions η_ψ^{NEv} arising from ABV-packets through our calculations in Table 14.3.2. We will examine the invariant distributions $\eta_{\psi,s}^{\text{NEv}}$, later.

14.3.3. Kazhdan-Lusztig conjecture. — Using the bijection of Section 14.1.4, we compare the multiplicity matrix from Section 14.1.3 with the normalised geometric

TABLE 14.2.4. $\mathbf{Evs} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{sreg}})$ on simple objects; see also Table 14.2.5. Here we use the notation $\mathbf{Evs}_i := \mathbf{Evs}_{C_i}$.

\mathcal{P}	$\mathbf{Evs}_0 \mathcal{P}$	$\mathbf{Evs}_1 \mathcal{P}$	$\mathbf{Evs}_2 \mathcal{P}$	$\mathbf{Evs}_3 \mathcal{P}$	$\mathbf{Evs}_4 \mathcal{P}$	$\mathbf{Evs}_5 \mathcal{P}$	$\mathbf{Evs}_6 \mathcal{P}$	$\mathbf{Evs}_7 \mathcal{P}$
$\mathcal{I}(\mathbb{1}_{C_0})$	++	0	0	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_1})$	0	++	0	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_2})$	--	0	--	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_3})$	0	0	0	++	0	0	0	0
$\mathcal{I}(\mathcal{L}_{C_3})$	0	0	++	--	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_4})$	0	0	0	0	+	0	0	0
$\mathcal{I}(\mathbb{1}_{C_5})$	0	0	0	0	0	+	0	0
$\mathcal{I}(\mathbb{1}_{C_6})$	0	--	0	0	0	0	--	0
$\mathcal{I}(\mathbb{1}_{C_7})$	0	0	0	0	0	0	0	++
$\mathcal{I}(\mathcal{L}_{C_7})$	0	0	0	0	0	0	++	--
$\mathcal{I}(\mathcal{F}_{C_2})$	0	0	+-	0	0	0	0	0
$\mathcal{I}(\mathcal{F}_{C_4})$	-+	-+	0	0	-	0	0	0
$\mathcal{I}(\mathcal{F}_{C_6})$	0	0	0	0	0	0	+-	0
$\mathcal{I}(\mathcal{F}_{C_7})$	0	0	0	-+	0	-	0	-+
$\mathcal{I}(\mathcal{E}_{C_7})$	+-	-+	+-	-+	-	-	-+	+-

multiplicity matrix from Section 14.2.3. Since ${}^t m_{\text{rep}} = m'_{\text{geo}}$, this confirms the Kazhdan-Lusztig conjecture in this example.

14.3.4. Aubert duality and Fourier transform. — To verify (150), use Vogan’s bijection from Section 14.3.1 to compare Aubert duality from Section 14.1.5 with the Fourier transform from Section 14.2.4.

14.3.5. Normalisation. — To verify (152), compare the twisting characters χ_ψ of A_ψ from Section 14.1.5 with the restriction \mathcal{T}_ψ to $T_{C_\psi}^*(V)_{\text{reg}}$ of the \mathcal{T} from Section 14.2.8. In both cases the character is trivial except on A_{ψ_2} and A_{ψ_6} , where it is the character $(--)$, using notation from Section 14.1.4. Using this notation, here is another perspective on \mathcal{T} , where for each $C \subseteq V$ we display the corresponding character of A_C^{mic} . As a provocation, we also display the parity of the eccentricities of the orbits C .

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
\mathcal{T}_C	++	++	--	++	+	+	--	++
$(-1)^{e_C}$	1	1	-1	1	1	1	-1	1

14.3.6. ABV-packets that are not Arthur packets. — We conclude Section 14.3 by drawing attention to the two ABV-packets $\Pi_{\text{pure}, \phi_1}^{\text{ABV}}(G/F)$ and $\Pi_{\text{pure}, \phi_3}^{\text{ABV}}(G/F)$ that are not Arthur packets, as ϕ_1 and ϕ_3 are not of Arthur type. While the following two admissible homomorphisms $L_F \times \text{SL}(2, \mathbb{C}) \rightarrow {}^L G$ are not Arthur parameters because

TABLE 14.2.5. ${}^p\text{NEv} : \text{Per}_{H_\lambda}(V_\lambda) \rightarrow \text{Per}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{reg}})$ on simple objects, for $\lambda : W_F \rightarrow {}^L G$ given at the beginning of Section 14. See also Table 14.5.7

$\text{Per}_H(V)$	$\xrightarrow{{}^p\text{NEv}}$	$\text{Per}_H(T_H^*(V)_{\text{reg}})$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_4})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_5})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_1})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	\mapsto	$\mathcal{IC}(\mathbb{1}_{\mathcal{O}_7})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{L}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_2})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_6})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{F}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	\mapsto	$\mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_4})$ $\oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_2}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0})$

they are not bounded on W_F ,

$$\begin{aligned}\psi_1(w, x, y) &:= \nu_2(y) \oplus (\nu_2^2(d_w) \otimes \nu_2(x)), \\ \psi_3(w, x, y) &:= \nu_2(x) \oplus (\nu_2^2(d_w) \otimes \nu_2(y)),\end{aligned}$$

they do behave like Arthur parameters in other regards, as we now explain. First $\phi_{\psi_1} = \phi_1$ and $\phi_{\psi_3} = \phi_3$. We note too that ψ_3 is the Aubert dual of ψ_1 . Let us define

$$\Pi_{\text{pure}, \psi_1}(G/F) := \Pi_{\text{pure}, \phi_1}^{\text{ABV}}(G/F) \quad \text{and} \quad \Pi_{\text{pure}, \psi_3}(G/F) := \Pi_{\text{pure}, \phi_3}^{\text{ABV}}(G/F).$$

Then $\Pi_{\text{pure}, \psi_1}(G/F)$ and $\Pi_{\text{pure}, \psi_3}(G/F)$ define the following pseudo-Arthur packets for G_1 and G :

$$\begin{aligned}\Pi_{\psi_1}(G(F)) &:= \{\pi(\phi_1), \pi(\phi_6, +)\}, \\ \Pi_{\psi_3}(G(F)) &:= \{\pi(\phi_3, +), \pi(\phi_3, -)\},\end{aligned}$$

and

$$\begin{aligned}\Pi_{\psi_1}(G_1(F)) &:= \{\pi(\phi_4, -), \pi(\phi_7, +-)\}, \\ \Pi_{\psi_3}(G_1(F)) &:= \{\pi(\phi_7, -+), \pi(\phi_7, +-)\}.\end{aligned}$$

TABLE 14.2.6. Comparing this table with Table 14.2.2 verifies (141) in this example. Recall the notation $\mathcal{O}_i := T_{C_i}^*(V)_{\text{sreg}}$ and that $\mathcal{L}_{\mathcal{O}_i}^{\natural}$ denotes the extension by zero of $\mathcal{L}_{\mathcal{O}_i}^{\natural}$ from $T_{C_i}^*(V)_{\text{sreg}}$ to $T_H^*(V)_{\text{sreg}}$.

$\text{Per}_H(V)$	$\xrightarrow{\text{NE}_6}$	$\text{Loc}_H(T_H^*(V)_{\text{sreg}})$	$\xrightarrow{a_*}$	$\text{Loc}_H(T_H^*(V^*)_{\text{sreg}})$	$\xleftarrow{\text{E}_6^*}$	$\text{Per}_H(V^*)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	\mapsto	$\mathbb{1}_{\mathcal{O}_0}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_0^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_1})$	\mapsto	$\mathbb{1}_{\mathcal{O}_1}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_1^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_1^*})$
$\mathcal{IC}(\mathbb{1}_{C_2})$	\mapsto	$\mathbb{1}_{\mathcal{O}_2}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_0}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_2^*}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_0^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{L}_{C_0^*})$
$\mathcal{IC}(\mathbb{1}_{C_3})$	\mapsto	$\mathbb{1}_{\mathcal{O}_3}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_3^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_3^*})$
$\mathcal{IC}(\mathcal{L}_{C_3})$	\mapsto	$\mathcal{L}_{\mathcal{O}_3}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_2}^{\natural}$	\mapsto	$\mathcal{L}_{\mathcal{O}_3^*}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_2^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_2^*})$
$\mathcal{IC}(\mathbb{1}_{C_4})$	\mapsto	$\mathbb{1}_{\mathcal{O}_4}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_4^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_4^*})$
$\mathcal{IC}(\mathbb{1}_{C_5})$	\mapsto	$\mathbb{1}_{\mathcal{O}_5}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_5^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_5^*})$
$\mathcal{IC}(\mathbb{1}_{C_6})$	\mapsto	$\mathbb{1}_{\mathcal{O}_6}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_1}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_6^*}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_1^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{L}_{C_1^*})$
$\mathcal{IC}(\mathbb{1}_{C_7})$	\mapsto	$\mathbb{1}_{\mathcal{O}_7}^{\natural}$	\mapsto	$\mathbb{1}_{\mathcal{O}_7^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_7^*})$
$\mathcal{IC}(\mathcal{L}_{C_7})$	\mapsto	$\mathcal{L}_{\mathcal{O}_7}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_6}^{\natural}$	\mapsto	$\mathcal{L}_{\mathcal{O}_7^*}^{\natural} \oplus \mathcal{L}_{\mathcal{O}_6^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathbb{1}_{C_6^*})$
$\mathcal{IC}(\mathcal{F}_{C_2})$	\mapsto	$\mathcal{E}_{\mathcal{O}_2}^{\natural}$	\mapsto	$\mathcal{E}_{\mathcal{O}_2^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{F}_{C_2^*})$
$\mathcal{IC}(\mathcal{F}_{C_4})$	\mapsto	$\mathcal{F}_{\mathcal{O}_4}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_1}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_0}^{\natural}$	\mapsto	$\mathcal{F}_{\mathcal{O}_4^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_1^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_0^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{F}_{C_0^*})$
$\mathcal{IC}(\mathcal{F}_{C_6})$	\mapsto	$\mathcal{F}_{\mathcal{O}_6}^{\natural}$	\mapsto	$\mathcal{F}_{\mathcal{O}_6^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{F}_{C_6^*})$
$\mathcal{IC}(\mathcal{F}_{C_7})$	\mapsto	$\mathcal{F}_{\mathcal{O}_7}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_5}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_3}^{\natural}$	\mapsto	$\mathcal{F}_{\mathcal{O}_7^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_5^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_3^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{F}_{C_5^*})$
$\mathcal{IC}(\mathcal{E}_{C_7})$	\mapsto	$\mathcal{E}_{\mathcal{O}_7}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_6}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_5}^{\natural}$ $\oplus \mathcal{E}_{\mathcal{O}_4}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_3}^{\natural} \oplus$ $\mathcal{F}_{\mathcal{O}_2}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_1}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_0}^{\natural}$	\mapsto	$\mathcal{E}_{\mathcal{O}_7^*}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_6^*}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_5^*}^{\natural}$ $\oplus \mathcal{E}_{\mathcal{O}_4^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_3^*}^{\natural} \oplus$ $\mathcal{F}_{\mathcal{O}_2^*}^{\natural} \oplus \mathcal{F}_{\mathcal{O}_1^*}^{\natural} \oplus \mathcal{E}_{\mathcal{O}_0^*}^{\natural}$	\leftarrow	$\mathcal{IC}(\mathcal{E}_{C_0^*})$

TABLE 14.2.7. Arthur sheaves

Arthur sheaves	pure packet sheaves	coronal sheaves
\mathcal{A}_{C_0}	$\mathcal{IC}(\mathbb{1}_{C_0}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_1}	$\mathcal{IC}(\mathbb{1}_{C_1}) \oplus$	$\mathcal{IC}(\mathbb{1}_{C_6}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_2}	$\mathcal{IC}(\mathbb{1}_{C_2}) \oplus \mathcal{IC}(\mathcal{F}_{C_2}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_3}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_3}	$\mathcal{IC}(\mathbb{1}_{C_3}) \oplus \mathcal{IC}(\mathcal{L}_{C_3}) \oplus$	$\mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_4}	$\mathcal{IC}(\mathbb{1}_{C_4}) \oplus \mathcal{IC}(\mathcal{F}_{C_4}) \oplus$	$\mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_5}	$\mathcal{IC}(\mathbb{1}_{C_5}) \oplus$	$\mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_6}	$\mathcal{IC}(\mathbb{1}_{C_6}) \oplus \mathcal{IC}(\mathcal{F}_{C_6}) \oplus$	$\mathcal{IC}(\mathcal{L}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$
\mathcal{A}_{C_7}	$\mathcal{IC}(\mathbb{1}_{C_7}) \oplus \mathcal{IC}(\mathcal{L}_{C_7}) \oplus \mathcal{IC}(\mathcal{F}_{C_7}) \oplus \mathcal{IC}(\mathcal{E}_{C_7})$	

TABLE 14.3.1. Bijection between equivalence classes of irreducible admissible representations of pure rational forms of $\mathrm{SO}(7)$ and isomorphism classes of simple perverse sheaves on V_λ

$\mathrm{Per}_{H_\lambda}(V_\lambda)_{\mathrm{iso}}^{\mathrm{simple}}$	$\Pi_{\mathrm{pure},\lambda}(G/F)$
$\mathcal{IC}(\mathbb{1}_{C_0})$	$[\pi(\phi_0), 0]$
$\mathcal{IC}(\mathbb{1}_{C_1})$	$[\pi(\phi_1), 0]$
$\mathcal{IC}(\mathbb{1}_{C_2})$	$[\pi(\phi_2, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_3})$	$[\pi(\phi_3, +), 0]$
$\mathcal{IC}(\mathcal{L}_{C_3})$	$[\pi(\phi_3, -), 0]$
$\mathcal{IC}(\mathbb{1}_{C_4})$	$[\pi(\phi_4, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_5})$	$[\pi(\phi_5), 0]$
$\mathcal{IC}(\mathbb{1}_{C_6})$	$[\pi(\phi_6, +), 0]$
$\mathcal{IC}(\mathbb{1}_{C_7})$	$[\pi(\phi_7, ++), 0]$
$\mathcal{IC}(\mathcal{L}_{C_7})$	$[\pi(\phi_7, --), 0]$
$\mathcal{IC}(\mathcal{F}_2)$	$[\pi(\phi_2, -), 1]$
$\mathcal{IC}(\mathcal{F}_4)$	$[\pi(\phi_4, -), 1]$
$\mathcal{IC}(\mathcal{F}_6)$	$[\pi(\phi_6, -), 1]$
$\mathcal{IC}(\mathcal{F}_7)$	$[\pi(\phi_7, -+), 1]$
$\mathcal{IC}(\mathcal{E}_7)$	$[\pi(\phi_7, +-), 1]$

TABLE 14.3.2. Stable virtual representations arising from ABV-packets

ABV-packets	pure L-packet representations	coronal representations
$\eta_{\psi_0}^{\mathrm{NE}}$	$+\pi(\phi_0)$	$+\pi(\phi_2, +) + \pi(\phi_4, -) + \pi(\phi_7, +-)$
$\eta_{\phi_1}^{\mathrm{NE}}$	$+\pi(\phi_1)$	$+\pi(\phi_4, -) + \pi(\phi_6, +) + \pi(\phi_7, +-)$
$\eta_{\psi_2}^{\mathrm{NE}}$	$+\pi(\phi_2, +) - \pi(\phi_2, -)$	$-\pi(\phi_3, -) + \pi(\phi_7, +-)$
$\eta_{\phi_3}^{\mathrm{NE}}$	$+\pi(\phi_3, +) + \pi(\phi_3, -)$	$-\pi(\phi_7, -+) - \pi(\phi_7, +-)$
$\eta_{\psi_4}^{\mathrm{NE}}$	$+\pi(\phi_4, +) - \pi(\phi_4, -)$	$-\pi(\phi_7, +-)$
$\eta_{\psi_5}^{\mathrm{NE}}$	$+\pi(\phi_5)$	$+\pi(\phi_7, -+) + \pi(\phi_7, +-)$
$\eta_{\psi_6}^{\mathrm{NE}}$	$+\pi(\phi_6, +) - \pi(\phi_6, -)$	$-\pi(\phi_7, --) + \pi(\phi_7, +-)$
$\eta_{\psi_7}^{\mathrm{NE}}$	$+\pi(\phi_7, ++)$	$+\pi(\phi_7, --) - \pi(\phi_7, -+) - \pi(\phi_7, +-)$

Aubert duality defines a bijection between $\Pi_{\psi_3}(G(F))$ and $\Pi_{\psi_1}(G(F))$ and between $\Pi_{\psi_3}(G_1(F))$ and $\Pi_{\psi_1}(G_1(F))$. Moreover, it follows from the Kazhdan-Lusztig conjecture, which we have already established for this example in Section 14.3.3, that the associated distributions

$$\begin{aligned}\Theta_{\psi_1}^G &:= \text{trace } \pi(\phi_1) + \text{trace } \pi(\phi_6, +) \\ \Theta_{\psi_3}^G &:= \text{trace } \pi(\phi_3, +) + \text{trace } \pi(\phi_3, -)\end{aligned}$$

and

$$\begin{aligned}\Theta_{\psi_1}^{G_1} &:= -(-\text{trace } \pi(\phi_4, -) - \text{trace } \pi(\phi_7, +-)) \\ \Theta_{\psi_3}^{G_1} &:= -(+\text{trace } \pi(\phi_7, -+) + \text{trace } \pi(\phi_7, +-))\end{aligned}$$

are stable. Moreover, using the characters of microlocal fundamental groups arising from our calculation of the functor Ev_{C_1} and Ev_{C_3} we may define $\Theta_{\psi_1, s}^G$, $\Theta_{\psi_1, s}^{G_1}$, $\Theta_{\psi_1, s}^G$ and $\Theta_{\psi_1, s}^{G_1}$. It follows from Section 14.1.6 that these distributions coincide with the endoscopic transfer of stable distributions from an elliptic endoscopic group G' ; those stable distributions on $G'(F)$ also arise from ABV-packets that are not Arthur packets. In these regards, the pseudo-Arthur packets $\Pi_{\psi_1}(G(F))$, $\Pi_{\psi_1}(G_1(F))$, $\Pi_{\psi_3}(G(F))$, and $\Pi_{\psi_3}(G_1(F))$ behave like Arthur packets.

14.4. Endoscopy and equivariant restriction of perverse sheaves. — In this section we will calculate both sides of (154) for $G = \text{SO}(7)$ and the elliptic endoscopic $G' = \text{SO}(5) \times \text{SO}(3)$, which already appeared in Section 14.1.6. This will illustrate how the Langlands-Shelstad lift of $\Theta_{\psi'}$ on $G'(F)$ to $\Theta_{\psi, s}$ on $G(F)$ is related to equivariant restriction of perverse sheaves from V to the Vogan variety V' for G' ; see Section 14.1.6 for ψ' .

The endoscopic datum for G' includes $s \in H$ given by

$$s := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$Z_{\widehat{G}}(s) = \left\{ \left(\begin{array}{c|c|c} A & 0 & B \\ \hline 0 & E & 0 \\ \hline C & 0 & D \end{array} \right) \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(4), E \in \text{Sp}(2) \right\} \cong \text{Sp}(4) \times \text{Sp}(4),$$

so $\widehat{G}' = Z_{\widehat{G}}(s)$.

14.4.1. Endoscopic Vogan variety. — The infinitesimal parameter $\lambda : W_F \rightarrow {}^L G$ factors through $\epsilon : {}^L G' \hookrightarrow {}^L G$ to define $\lambda' : W_F \rightarrow {}^L G'$ by

$$\lambda'(w) = \left(\begin{pmatrix} |w|^{3/2} & 0 & 0 & 0 \\ 0 & |w|^{1/2} & 0 & 0 \\ 0 & 0 & |w|^{-1/2} & 0 \\ 0 & 0 & 0 & |w|^{-3/2} \end{pmatrix}, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right).$$

To simplify notation below, let us set $G^{(1)} := \mathrm{SO}(3)$ and $G^{(2)} := \mathrm{SO}(5)$ and define $\lambda^{(1)} : W_F \rightarrow {}^L G^{(1)}$ and $\lambda^{(2)} : W_F \rightarrow {}^L G^{(2)}$ accordingly. Also set

$$H^{(1)} := Z_{\widehat{G}^{(1)}}(\lambda^{(1)}) \quad \text{and} \quad H^{(2)} := Z_{\widehat{G}^{(2)}}(\lambda^{(2)})$$

and $V^{(1)} := V_{\lambda^{(1)}}$ and $V^{(2)} := V_{\lambda^{(2)}}$. Then,

$$H' = H^{(1)} \times H^{(2)} \quad \text{and} \quad V' = V^{(2)} \times V^{(2)},$$

with the action of $H^{(1)}$ on $V^{(1)}$ given in Section 10 and the action of $H^{(2)}$ on $V^{(2)}$ given in Section 12. It follows that, with reference to Sections 10 and 12, V' is stratified into eight H' -orbits:

$$\begin{array}{cccc} C_{ux} \times C_y & C_x \times C_y & C_u \times C_y & C_0 \times C_y \\ C_{ux} \times C_0 & C_x \times C_0 & C_u \times C_0 & C_0 \times C_0. \end{array}$$

For all H' -orbits $C' \subset V'$, the microlocal fundamental group $A_{C'}^{\mathrm{mic}}$ is canonically isomorphic to the centre $Z(\widehat{G}') = Z(\widehat{G}^{(2)}) \times Z(\widehat{G}^{(1)})$, because we have chosen G' so that the unramified infinitesimal parameter λ' is regular semisimple at Fr. Consequently, the image of $Z(\widehat{G}')$ under $\epsilon : \widehat{G}' \hookrightarrow \widehat{G}$ is the group $S[2]$ introduced in Section 14.1.4.

14.4.2. Restriction. — We now describe the restriction functor $\mathrm{D}_H(V) \rightarrow \mathrm{D}_{H'}(V')$ on simple perverse sheaves, after passing to Grothendieck groups.

$$\begin{array}{ll} \mathrm{res} : \mathrm{Per}_H(V) & \longrightarrow \mathrm{KPer}_{H'}(V') \\ \mathcal{IC}(\mathbb{1}_{C_0}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[0] \\ \mathcal{IC}(\mathbb{1}_{C_1}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[1] \\ \mathcal{IC}(\mathbb{1}_{C_2}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_0})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[1] \\ \mathcal{IC}(\mathbb{1}_{C_3}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_y})[1] \\ \mathcal{IC}(\mathcal{L}_{C_3}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[1] \\ \mathcal{IC}(\mathbb{1}_{C_4}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[1] \\ \mathcal{IC}(\mathbb{1}_{C_5}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_x} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_y})[2] \\ & \quad \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathbb{1}_{C_0})[2] \\ \mathcal{IC}(\mathbb{1}_{C_6}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_{ux}} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[2] \\ \mathcal{IC}(\mathbb{1}_{C_7}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_{ux}} \boxtimes \mathbb{1}_{C_y})[2] \\ \mathcal{IC}(\mathcal{L}_{C_7}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathbb{1}_{C_0})[2] \end{array}$$

and

$$\begin{array}{ll} \mathcal{IC}(\mathcal{F}_{C_2}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[1] \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathcal{E}_{C_y})[1] \\ \mathcal{IC}(\mathcal{F}_{C_4}) & \mapsto \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[2] \\ \mathcal{IC}(\mathcal{F}_{C_6}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathbb{1}_{C_0})[2] \oplus \mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[2] \\ \mathcal{IC}(\mathcal{F}_{C_7}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})[4] \\ & \quad \oplus \mathcal{IC}(\mathbb{1}_{C_0} \boxtimes \mathcal{E}_{C_y})[4] \\ \mathcal{IC}(\mathcal{E}_{C_7}) & \mapsto \mathcal{IC}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y})[2] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})[2] \end{array}$$

14.4.3. Restriction and vanishing cycles. — Although the inclusion $V' \hookrightarrow V$ induces a map of conormal bundles $\epsilon : T_{H'}^*(V') \hookrightarrow T_H^*(V)$, this does not restrict to a map of

regular conormal bundles. Instead, we have

$$\begin{aligned}
 T_{C_0}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_0 \times C_0}^*(V')_{\text{reg}} \\
 T_{C_1}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_u \times C_0}^*(V')_{\text{reg}} \\
 T_{C_2}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_0 \times C_y}^*(V')_{\text{reg}} \\
 T_{C_3}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_x \times C_y}^*(V')_{\text{reg}} \\
 T_{C_4}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= \emptyset \\
 T_{C_5}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= \emptyset \\
 T_{C_6}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_{ux} \times C_0}^*(V')_{\text{reg}} \\
 T_{C_7}^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}} &= T_{C_{ux} \times C_y}^*(V')_{\text{reg}}
 \end{aligned}$$

Thus, the hypothesis for (153) is met only for $(x', \xi') \in T_{H'}^*(V')_{\text{reg}}$ from the list of regular conormal bundles appearing on the right-hand side of these equations.

We now prove a few more interesting cases of (153).

The case $\mathcal{P} = \mathcal{I}(\mathcal{E}_{C_7})$. — From Section 14.4.2 we see that, in the Grothendieck group of $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$,

$$\begin{aligned}
 & {}^p\text{Ev}'(\mathcal{I}(\mathcal{E}_{C_7})|_{V'}) \\
 & \equiv {}^p\text{Ev}'(\mathcal{I}(\mathcal{L}_{C_{ux}} \boxtimes \mathcal{E}_{C_y}) \oplus \mathcal{I}(\mathcal{L}_{C_x} \boxtimes \mathcal{E}_{C_y})) \\
 & = \left({}^p\text{Ev}^{(2)} \mathcal{I}(\mathcal{L}_{C_{ux}}) \boxtimes {}^p\text{Ev}^{(1)} \mathcal{I}(\mathcal{E}_{C_y}) \right) \oplus \left({}^p\text{Ev}^{(2)} \mathcal{I}(\mathcal{L}_{C_x}) \boxtimes {}^p\text{Ev}^{(1)} \mathcal{I}(\mathcal{E}_{C_y}) \right) \\
 & = ((\mathcal{I}(\mathcal{L}_{\mathcal{O}_{ux}}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_u})) \boxtimes (\mathcal{I}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_0}))) \\
 & \quad \oplus ((\mathcal{I}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_0})) \boxtimes (\mathcal{I}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_0}))) \\
 & = \mathcal{I}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \\
 & \quad \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\
 & \quad \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{I}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_0}).
 \end{aligned}$$

On the other hand, recall from Section 14.2.7 that

$$\begin{aligned}
 {}^p\text{Ev} \mathcal{I}(\mathcal{E}_{C_7}) &= \mathcal{I}(\mathcal{E}_{\mathcal{O}_7}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_6}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_5}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_4}) \\
 & \quad \oplus \mathcal{I}(\mathcal{F}_{\mathcal{O}_3}) \oplus \mathcal{I}(\mathcal{F}_{\mathcal{O}_2}) \oplus \mathcal{I}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{I}(\mathcal{E}_{\mathcal{O}_0}).
 \end{aligned}$$

We can now easily calculate both sides of (153) on all six components of $T_H^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}}$.

$(C_0 \times C_0)$. If $(x', \xi') \in T_{C_0 \times C_0}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_0}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned}
 & (-1)^{\dim(C_0 \times C_0)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{E}_{C_7})|_{V'} \right)_{(x', \xi')} \\
 & = (-1)^0 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\
 & = (-)(+1, -1) \\
 & = -1,
 \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned}
 & (-1)^{\dim C_0} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{I}(\mathcal{E}_{C_7}))_{(x, \xi)} \\
 & = (-1)^{\dim C_0} \text{trace}_{a_s} \mathcal{I}(\mathcal{E}_{\mathcal{O}_0}) \\
 & = \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{E}_{\mathcal{O}_0}) \\
 & = (+)(+1, -1) \\
 & = -1.
 \end{aligned}$$

This confirms (153) on $T_{C_0 \times C_0}^*(V')_{\text{reg}}$.
 $(C_u \times C_0)$. If $(x', \xi') \in T_{C_u \times C_0}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_1}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_u \times C_0)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{E}_{C_7})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^1 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{L}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\ &= -(-)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_1} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{E}_{C_7}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_1} \text{trace}_{a_s} \mathcal{I}(\mathcal{F}_{\mathcal{O}_1}) \\ &= (-1)^2 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{F}_{\mathcal{O}_1}) \\ &= (-+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C_0 \times C_0}^*(V')_{\text{reg}}$.
 $(C_0 \times C_y)$. If $(x', \xi') \in T_{C_0 \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_2}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_0 \times C_y)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{E}_{C_7})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^1 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \\ &= -(-)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_2} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{E}_{C_7}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_2} \text{trace}_{a_s} \mathcal{I}(\mathcal{F}_{\mathcal{O}_2}) \\ &= (-1)^2 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{F}_{\mathcal{O}_2}) \\ &= (-+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C_0 \times C_y}^*(V')_{\text{reg}}$.
 $(C_x \times C_y)$. If $(x', \xi') \in T_{C_x \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_3}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_x \times C_y)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{E}_{C_7})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^2 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \\ &= (-)(+1, -1) \\ &= -1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_3} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{E}_{C_7}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_3} \text{trace}_{a_s} \mathcal{I}(\mathcal{F}_{\mathcal{O}_3}) \\ &= (-1)^3 \text{trace}_{(+1, -1)} \mathcal{I}(\mathcal{F}_{\mathcal{O}_3}) \\ &= -(-+)(+1, -1) \\ &= -1. \end{aligned}$$

This confirms (153) on $T_{C_x \times C_y}^*(V')_{\text{reg}}$.
 $(C_{ux} \times C_0)$. If $(x', \xi') \in T_{C_{ux} \times C_0}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_6}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_{ux} \times C_0)} \text{trace}_{a'_s} ({}^p\text{Ev}' \mathcal{IC}(\mathcal{E}_{C_7})|_{V'})_{(x', \xi')} \\ &= (-1)^2 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\ &= (-)(+1, -1) \\ &= -1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_6} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{IC}(\mathcal{E}_{C_7}))_{(x, \xi)} \\ &= (-1)^{\dim C_6} \text{trace}_{a_s} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \\ &= (-1)^4 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_6}) \\ &= (+)(+1, -1) \\ &= -1. \end{aligned}$$

This confirms (153) on $T_{C_{ux} \times C_0}^*(V')_{\text{reg}}$.
 $(C_{ux} \times C_y)$. If $(x', \xi') \in T_{C_{ux} \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_7}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_{ux} \times C_y)} \text{trace}_{a'_s} ({}^p\text{Ev}' \mathcal{IC}(\mathcal{E}_{C_7})|_{V'})_{(x', \xi')} \\ &= (-1)^3 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_{ux}} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \\ &= -(-)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_7} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{IC}(\mathcal{E}_{C_7}))_{(x, \xi)} \\ &= (-1)^{\dim C_7} \text{trace}_{a_s} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \\ &= (-1)^5 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{E}_{\mathcal{O}_7}) \\ &= -(+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C_{ux} \times C_y}^*(V')_{\text{reg}}$.

This confirms (154) for $\mathcal{P} = \mathcal{IC}(\mathcal{E}_{C_7})$.

The case $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$. — Then from Section 14.4.2 we see that, in the Grothendieck group of $\text{Per}_{H'}(T_{H'}^*(V')_{\text{reg}})$,

$$\begin{aligned} & {}^p\text{Ev}'(\mathcal{IC}(\mathcal{F}_{C_4})|_{V'}) \\ &\equiv {}^p\text{Ev}'(\mathcal{IC}(\mathbb{1}_{C_u} \boxtimes \mathcal{E}_{C_y})[1] \oplus \mathcal{IC}(\mathcal{L}_{C_x} \boxtimes \mathbb{1}_{C_0})) \\ &= \left({}^p\text{Ev}^{(2)} \mathcal{IC}(\mathbb{1}_{C_u}) \boxtimes {}^p\text{Ev}^{(1)} \mathcal{IC}(\mathcal{E}_{C_y}) \right) \oplus \left({}^p\text{Ev}^{(2)} \mathcal{IC}(\mathcal{L}_{C_x}) \boxtimes {}^p\text{Ev}^{(1)} \mathcal{IC}(\mathbb{1}_{C_0}) \right) \\ &= (\mathcal{IC}(\mathbb{1}_{\mathcal{O}_u}) \boxtimes (\mathcal{IC}(\mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathcal{E}_{\mathcal{O}_0}))) \oplus ((\mathcal{IC}(\mathcal{L}_{\mathcal{O}_x}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0})) \boxtimes \mathcal{IC}(\mathbb{1}_{\mathcal{O}_0})) \\ &= \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_y}) \oplus \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_x} \boxtimes \mathcal{L}_{\mathcal{O}_0}) \oplus \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}). \end{aligned}$$

On the other hand, recall from Section 14.2.7 that

$${}^p\text{Ev} \mathcal{IC}(\mathcal{F}_{C_4}) = \mathcal{IC}(\mathcal{F}_{\mathcal{O}_4}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \oplus \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}).$$

We can now easily calculate both sides of (153) on all six components of $T_H^*(V)_{\text{reg}} \cap T_{H'}^*(V')_{\text{reg}}$.

$(C_0 \times C_0)$. If $(x', \xi') \in T_{C_0 \times C_0}^*(V')_{\text{reg}}$ then the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_0 \times C_0)} \text{trace}_{a'_s} ({}^p\text{Ev}' \mathcal{IC}(\mathcal{F}_{C_4})|_{V'})_{(x', \xi')} \\ &= (-1)^0 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{L}_{\mathcal{O}_0} \boxtimes \mathbb{1}_{\mathcal{O}_0}) \\ &= (-+)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_0} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{IC}(\mathcal{F}_{C_4}))_{(x, \xi)} \\ &= (-1)^{\dim C_0} \text{trace}_{a_s} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\ &= \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_0}) \\ &= (-+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C_0 \times C_0}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_u \times C_0)$. If $(x', \xi') \in T_{C_u \times C_0}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_1}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_u \times C_0)} \text{trace}_{a'_s} ({}^p\text{Ev}' \mathcal{IC}(\mathcal{F}_{C_4})|_{V'})_{(x', \xi')} \\ &= (-1)^1 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathbb{1}_{\mathcal{O}_u} \boxtimes \mathcal{E}_{\mathcal{O}_0}) \\ &= -(+-)(+1, -1) \\ &= +1, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_1} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{IC}(\mathcal{F}_{C_4}))_{(x, \xi)} \\ &= (-1)^{\dim C_1} \text{trace}_{a_s} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \\ &= (-1)^2 \text{trace}_{(+1, -1)} \mathcal{IC}(\mathcal{F}_{\mathcal{O}_1}) \\ &= (-+)(+1, -1) \\ &= +1. \end{aligned}$$

This confirms (153) on $T_{C_u \times C_0}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_0 \times C_y)$. If $(x', \xi') \in T_{C_0 \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_2}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_0 \times C_y)} \text{trace}_{a'_s} ({}^p\text{Ev}' \mathcal{IC}(\mathcal{F}_{C_4})|_{V'})_{(x', \xi')} \\ &= (-1)^1 \text{trace}_{(+1, -1)} 0 \\ &= 0, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_2} \text{trace}_{a_s} ({}^p\text{Ev} \mathcal{IC}(\mathcal{F}_{C_4}))_{(x, \xi)} \\ &= (-1)^{\dim C_2} \text{trace}_{a_s} 0 \\ &= 0. \end{aligned}$$

This confirms (153) on $T_{C_0 \times C_y}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{IC}(\mathcal{F}_{C_4})$.

$(C_x \times C_y)$. If $(x', \xi') \in T_{C_x \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_3}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_x \times C_y)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{F}_{C_4})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^2 \text{trace}_{(+1, -1)} 0 \\ &= 0, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_3} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{F}_{C_4}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_3} \text{trace}_{a_s} 0 \\ &= 0. \end{aligned}$$

This confirms (153) on $T_{C_x \times C_y}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{I}(\mathcal{F}_{C_4})$.

$(C_{ux} \times C_0)$. If $(x', \xi') \in T_{C_{ux} \times C_0}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_6}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_{ux} \times C_0)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{F}_{C_4})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^2 \text{trace}_{(+1, -1)} 0 \\ &= 0, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_6} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{F}_{C_4}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_6} \text{trace}_{a_s} 0 \\ &= 0. \end{aligned}$$

This confirms (153) on $T_{C_{ux} \times C_0}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{I}(\mathcal{F}_{C_4})$.

$(C_{ux} \times C_y)$. If $(x', \xi') \in T_{C_{ux} \times C_y}^*(V')_{\text{reg}}$ then $(x, \xi) \in T_{C_7}^*(V)_{\text{reg}}$. In this case the left-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim(C_{ux} \times C_y)} \text{trace}_{a'_s} \left({}^p\text{Ev}' \mathcal{I}(\mathcal{E}_{C_7})|_{V'} \right)_{(x', \xi')} \\ &= (-1)^3 \text{trace}_{(+1, -1)} 0 \\ &= 0, \end{aligned}$$

while the right-hand side of (153) is

$$\begin{aligned} & (-1)^{\dim C_7} \text{trace}_{a_s} \left({}^p\text{Ev} \mathcal{I}(\mathcal{F}_{C_4}) \right)_{(x, \xi)} \\ &= (-1)^{\dim C_7} \text{trace}_{a_s} 0 \\ &= 0. \end{aligned}$$

This confirms (153) on $T_{C_{ux} \times C_y}^*(V')_{\text{reg}}$ for $\mathcal{P} = \mathcal{I}(\mathcal{F}_{C_4})$.

This confirms (154) for $\mathcal{P} = \mathcal{I}(\mathcal{F}_{C_4})$.

14.5. Tables for the $\mathrm{SO}(7)$ example. — Here we gather together all the main results of the calculations performed in Section 14.

TABLE 14.5.1. Arthur packets for representations of $G(F)$ and $G_1(F)$ with infinitesimal parameter λ . For typographic reasons, here we use the abbreviated notation $\pi_i := \pi(\phi_i)$, $\pi_i^\pm := \pi(\phi_i, \pm)$ and $\pi_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$.

$\Pi_{\phi_i}(G(F))$	$\Pi_{\psi_i}(G(F))$	$\Pi_{\phi_i}(G_1(F))$	$\Pi_{\psi_i}(G_1(F))$
$\{\pi_0\}$	$\{\pi_0, \pi_2^-\}$	\emptyset	$\{\pi_4^-, \pi_7^{+-}\}$
$\{\pi_1\}$	undefined	\emptyset	undefined
$\{\pi_2^+\}$	$\{\pi_2^+, \pi_3^-\}$	$\{\pi_2^-\}$	$\{\pi_2^-, \pi_7^{+-}\}$
$\{\pi_3^+, \pi_3^-\}$	undefined	\emptyset	undefined
$\{\pi_4^+\}$	$\{\pi_4^+\}$	$\{\pi_4^-\}$	$\{\pi_4^-, \pi_7^{+-}\}$
$\{\pi_5\}$	$\{\pi_5\}$	\emptyset	$\{\pi_7^{-+}, \pi_7^{+-}\}$
$\{\pi_6^+\}$	$\{\pi_6^+, \pi_7^{--}\}$	$\{\pi_6^-\}$	$\{\pi_7^{+-}\}$
$\{\pi_7^{++}, \pi_7^{--}\}$	$\{\pi_7^{++}, \pi_7^{--}\}$	$\{\pi_7^{-+}, \pi_7^{+-}\}$	$\{\pi_7^{-+}, \pi_7^{+-}\}$

TABLE 14.5.2. ABV-packets for representations of $G(F)$ and $G_1(F)$ with infinitesimal parameter λ . Comparing this table with Table 14.5 shows that all Arthur packets for admissible representations with infinitesimal parameter λ are recovered from ABV-packets. Again we use the abbreviated notation $\pi_i := \pi(\phi_i)$, $\pi_i^\pm := \pi(\phi_i, \pi)$ and $\pi_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$.

$\Pi_{\phi_i}(G(F))$	$\Pi_{\phi_i}^{\mathrm{ABV}}(G(F))$	$\Pi_{\phi_i}(G_1(F))$	$\Pi_{\phi_i}^{\mathrm{ABV}}(G_1(F))$
$\{\pi_0\}$	$\{\pi_0, \pi_2^-\}$	\emptyset	$\{\pi_4^-, \pi(\phi_7^{+-})\}$
$\{\pi_1\}$	$\{\pi_1, \pi_6^+\}$	\emptyset	$\{\pi_4^-, \pi_7^{+-}\}$
$\{\pi_2^+\}$	$\{\pi_2^+, \pi_3^-\}$	$\{\pi_2^-\}$	$\{\pi_2^-, \pi_7^{+-}\}$
$\{\pi_3^+, \pi_3^-\}$	$\{\pi_3^+, \pi_3^-\}$	\emptyset	$\{\pi_7^{-+}, \pi_7^{+-}\}$
$\{\pi_4^+\}$	$\{\pi_4^+\}$	$\{\pi_4^-\}$	$\{\pi_4^-, \pi_7^{+-}\}$
$\{\pi_5\}$	$\{\pi_5\}$	\emptyset	$\{\phi_7^{-+}, \pi_7^{+-}\}$
$\{\pi_6^+\}$	$\{\pi_6^+, \pi_7^{--}\}$	$\{\pi_6^-\}$	$\{\phi_7^{+-}\}$
$\{\pi_7^{++}, \pi_7^{--}\}$	$\{\pi_7^{++}, \pi_7^{--}\}$	$\{\pi_7^{-+}, \pi_7^{+-}\}$	$\{\pi_7^{-+}, \pi_7^{+-}\}$

TABLE 14.5.3. Pure Arthur packets, decomposed into pure L-packets $\Pi_{\text{pure},\phi}(G/F)$ and the remaining coronal representations. The notation $[\pi, \delta]$ is explained in Section 2.7 and recalled in Section 8.1.2. For typographic reasons, here we use the abbreviated notation $\pi_i := \pi(\phi_i)$, $\pi_i^\pm := \pi(\phi_i, \pi)$ and $\pi_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$.

pure Arthur packet	pure L-packet	coronal representations
$\Pi_{\text{pure},\psi_0}(G/F)$	$[\pi_0, 0]$,	$[\pi_2^+, 0], [\pi_4^-, 1], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\psi_2}(G/F)$	$[\pi_2^+, 0], [\pi_2^-, 1]$,	$[\pi_3^-, 0], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\psi_4}(G/F)$	$[\pi_4^+, 0], [\pi_4^-, 1]$,	$[\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\psi_5}(G/F)$	$[\pi_5, 0]$,	$[\pi_7^{-+}, 1], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\psi_6}(G/F)$	$[\pi_6^+, 0], [\pi_6^-, 1]$,	$[\pi_7^{--}, 0], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\psi_7}(G/F)$	$[\pi_7^{++}, 0], [\pi_7^{--}, 0], [\pi_7^{-+}, 1], [\pi_7^{+-}, 1]$	

TABLE 14.5.4. ABV-packets $\Pi_{\text{pure},\phi}^{\text{ABV}}(G/F)$, decomposed into pure L-packets $\Pi_{\text{pure},\phi}(G/F)$ and the remaining coronal representations. Comparing this table with Table 14.5 verifies Conjecture 1(a) in this case. The same comparison shows that not all ABV-packets are pure Arthur packets. The notation $[\pi, \delta]$ is explained in Section 2.7 and recalled in Section 8.1.2. For typographic reasons, we use the abbreviated notation $\pi_i := \pi(\phi_i)$, $\pi_i^\pm := \pi(\phi_i, \pi)$ and $\pi_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$.

ABV-packet	pure L-packet	coronal representations
$\Pi_{\text{pure},\phi_0}^{\text{ABV}}(G/F)$	$[\pi_0, 0]$	$[\pi_2^+, 0], [\pi_4^-, 1], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_1}^{\text{ABV}}(G/F)$	$[\pi(\phi_1), 0]$	$[\pi_4^-, 1], [\pi_6^+, 0], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_2}^{\text{ABV}}(G/F)$	$[\pi_2^+, 0], [\pi_2^-, 1]$	$[\pi_3^-, 0], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_3}^{\text{ABV}}(G/F)$	$[\pi_3^+, 0], [\pi_3^-, 0]$	$[\pi_7^{-+}, 1], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_4}^{\text{ABV}}(G/F)$	$[\pi_4^+, 0], [\pi_4^-, 1]$	$[\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_5}^{\text{ABV}}(G/F)$	$[\pi_5, 0]$	$[\pi_7^{-+}, 1], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_6}^{\text{ABV}}(G/F)$	$[\pi_6^+, 0], [\pi_6^-, 1]$	$[\pi_7^{--}, 0], [\pi_7^{+-}, 1]$
$\Pi_{\text{pure},\phi_7}^{\text{ABV}}(G/F)$	$[\pi_7^{++}, 0], [\pi_7^{--}, 0], [\pi_7^{-+}, 1], [\pi_7^{+-}, 1]$	

TABLE 14.5.5. Multiplicities of admissible representations in standard modules. We use the abbreviated notation $\pi_i := \pi(\phi_i)$, $\pi_i^\pm := \pi(\phi_i, \pi)$ and $\pi_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$ and we also set $M_i := M(\phi_i)$, $M_i^\pm := \pi(\phi_i, \pi)$ and $M_i^{\pm\pm} := \pi(\phi_i, \pm\pm)$.

[illegible]

TABLE 14.5.6. The normalised geometric multiplicity matrix. The table records the multiplicity of \mathcal{L}_C in $\mathcal{L}'_{C'}|_C$. Comparing this table with Table 14.5 verifies the Kazhdan-Lusztig conjecture in this case; see also Table 14.2.1. Recall the notation $\mathcal{L}^\sharp := \mathcal{IC}(\mathcal{L}_C)[- \dim C]$.

[illegible]

TABLE 14.5.7. $\mathbf{NEvs} : \mathbf{Per}_{H_\lambda}(V_\lambda) \rightarrow \mathbf{Loc}_{H_\lambda}(T_{H_\lambda}^*(V_\lambda)_{\text{sreg}})$ on simple objects.
See also Table 14.2.5. Here we use the notation $\mathbf{NEvs}_i := \mathbf{NEvs}_{C_i}$.

\mathcal{P}	$\mathbf{NEvs}_0 \mathcal{P}$	$\mathbf{NEvs}_1 \mathcal{P}$	$\mathbf{NEvs}_2 \mathcal{P}$	$\mathbf{NEvs}_3 \mathcal{P}$	$\mathbf{NEvs}_4 \mathcal{P}$	$\mathbf{NEvs}_5 \mathcal{P}$	$\mathbf{NEvs}_6 \mathcal{P}$	$\mathbf{NEvs}_7 \mathcal{P}$
$\mathcal{I}(\mathbb{1}_{C_0})$	++	0	0	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_1})$	0	++	0	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_2})$	--	0	++	0	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_3})$	0	0	0	++	0	0	0	0
$\mathcal{I}(\mathcal{L}_{C_3})$	0	0	--	--	0	0	0	0
$\mathcal{I}(\mathbb{1}_{C_4})$	0	0	0	0	+	0	0	0
$\mathcal{I}(\mathbb{1}_{C_5})$	0	0	0	0	0	+	0	0
$\mathcal{I}(\mathbb{1}_{C_6})$	0	--	0	0	0	0	++	0
$\mathcal{I}(\mathbb{1}_{C_7})$	0	0	0	0	0	0	0	++
$\mathcal{I}(\mathcal{L}_{C_7})$	0	0	0	0	0	0	--	--
$\mathcal{I}(\mathcal{F}_{C_2})$	0	0	+-	0	0	0	0	0
$\mathcal{I}(\mathcal{F}_{C_4})$	-+	-+	0	0	-	0	0	0
$\mathcal{I}(\mathcal{F}_{C_6})$	0	0	0	0	0	0	-+	0
$\mathcal{I}(\mathcal{F}_{C_7})$	0	0	0	-+	0	-	0	-+
$\mathcal{I}(\mathcal{E}_{C_7})$	+-	-+	-+	-+	-	-	+-	+-

TABLE 14.5.8. The characters $\langle \cdot, \pi \rangle_\psi$ of A_ψ . Comparing this table with Table 14.5.7 verifies Conjecture 1 in this example.

π	$\langle \cdot, \pi \rangle_{\psi_0}$	$\langle \cdot, \pi \rangle_{\psi_2}$	$\langle \cdot, \pi \rangle_{\psi_4}$	$\langle \cdot, \pi \rangle_{\psi_5}$	$\langle \cdot, \pi \rangle_{\psi_6}$	$\langle \cdot, \pi \rangle_{\psi_7}$
$\pi(\phi_0)$	++	0	0	0	0	0
$\pi(\phi_2, +)$	--	++	0	0	0	0
$\pi(\phi_3, -)$	0	--	0	0	0	0
$\pi(\phi_4, +)$	0	0	+	0	0	0
$\pi(\phi_5)$	0	0	0	+	0	0
$\pi(\phi_6, +)$	0	0	0	0	++	0
$\pi(\phi_7, ++)$	0	0	0	0	0	++
$\pi(\phi_7, --)$	0	0	0	0	--	--
$\pi(\phi_2, -)$	0	+-	0	0	0	0
$\pi(\phi_4, -)$	-+	0	-	0	0	0
$\pi(\phi_6, -)$	0	0	0	0	-+	0
$\pi(\phi_7, -+)$	0	0	0	-	0	-+
$\pi(\phi_7, +-)$	+-	-+	-	-	+-	+-

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