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APPEARANCE OF THE KASHIWARA-SAITO SINGULARITY IN THE REPRESENTATION THEORY OF p-ADIC GL_{16}

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Abstract. In 1993 David Vogan proposed a basis for the vector space of stable distributions on p-adic groups using the microlocal geometry of moduli spaces of Langlands parameters. In the case of general linear groups, distribution characters of irreducible admissible representations, taken up to equivalence, form a basis for the vector space of stable distributions. In this paper we show that these two bases, one putative, cannot be equal. Specifically, we use the Kashiwara-Saito singularity to find a non-Arthur type irreducible admissible representation of p-adic GL_{16} whose ABV-packet, as defined in [CFM+21], contains exactly one other representation; remarkably, this other admissible representation is of Arthur type. In the course of this study we strengthen the main result concerning the Kashiwara-Saito singularity in [KS97]. The irreducible admissible representations in this paper illustrate a fact we found surprising: for general linear groups, while all A-packets are singletons, some ABV-packets are not.

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0. Introduction

0.1. **Background.** The local Langlands correspondence for a p-adic group $G(F)$ is generally stated, or more precisely, conjectured, as a bijection between sets: on the one hand, the set $\Pi(G(F))$ of equivalence classes of irreducible admissible representations of $G(F)$; and on the other hand, the set $\Xi({}^LG)$ of equivalence classes of pairs $\xi = (\phi, \epsilon)$, called enhanced Langlands parameters, where $\phi : W'_F \to {}^LG$ is a Langlands parameter and ϵ is an irreducible representation of a finite group attached to ϕ. This bijection should satisfy a list of properties, largely derived from compatibility with class field theory and the principle of functoriality together with certain normalizing choices. Since $\Pi(G(F))$ is the set of equivalence classes of simple objects in a category, namely, the category of smooth representations of $G(F)$, it is natural to ask if we can identify $\Phi(G(F))$ with the set of equivalence classes of simple objects in a category too and then seek a relation between these categories that would recover the bijection above on simple objects. In 1993 David Vogan observed that simple objects in the category $\mathsf{Per}_{\widehat{G}}(X_\lambda)$ of equivariant perverse sheaves on the moduli space X_λ of Langlands parameters with a common "infinitesimal parameter" λ, are naturally identified with pairs (ϕ, ϵ) as above, when the finite group attached to ϕ is properly interpreted and when the corresponding representations π are enlarged to include $G(F)$ and its pure inner forms. When viewed from this perspective, the local Langlands correspondence takes the form of a bijection between the set $\Xi_\lambda({}^LG)$ of equivalence classes of objects in $\mathsf{Per}_{\widehat{G}}(X_\lambda)$ and the set of simple objects $\Pi_\chi(G/F)$ in the category $\mathsf{Rep}_\chi(G/F)$ of smooth representations of $G(F)$, and its pure inner forms, with matching "infinitesimal character" χ. There is considerable evidence that the categories $\mathsf{Rep}_\chi(G/F)$ and $\mathsf{Per}_{\widehat{G}}(X_\lambda)$ are more closely related than simply by a bijection between their simple objects – at the very least we expect that their derived categories should be equivalent – and this evidence motivates a closer study of X_λ.

The Langlands correspondence determines a partition of $\Pi(G(F))$ into finite sets $\Pi_\phi(G(F))$, called L-packets, consisting of those irreducible admissible representations π that correspond to pairs $(\phi,\epsilon) \in \Xi({}^LG)$ for this fixed ϕ. Arthur packets $\Pi_\psi(G(F))$, also known as A-packets, are also finite sets of irreducible representations, where ψ is an Arthur parameter, but these packets are not disjoint; it is expected that their union is the set of unitary representations of $G(F)$. Every A-packet $\Pi_\psi(G(F))$ contains a distinguished L-packet $\Pi_{\phi_\psi}(G(F))$ through a simple injection $\psi \mapsto \phi_\phi$ from Arthur parameters to Langlands parameters of Arthur type; we refer to $\Pi_\psi(G(F)) \setminus \Pi_{\phi_\psi}(G(F))$ as the *corona* of $\Pi_{\phi_\psi}(G(F))$.

Vogan's perspective on the local Langlands correspondence suggests a beautiful geometric approach to A-packets and in fact defines a corona for every L-packet, not just those of Arthur type. Following [Vog93], the ABV-packet $\Pi^{\text{ABV}}_\phi(G(F))$ is defined as the set of irreducible representations of $G(F)$ for which the characteristic cycles of the corresponding simple perverse sheaf contains the conormal bundle of the $\widehat{G}$-orbit of ϕ; this notion is revisited in [CFM+21] where it is cast in terms of vanishing cycles.

Apocryphally, ABV-packets are rather difficult to calculate; see [BST19, Introduction], for example. In [CFZb] and [CFZa] we found the ABV-packets for all unipotent representations of the exceptional group G_2, building on techniques developed in [CFM+21], where we also calculated over 50 examples of ABV-packets for classical groups. In this paper we continue our exploration by finding a curious ABV-packet for $\mathrm{GL}_{16}(F)$, using a method that lends itself to algorithmic implementation and symbolic computation.

0.2. **Main result.** Since all A-packets for general linear groups are singletons, and since ABV-packets are expected to generalize A-packets, it is reasonable to ask: *Are all ABV-packets singletons for p-adic general linear groups?* In this paper we find an unramified Langlands parameter ϕ_{KS} for $\mathrm{GL}_{16}(F)$ for which the ABV-packet has exactly two representations. More precisely, we find two unipotent, non-tempered, irreducible representations, π_{KS} and π_ψ, such that

$$\Pi^{\text{ABV}}_{\phi_{\mathrm{KS}}}(\mathrm{GL}_{16}(F)) = \{\pi_{\mathrm{KS}}, \pi_\psi\}, \tag{1}$$

where ϕ_{KS} is the Langlands parameter for π_{KS}. We show that π_{KS} is not of Arthur type while its coronal representation, π_ψ, is of Arthur type. The theory of ABV-packets matches the representations π_{KS} and π_ψ with constant and quadratic characters, respectively, of a group $A^{\text{ABV}}_{\phi_{\mathrm{KS}}}$, studied in Section 2.3.

After extensive searching in lower rank general linear groups, we expect that π_{KS} is the simplest example of an admissible representation of a p-adic general linear group with a corona. Many examples of coronal representations for orthogonal groups appear in [CFM+21] and for the exceptional group $G_2(F)$ in [CFZa] and [CFZb].

0.3. **Geometry.** The main result, above, is a consequence of a geometric calculation at the heart of this paper. Before describing that calculation, we begin by recalling a fact from geometry. Let V be a prehomogenous vector space, that is, let V be a finite-dimensional vector space over a field, equipped with the action of an affine algebraic group H for which there is a dense Zariski-open H-orbit. Now let Λ be the conormal variety for V with this action, which is to say, set

$$\Lambda := \{(x,y) \in V \times V^* \mid [x,y] = 0\},$$

where $[\ ,\ ] : V \times V^* \to \operatorname{Lie} H$ is the moment map for V with H-action. For every $x \in V$, the fibre Λ_x of the projection $\Lambda \to V$ is also a finite-dimensional vector space, equipped with the natural action of the affine algebraic group $Z_H(x)$, so it is natural to ask: Does Λ_x have a dense Zariski-open $Z_H(x)$-orbit, or in other words, *is* Λ *a bundle of prehomogeneous vector spaces?* In this paper we show that even when V is the representation variety of a quiver of type A_5, the answer is no. This paper is largely an exploration of the consequence of this fact for p-adic representation theory, as it relates directly to Arthur parameters and A-packets.

The geometric calculation at the heart of this paper concerns the moduli space X of Langlands parameters with a common infinitesimal parameter, λ, in the sense of Sections 0.1 and 1.4. We find a prehomogeneous vector space V with $H := Z_{\widehat{G}}(\lambda)$-action such that

$$X = \widehat{G} \times_H V$$

and we find the elements $x_\psi, x_{\mathrm{KS}} \in V$ that correspond to the Langlands parameters ϕ_ψ and ϕ_{KS}, respectively; again see Section 1.4. In this paper we find that Λ, the conormal bundle to V, is not a bundle of prehomogenous vector spaces by showing that $\Lambda_{x_{\mathrm{KS}}}$ is not a prehomogeneous vector space.

We make a fairly detailed study of the H-orbits C_ψ of x_ψ and C_{KS} of x_{KS}. We find that C_ψ has co-dimension 8 in V and that C_{KS} has co-dimension 8 in the closure $\overline{C}_\psi$ of C_ψ in V. We show that the conormal bundle

$$\Lambda_{C_{\mathrm{KS}}} = \{(x,y) \in V \times V^* \mid x \in C_{\mathrm{KS}}, [x,y] = 0\}$$

has no open H-orbit. We use this fact to conclude that ϕ_{KS} is not of Arthur type.

In fact, we find a connected, dense open H-stable sub-bundle $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \subset \Lambda_{C_{\mathrm{KS}}}$ whose equivariant fundamental group, $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$, is non-trivial. This is another surprise: the groups A^{ABV}_{ϕ} are trivial for all Langlands parameters ϕ of Arthur type for general linear groups, but if ϕ is not of Arthur type, we observe that these groups can be non-trivial even for general linear groups. For classical groups, it is easy to find examples when the group A^{ABV}_{ϕ} is non-trivial; many appear in [CFM+21, Part II].

The main geometric result of this paper refers to the functor

$$\mathsf{Evs}_{C_{\mathrm{KS}}} : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}})$$

which is a special case of the functor defined in [CFM+21, Section 7.9]. Theorem 3.1, implies that the rank of the local system $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_C)$ is 1 for $C = C_{\mathrm{KS}}$ and $C = C_\psi$ and otherwise the local system $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_C)$ is 0. In fact, Theorem 3.1 gives more information, since we find the local systems $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$ and $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_\psi})$; the former is constant, the latter is quadratic. Theorem 3.1 implies Corollary 2.4.1.

0.4. **Consequences.** The singularity we are studying was introduced in [KS97, Section 7]. In [KS97, Theorem 7.2.2], it is shown that structure of the singularity is fundamentally the same as the singularity which other sources, such as [BST19, Section 3.4] and [Wil15], refer to as the Kashiwara-Saito singularity. Specifically, a transverse slice through $\overline{C}_\psi$ at x_{KS} reveals the Kashiwara-Saito singularity in the moduli space of Langlands parameters with the same infinitesimal parameter as ψ.

It follows from Theorem 3.1 that the characteristic cycles of the equivariant perverse sheaf $\mathcal{IC}(\mathbb{1}_{C_\psi})$ contains $\Lambda_{C_{\mathrm{KS}}}$ and Λ_{C_ψ}. That result is proved in [KS97, Theorem 7.2.1] where it is used to provide a counter example to a conjecture of Lusztig

[Lus91, 13.7]. Our proof of Theorem 3.1 gives an independent proof of [KS97, Theorem 7.2.1] using techniques developed here and in [CFM+21]. Moreover, Theorem 3.1 is stronger than [KS97, Theorem 7.2.1], since the latter does not establish that the multiplicities of $\Lambda_{C_{\text{KS}}}$ and Λ_{C_ψ} in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ are both 1, which also follows from Theorem 3.1. See Remark 3.2 for more on the relation between Theorem 3.1 and [KS97, Theorem 7.2.1].

The main result of this paper also has the following interesting interpretation related to another conjecture by Vogan. To state this conjecture, we must build distributions on $G(F)$ from ABV-packets. These distributions are defined using a function $\Pi_\phi^{\text{ABV}}(G(F)) \to \mathsf{Rep}(A_\phi^{\text{ABV}})$ that attaches irreducible representations of a finite group A_ϕ^{ABV} attached to the Langlands parameter ϕ. The group A_ϕ^{ABV} is the equivariant fundamental group of an open, dense variety in the conormal bundle above ϕ in the moduli space of Langlands parameters. Using this, in [CFM+21] we define *packet coefficients* $\langle a, \pi\rangle$, for $a \in A_\phi^{\text{ABV}}$ and $\pi \in \Pi_\phi^{\text{ABV}}(G(F))$. Consider the distribution

$$\Theta_\phi := e(G) \sum_{\pi\in\Pi_\phi^{\text{ABV}}(G(F))} (-1)^{\dim(\phi)-\dim(\phi_\pi)}\, \text{trace}\langle 1, \pi\rangle\ \Theta_\pi, \tag{2}$$

where $e(G)$ is the Kottwitz invariant of the group G and $\dim(\phi_\pi) - \dim(\phi)$ is the relative dimension of the orbit of ϕ in the closure of the orbit of the Langlands parameter ϕ_π for π in the moduli space of Langlands parameters; see [CFM+21] and [CFZa] and [CFZb] for the theory and examples of these distributions. Vogan's [Vog93, Conjecture 8.15'], revisited in [CFM+21, Conjecture 2], posits that the distributions Θ_ϕ are stable and, moreover, form a basis for the space of stable distributions on $G(F)$, letting ϕ run over conjugacy classes of Langlands parameters. It follows from the main result of this paper, Corollary 2.4.1, that

$$\Theta_{\phi_{\text{KS}}} = \Theta_{\pi_{\text{KS}}} + \Theta_{\pi_\psi}. \tag{3}$$

Since $\Theta_{\pi_{\text{KS}}}$ and Θ_{π_ψ} are stable distributions on $\text{GL}_{16}(F)$, this means that *Vogan's putative basis for the vector space of stable distributions is not the same as the basis given by characters of irreducible admissible representations, for general linear groups.* This does not in any way contradict Vogan's conjecture, but we found the result surprising. We further remark that, since $\Theta_{\phi_\psi} = \Theta_{\pi_\psi}$, the transition matrix comparing Vogan's basis for this part of the vector space of stable distributions is unipotent.

This paper is almost entirely an exploration of the consequences for representation theory of geometric facts that have come as a surprise to the authors: not only are the characteristic cycles for conormal bundles of quiver representation varieties of type A_n sometimes *reducible*, contrary to a "hope" of Lusztig [Lus91, 13.7], but the microlocal equivariant fundamental groups of these bundles are sometimes *non-trivial* and, moreover, the microlocal vanishing cycles of equivariant perverse sheaves on these quiver representation varieties are sometimes non-constant, corresponding to *non-trivial representations* of these groups. This paper rests on an example of each of these surprises.

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1. The Kashiwara-Saito representation of $\mathrm{GL}_{16}(F)$

In this section we introduce the Kashiwara-Saito representation π_{KS} and its coronal representation π_ψ. We give the standard modules for these two representations, as well as their L-parameters. While both π_{KS} and π_ψ are unipotent representations, neither are tempered and only π_ψ is of Arthur type, as we will see in Section 2.

1.1. A self-dual representation of $\mathrm{GL}_{16}(F)$ of Arthur type.

Let F be a p-adic field and consider the Arthur parameter for $\mathrm{GL}_8(F)$ defined by

$$\begin{array}{rcl} \psi_0 : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) & \xrightarrow{\psi_0} & \mathrm{GL}_8(\mathbb{C}) \\ (w,x,y) & \mapsto & \mathrm{Sym}^3(x) \otimes \mathrm{Sym}^1(y), \end{array}$$

where Sym^n is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $n+1$. Let ϕ_0 be the Langlands parameter defined by $\psi_0(w,x) := \psi_0(w,x,\mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$, so

$$\phi_0(w,x) = |w|^{1/2}\,\mathrm{Sym}^3(x) \oplus |w|^{-1/2}\,\mathrm{Sym}^3(x).$$

Now consider the dual Arthur parameter $\hat{\psi}_0$ for $\mathrm{GL}_8(F)$ defined by

$$\hat{\psi}_0(w,x,y) = \mathrm{Sym}^1(x) \otimes \mathrm{Sym}^3(y).$$

The Langlands parameter for $\hat{\psi}_0$ is

$$\begin{array}{rcl} \hat{\phi}_0(w,x) & = & |w|^{3/2}\,\mathrm{Sym}^1(x) \oplus |w|^{1/2}\,\mathrm{Sym}^1(x) \\ & & \oplus\ |w|^{-1/2}\,\mathrm{Sym}^1(x) \oplus |w|^{-3/2}\,\mathrm{Sym}^1(x). \end{array}$$

Now let ψ be the Arthur parameter for $\mathrm{GL}_{16}(F)$ defined by $\psi = r \circ (\psi_0 \boxtimes \hat{\psi}_0)$, where $r : \mathrm{GL}_8 \times \mathrm{GL}_8 \to \mathrm{GL}_{16}$ is a natural diagonal block representation; we can also write $\psi = \psi_0 \oplus \hat{\psi}_0$, so

$$\psi(w,x,y) = \mathrm{Sym}^3(x) \otimes \mathrm{Sym}^1(y) \oplus \mathrm{Sym}^1(x) \otimes \mathrm{Sym}^3(y).$$

Then the Langlands parameter ϕ_ψ for ψ is $\phi_\psi = \phi_0 \oplus \hat{\phi}_0$, so

$$\begin{array}{rcl} \phi_\psi(w,x) & = & |w|^{3/2}\,\mathrm{Sym}^1(x) \oplus |w|^{1/2}\,\mathrm{Sym}^3(x) \oplus |w|^{1/2}\,\mathrm{Sym}^1(x) \\ & & \oplus\ |w|^{-1/2}\,\mathrm{Sym}^1(x) \oplus |w|^{-1/2}\,\mathrm{Sym}^3(x) \oplus |w|^{-3/2}\,\mathrm{Sym}^1(x). \end{array}$$

Let π_ψ be the irreducible admissible representation of $\mathrm{GL}_{16}(F)$ with Langlands parameter ϕ_ψ.

To describe the standard module for π_ψ, set

$$M_\psi := \mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2$$

and set

$$\begin{array}{rcl} \sigma_\psi & := & \left(|\det|^{3/2} \otimes \mathrm{St}_{\mathrm{GL}_2(F)}\right) \boxtimes \left(|\det|^{1/2} \otimes \mathrm{St}_{\mathrm{GL}_4(F)}\right) \\ & & \boxtimes\ \left(|\det|^{1/2} \otimes \mathrm{St}_{\mathrm{GL}_2(F)}\right) \boxtimes \left(|\det|^{-1/2} \otimes \mathrm{St}_{\mathrm{GL}_2(F)}\right) \\ & & \boxtimes\ \left(|\det|^{-1/2} \otimes \mathrm{St}_{\mathrm{GL}_4(F)}\right) \boxtimes \left(|\det|^{-3/2} \otimes \mathrm{St}_{\mathrm{GL}_2(F)}\right). \end{array}$$

Let P_ψ be the standard parabolic subgroup of GL_{16} with Levi subgroup M_ψ. Then π_ψ is the unique irreducible quotient of

$$\mathrm{Ind}_{P_\psi(F)}^{\mathrm{GL}_{16}(F)}(\sigma_\psi).$$

Note that, by construction, the irreducible admissible representation π_ψ is of Arthur type and equivalent to its Zelevinsky dual.

1.2. **The Kashiwara-Saito representation of** $\mathrm{GL}_{16}(F)$**.** Consider the Langlands parameter ϕ_1 for $\mathrm{GL}_8(F)$ defined by

$$\phi_1(w,x) := \mathrm{Sym}^2(x) \oplus |w|^{3/2}\,\mathrm{Sym}^1(x) \oplus |w|^{-3/2}\,\mathrm{Sym}^1(x) \oplus \mathrm{Sym}^0(x).$$

We apologize for writing $\mathrm{Sym}^0(x)$ for the trivial representation. Set

$$\phi_{\mathrm{KS}} := r \circ (\phi_1 \boxtimes \phi_1) = \phi_1 \oplus \phi_1,$$

for : $\mathrm{GL}_8 \times \mathrm{GL}_8 \to \mathrm{GL}_{16}$. Thus,

$$\begin{array}{rcl} \phi_{\mathrm{KS}}(w,x) & = & \oplus |w|^{3/2}\left(\mathrm{Sym}^1(x) \oplus \mathrm{Sym}^1(x)\right) \\ & & \oplus \ \left(\mathrm{Sym}^2(x) \oplus \mathrm{Sym}^2(x)\right) \oplus (\mathrm{Sym}^0(x) \oplus \mathrm{Sym}^0(x)) \\ & & \oplus\ |w|^{-3/2}(\mathrm{Sym}^1(x) \oplus \mathrm{Sym}^1(x)). \end{array}$$

Let π_{KS} be the irreducible admissible representation of $\mathrm{GL}_{16}(F)$ with this Langlands parameter. We refer to π_{KS} as the *Kashiwara-Saito representation* of $\mathrm{GL}_{16}(F)$; this representation is self-dual.

We find the standard module for π_{KS}. Set

$$\begin{array}{rcl} \sigma_{\mathrm{KS}} & := & |\det|^{3/2}\mathrm{Ind}_{P_4(F)}^{\mathrm{GL}_4(F)}\left(\mathrm{St}_{\mathrm{GL}_2(F)} \boxtimes \mathrm{St}_{\mathrm{GL}_2(F)}\right) \\ & & \boxtimes\ \mathrm{Ind}_{P_8(F)}^{\mathrm{GL}_8(F)}\left(\mathrm{St}_{\mathrm{GL}_3(F)} \boxtimes \mathrm{St}_{\mathrm{GL}_3(F)} \boxtimes \mathrm{St}_{\mathrm{GL}_1(F)} \boxtimes \mathrm{St}_{\mathrm{GL}_1(F)}\right) \\ & & \boxtimes\ |\det|^{-3/2}\mathrm{Ind}_{P_4(F)}^{\mathrm{GL}_4(F)}\left(\mathrm{St}_{\mathrm{GL}_2(F)} \boxtimes \mathrm{St}_{\mathrm{GL}_2(F)}\right). \end{array}$$

Then σ_{KS} is tempered and lies in the positive Weyl chamber for the standard parabolic with Levi subgroup $M_{\mathrm{KS}}(F)$ where $M_{\mathrm{KS}} = \mathrm{GL}_4 \times \mathrm{GL}_8 \times \mathrm{GL}_4$. The standard module for π_{KS} is

$$\mathrm{Ind}_{P_{\mathrm{KS}}(F)}^{\mathrm{GL}_{16}(F)}(\sigma_{\mathrm{KS}}),$$

where P_{KS} is the standard parabolic with Levi subgroup M_{KS}.

1.3. **Multisegments.** Following [Zel80, Section 3.1], a segment is a finite set

$$[\rho, \nu^k\rho] := \{\rho, \nu\rho, \nu^2\rho, \ldots, \nu^k\rho\},$$

where ρ is an (equivalence class of) irreducible cuspidal representation of $\mathrm{GL}_n(F)$, for some positive integer n, and ν is the character of $\mathrm{GL}_n(F)$ defined by $\nu(g) = |\det(g)|$. A multisegment $\underline{m} = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$, is a multiset of segments, without fixing n. As explained in [Zel80, Theorem 6.1], there is a natural bijection between multisegments and (equivalence classes of) irreducible admissible representations of $\mathrm{GL}_n(F)$, allowing n to range over all positive integers.

After adapting the theory by consistently replacing irreducible submodules with irreducible quotients, the bijection of [Zel80, Theorem 6.1] attaches the following multisegments to the representations appearing in Sections 1.1 and 1.2: the multisegment for π_ψ is

$$\underline{m}_\psi = \{[\nu^{-2}, \nu^{-1}], [\nu^{-2}, \nu^{1}], [\nu^{-1}, \nu^{0}], [\nu^{-1}, \nu^{2}], [\nu^{0}, \nu^{1}], [\nu^{1}, \nu^{2}]\},$$

and this is self-dual; the multisegment for π_{KS} is

$$\underline{m}_{\mathrm{KS}} = \{2[\nu^{-2},\nu^{-1}], 2[\nu^{-1},\nu^{1}], 2\{\nu^{0}\}, 2[\nu^{1},\nu^{2}]\},$$

which is also self-dual.

1.4. **Moduli spaces of Langlands parameters.** The infinitesimal parameters of π_ψ and π_{KS}, in the sense of [CFM+21, Section 4.1], are equal and henceforth denoted by $\lambda : W_F \to \mathrm{GL}_{16}(\mathbb{C})$, defined by

$$\lambda(w) := 2|w|^{-2} \oplus 4|w|^{-1} \oplus 4|w|^{0} \oplus 4|w|^{1} \oplus 2|w|^{2},$$

where the coefficients 2, 4, 4, 4 and 2 denote multiplicities. When viewed as a Langlands parameter for $\mathrm{GL}_{16}(F)$, the multisegment for λ is

$$\underline{m}_{\lambda} = \{2\{\nu^{-2}\}, 4\{\nu^{-1}\}, 4\{\nu^{0}\}, 4\{\nu^{1}\}, 2\{\nu^{2}\}\}.$$

The calculation of $\Pi^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}(\mathrm{GL}_{16}(F))$ begins with the moduli space of Langlands parameters with infinitesimal parameter λ. We now describe that moduli space. Following [Vog93] and [CFM+21], consider

$$V := \{x \in \mathfrak{gl}_{16}(\mathbb{C}) \mid \mathrm{Ad}(\lambda(\mathrm{Fr}))x = qx\}$$

on which

$$H := Z_{\mathrm{GL}_{16}(\mathbb{C})}(\lambda(\mathrm{Fr}))$$

acts by conjugation in $\mathfrak{gl}_{16}$. Using [CFM+21, Section 4.3], we see that V is a moduli space of unramified Langlands parameters $\phi : W'_F \to \mathrm{GL}_{16}(\mathbb{C})$ such that $\phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})) = \lambda(w)$ and

$$X := \mathrm{GL}_{16}(\mathbb{C}) \times_H V$$

is a moduli space of unramified Langlands parameters ϕ for $\mathrm{GL}_{16}(F)$ that are $\mathrm{GL}_{16}(\mathbb{C})$-conjugate to λ. L-parameters are $\mathrm{GL}_{16}(\mathbb{C})$-conjugacy classes of Langlands parameters and are therefore identified with $\mathrm{GL}_{16}(\mathbb{C})$-orbits X, or equivalently, with H-orbits in V. Since the categories $\mathsf{Per}_{\widehat{G}}(X)$ and $\mathsf{Per}_H(V)$ are equivalent, as discussed in [CFM+21, Section 4.5], we henceforth dispense with X and work exclusively with V.

We now describe V in more detail. Let q be the cardinality of the residue field of F. The semisimple element $\lambda(\mathrm{Fr}) \in \mathrm{GL}_{16}(\mathbb{C})$ has eigenvalues $q^{-2}, q^{-1}, q^{0}, q^{1}, q^{2}$ occurring with multiplicities $2,4,4,4,2$, respectively. We label these eigenvalues $\lambda_0 := q^{-2}$, $\lambda_1 := q^{-1}$, $\lambda_2 := q^{0}$, $\lambda_3 := q^{1}$ and $\lambda_4 := q^{2}$. Using this convention,

$$V = \left\{ \begin{pmatrix} 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} x_4 \in \mathrm{Mat}_{2,4}(\mathbb{C}) \\ x_3, x_2 \in \mathrm{Mat}_{4,4}(\mathbb{C}) \\ x_1 \in \mathrm{Mat}_{4,2}(\mathbb{C}) \end{array} \right\}.$$

If we write E_{λ_i} for the eigenspace of $\lambda_{\mathrm{KS}}(\mathrm{Fr})$ with eigenvalue λ_i, then

$$V = \mathrm{Hom}(E_{\lambda_3}, E_{\lambda_4}) \times \mathrm{Hom}(E_{\lambda_2}, E_{\lambda_3}) \times \mathrm{Hom}(E_{\lambda_1}, E_{\lambda_2}) \times \mathrm{Hom}(E_{\lambda_0}, E_{\lambda_1}).$$

In other words, V is a representation variety for the quiver of type A_5:

$$\overset{\lambda_0}{\bullet} \xrightarrow{x_1} \overset{\lambda_1}{\bullet} \xrightarrow{x_2} \overset{\lambda_2}{\bullet} \xrightarrow{x_3} \overset{\lambda_3}{\bullet} \xrightarrow{x_4} \overset{\lambda_4}{\bullet}.$$

Consequently, for elements $x \in V$ we use the notation

$$x = (x_4, x_3, x_2, x_1) \in \mathrm{Mat}_{2,4}(\mathbb{C}) \times \mathrm{Mat}_{4,4}(\mathbb{C}) \times \mathrm{Mat}_{4,4}(\mathbb{C}) \times \mathrm{Mat}_{4,2}(\mathbb{C}).$$

Similarly,

$$H = \mathrm{Aut}(E_{\lambda_4}) \times \mathrm{Aut}(E_{\lambda_3}) \times \mathrm{Aut}(E_{\lambda_2}) \times \mathrm{Aut}(E_{\lambda_1}) \times \mathrm{Aut}(E_{\lambda_0}).$$

Let us use the notation

$$h = (h_4, h_3, h_2, h_1, h_0) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$$

for elements of H. From this we see $\dim V = 48$ and $\dim H = 56$. In this notation, the action $H \times V \to V$ is given by

$$(h_4, h_3, h_2, h_1, h_0) \cdot (x_4, x_3, x_2, x_1) := (h_4 x_4 h_3^{-1}, h_3 x_3 h_2^{-1}, h_2 x_2 h_1^{-1}, h_1 x_1 h_0^{-1}).$$

With this H-action, V is a prehomogenous vector space; the open, dense H-orbit in V is the set of $x \in V$ satisfying the conditions

$$\begin{array}{llll} \mathrm{rank}(x_4) = 2, & \mathrm{rank}(x_3) = 4, & \mathrm{rank}(x_2) = 4, & \mathrm{rank}(x_1) = 2, \\ \mathrm{rank}(x_4x_3) = 2, & \mathrm{rank}(x_3x_2) = 4, & \mathrm{rank}(x_2x_1) = 2, & \\ \mathrm{rank}(x_4x_3x_2) = 2, & \mathrm{rank}(x_3x_2x_1) = 2, & & \\ \mathrm{rank}(x_4x_3x_2x_1) = 2. & & & \end{array}$$

1.5. ***L*-parameters, rank triangles and multisegments.** By [CFM+21, Section 4.3] we have that L-parameters with infinitesimal parameter λ are in bijection with H-orbits in V. Recall that the open dense H-orbit in V is defined by the ranks of all possible combinations of the matrices x_4, x_3, x_2 and x_1 appearing in $x \in V$. More generally, every H-orbit C in V is determined by the ranks that appear as

$$r_{ij} := \mathrm{rank}(x_i \cdots x_j),$$

where $1 \leq j \leq i \leq 4$, for any $x \in C$; these equations exactly describe C as a variety. We arrange these ranks into a triangle to reflect the corresponding combinations of the x_i and refer to this as a *rank triangle*:

$$\begin{array}{ccccccccc} m_4 & & m_3 & & m_2 & & m_1 & & m_0 \\ \hline & r_{44} & & r_{33} & & r_{22} & & r_{11} & \\ & & r_{43} & & r_{32} & & r_{21} & & \\ & & & r_{42} & & r_{31} & & & \\ & & & & r_{41} & & & & \end{array}$$

From left to right, the values in the top row of the rank triangle correspond, respectively, to the multiplicities m_i of the eigenvalues $\lambda_4, \lambda_3, \lambda_2, \lambda_1$, and λ_0. The row below these eigenvalue multiplicites show ranks r_{ii} of x_i, which are subject only to the condition that every rank is less than the two eigenvalue multiplicites above it. The ranks r_{ij} with $i \neq j$ are subject to exactly two conditions: every rank is less than the two ranks above it, and

$$r_{ik} - r_{ij} \leq r_{lk} - r_{lj} \qquad l < i, k < j.$$

The set of H-orbits in V is naturally in bijection with rank triangles subject to these conditions. Of the 1138 H-orbits in V, we are mainly interested in C_{KS} and C_ψ; in Appendix A.1, we find that $\dim C_{\mathrm{KS}} = 32$ and $\dim C_\psi = 40$.

We now recall the bijection between rank triangles with eigenvalues $\lambda_0 = q^{-2}$, $\lambda_1 = q^{-1}$, $\lambda_2 = q^0$, $\lambda_3 = q^1$ and $\lambda_4 = q^2$ with respective multiplicities (top row) $2, 4, 4, 4, 2$ and multisegments with support $\underline{m}_\lambda$. Henceforth, in this paper by a segment Δ we mean a (non-empty) consecutive sequence of integers $\Delta = \{j, j+1, \dots, i-1, i\} =: [j, i]$, and by a multisegment $\underline{m}$ we mean a multiset of these segments.

TABLE 1.1. Rank triangles for the H-orbits C_ψ and C_{KS} in V corresponding to the admissible representations π_ψ and π_{KS}.

$$\begin{array}{ccccccccc} & & & & C_\psi & & & & \\ 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 3 & & 3 & & 2 & \\ & & 1 & & 2 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & C_{\mathrm{KS}} & & & & \\ 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 2 & & 2 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}$$

Every rank triangle determines a multisegment by the following inductive rule:

(1) Set $\underline{m} := \{\}$ to be the emptyset.
(2) Let r_{ij} be the lowest and most to the left non-zero entry in the rank triangle.
(3) Add to $\underline{m}$ the segment $[j, i]$.
(4) Decrease the quantity r_{kl} by one for each value of k, l with $i \geq k \geq l \geq j$.
(5) Repeat until all r_{ij} are zero.
(6) The multiset $\underline{m}$ is the multisegement determined by the rank triangle.

Thus, each single segment $\Delta = [j, i]$ gives rise to a rank triangle T_Δ with

$$r_{l,k} = \begin{cases} 1 & j \leq k \leq l \leq i \\ 0 & \text{otherwise} \end{cases}.$$

A multisegment $\underline{m}$ determines the triangle

$$T_\alpha = \sum_{\Delta \in \underline{m}} T_\Delta.$$

Table 1.1 lists the rank triangles for the multisegments $\underline{m}_\psi$ and $\underline{m}_{\mathrm{KS}}$.

The set of H-orbits in V carries a partial order defined by the Zariski topology: $C \leq C'$ means $C \subseteq \overline{C'}$. This partial order can be read easily from the corresponding rank triangles as follows: If the ranks for C are r_{ij} and C' are r'_{ij} then $C \leq C'$ if and only if $r_{ij} \leq r'_{ij}$ for all i, j.

Remark 1.1. In Section 2 we will show that ϕ_{KS} is not of Arthur type. Here we sketch an alternate argument: the bijection between H-orbits in V and multisegments with the same support as $\underline{m}_\lambda$ can be used to deduce that the Langlands parameter ϕ_{KS} is not Arthur type by directly inspecting the associated multisegment. To see this, suppose ϕ_{KS} is of Arthur type and suppose there is an Arthur parameter $\psi_{\mathrm{KS}} : W''_F \to \mathrm{GL}_{16}(\mathbb{C})$ of the form

$$\psi_{\mathrm{KS}}(w, x, y) = \bigoplus_i \mathrm{Sym}^{a_i}(x) \otimes \mathrm{Sym}^{b_i}(y)$$

such that $\phi_{\mathrm{KS}}(w, x) = \psi_{\mathrm{KS}}(w, x, d_w)$ where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$. We note that the Langlands parameter associated to $\mathrm{Sym}^a(x) \otimes \mathrm{Sym}^b(y)$ has multisegment of the form

$$\begin{aligned} \{&\{-(a+b)/2 + 0, -(a+b)/2 + 1, \ldots, -(a+b)/2 + a + 0\}, \\ &\{-(a+b)/2 + 1, -(a+b)/2 + 2, \ldots, -(a+b)/2 + a + 1\}, \\ &\ldots, \\ &\{-(a+b)/2 + b, -(a+b)/2 + b + 1, \ldots, -(a+b)/2 + a + b\}\} \end{aligned}$$

and so multisegments for Langlands parameters of Arthur type must be unions of multisegments of this shape. By inspection, ϕ_{KS} is not of this sort, so ψ_{KS} does not exist. While the argument in Section 2 that ϕ_{KS} is not of Arthur type is more complicated than the sketch above, the proof given there reveals properties of the moduli space V that will be crucial for the main result of this paper, Corollary 2.4.1, which follows from Theorem 3.1.

1.6. **Relation to the Kashiwara-Saito singularity.** Representations π_ψ and π_{KS} are reverse engineered from the Kashiwara-Saito singularity, as we now explain. In [KS97, Section 7], Kashiwara and Saito provide a counter-example to a conjecture articulated in [Lus91, Section 13.7]. The counter-example concerns the quiver representation variety V of type A_5 appearing in Section 1.4. They specify two elements of $b, b' \in V$, both given explicitly. The element b' satisfies the rank conditions for C_{KS}, while the element b satisfies the rank conditions for C_ψ; see [KS97, Lemma 7.2.2]. We pick $x_{\mathrm{KS}} = (x_4, x_3, x_2, x_1) \in C_{\mathrm{KS}}$ given by

$$x_4 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4}$$

where 1 is the 2×2 identity matrix, so it follows that x_{KS} is H-conjugate to b'. The representations π_ψ and π_{KS} are defined exactly so that their Langlands parameters correspond to the two orbits C_ψ and C_{KS}, respectively, in the quiver representation variety. To see this, observe that $\phi_{\mathrm{KS}}(1, e)$ is a unipotent element of $\mathrm{GL}_{16}(\mathbb{C})$ and it lies in the unipotent orbit corresponding to the partition $16 = 3+2+2+1+3+2+2+1$, which is the same partition that classifies the nilpotent orbit of $\mathfrak{gl}_{16}(\mathbb{C})$ that x_{KS} occupies. Using [CFM+21, Proposition 4.2.2], it follows that C_{KS} is the H-conjugacy class of V matching the $\mathrm{GL}_{16}(\mathbb{C})$-conjugacy class of ϕ_{KS}, which is the L-parameter for π_{KS}. In the same way, $\phi_\psi(1, e)$ is in the same conjugacy class as $\exp x_\psi$ as elements of $\mathfrak{gl}_{16}(\mathbb{C})$, classified by the partition $16 = 4 + 4 + 2 + 2 + 2 + 2$, so C_ψ is the H-conjugacy class of V matching the $\mathrm{GL}_{16}(\mathbb{C})$-conjugacy class of ϕ_ψ, which is the L-parameter for π_ψ.

2. Geometry of the Kashiwara-Saito representation

2.1. **The conormal bundle.** Using the Killing form for $\mathfrak{gl}_{16}(\mathbb{C})$, the dual variety V^* may be realized as

$$V^* := \{x \in \mathfrak{gl}_{16}(\mathbb{C}) \mid \mathrm{Ad}(\lambda(\mathrm{Fr}))x = q^{-1}x\},$$

which, in turn, is naturally identified with

$$V^* = \mathrm{Hom}(E_{\lambda_1}, E_{\lambda_0}) \times \mathrm{Hom}(E_{\lambda_2}, E_{\lambda_1}) \times \mathrm{Hom}(E_{\lambda_3}, E_{\lambda_2}) \times \mathrm{Hom}(E_{\lambda_4}, E_{\lambda_3}).$$

Then V^* is also a representation variety for the quiver of type A_5:

$$\overset{\lambda_0}{\bullet} \xleftarrow{y_1} \overset{\lambda_1}{\bullet} \xleftarrow{y_2} \overset{\lambda_2}{\bullet} \xleftarrow{y_3} \overset{\lambda_3}{\bullet} \xleftarrow{y_4} \overset{\lambda_4}{\bullet}.$$

We use the notation:

$$y = (y_1, y_2, y_3, y_4) \in \mathrm{Mat}_{2,4}(\mathbb{C}) \times \mathrm{Mat}_{4,4}(\mathbb{C}) \times \mathrm{Mat}_{4,4}(\mathbb{C}) \times \mathrm{Mat}_{4,2}(\mathbb{C})$$

for elements of V^*. In this notation the action $H \times V^* \to V^*$ is given by

$$(h_4, h_3, h_2, h_1, h_0) \cdot (y_1, y_2, y_3, y_4) := (h_0 y_1 h_1^{-1}, h_1 y_2 h_2^{-1}, h_2 y_3 h_3^{-1}, h_3 y_4 h_4^{-1}).$$

We note that V^* is a prehomogeneous vector space for this action.

The cotangent variety for V is

$$T^*(V) = V \times V^* \cong \left\{ \begin{pmatrix} 0 & x_4 & 0 & 0 & 0 \\ y_4 & 0 & x_3 & 0 & 0 \\ 0 & y_3 & 0 & x_2 & 0 \\ 0 & 0 & y_2 & 0 & x_1 \\ 0 & 0 & 0 & y_1 & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} x_4, y_1 \in \mathrm{Mat}_{2,4}(\mathbb{C}) \\ x_3, x_2, y_2, y_3 \in \mathrm{Mat}_{4,4}(\mathbb{C}) \\ x_1, y_4 \in \mathrm{Mat}_{4,2}(\mathbb{C}) \end{array} \right\}.$$

We use the following notation for elements of $T^*(V)$:

$$(x, y) = (x_4, x_3, x_2, x_1, y_1, y_2, y_3, y_4) \in T^*(V)$$

with x_i and y_j as above.

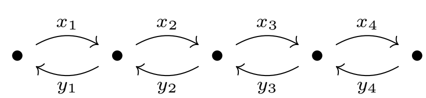
The conormal variety, $\Lambda \subset T^*(V)$, is defined as the kernel of $T^*(V) \to \mathrm{Lie}\, H$ given by

$$[x, y] := (x_4y_4, x_3y_3 - y_4x_4, x_2y_2 - y_3x_3, x_1y_1 - y_2x_2, -y_1x_1). \tag{5}$$

As the notation suggests, this is nothing more than the Lie bracket of x and y viewed as elements of $\mathfrak{gl}_{16}(\mathbb{C})$. Thus,

$$\Lambda = \left\{ \begin{pmatrix} 0 & x_4 & 0 & 0 & 0 \\ y_4 & 0 & x_3 & 0 & 0 \\ 0 & y_3 & 0 & x_2 & 0 \\ 0 & 0 & y_2 & 0 & x_1 \\ 0 & 0 & 0 & y_1 & 0 \end{pmatrix} \in T^*(V) \;\middle|\; \begin{array}{rl} x_4y_4 & = 0 \\ x_3y_3 & = y_4x_4 \\ x_2y_2 & = y_3x_3 \\ x_1y_1 & = y_2x_2 \\ 0 & = y_1x_1 \end{array} \right\}$$

Observe that Λ is the quiver representation variety for



with the relations specified above.

By [CFM+21, Proposition 6.3.1] for each orbit C,

$$\Lambda_C := \{(x, y) \in \Lambda \mid x \in C\}$$

is the conormal bundle to C. For $x \in C$, we write

$$\Lambda_x = \{y \in V^* \mid (x, y) \in \Lambda\}$$

for the fibre of the bundle Λ_C. By identifying $V^{**} \simeq V$, for C^* an H-orbit in V^* we may write

$$\Lambda_{C^*} = \{(x, y) \in \Lambda \mid y \in C^*\},$$

where Λ_{C^*} is the conormal bundle to C^*.

2.2. **The Kashiwara-Saito representation is not of Arthur type.** Let G be a connected, reductive algebraic group over a p-adic field F. A Langlands parameter $\phi : W'_F \to {}^LG$ is said to be of Arthur type if there is an Arthur parameter $\psi : W''_F \to {}^LG$ such that $\phi(w, x) = \psi(w, x, d_w)$ where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$. An irreducible admissible representation π of $G(F)$ is said to be of Arthur type if there is an Arthur parameter $\psi : W''_F \to {}^LG$ such that $\pi \in \Pi_\psi(G(F))$. For general linear groups G, since A-packets are L-packets, these two notions agree: π is of Arthur type if and only if its Langlands parameter ϕ_π is of Arthur type. In general, it is not true that the Langlands parameters of Arthur type representations are of Arthur type. In fact, in [CFM+21] we provide an example of a representation of

Arthur type whose Langlands parameter is not of Arthur type; see [CFZb, Remark 2.6] for an explanation of this example.

Theorem 2.1. *The conormal bundle* $\Lambda_{C_{\mathrm{KS}}}$ *does not have an open* H*-orbit and, consequently, the Kashiwara-Saito representation* π_{KS} *is not of Arthur type. Its coronal representation,* π_ψ*, is of Arthur type.*

Proof. It is clear that π_ψ is of Arthur type since, in Section 1.1, we exhibited an Arthur parameter ψ whose Langlands parameter ϕ_ψ is the Langlands parameter for π_ψ. We will prove that π_{KS} is not of Arthur type by contradiction.

First we must recall some notions. For every H-orbit C in V, define

$$\Lambda_C^{\mathrm{reg}} := \Lambda_C \setminus \bigcup_{C'>C} \overline{\Lambda_{C'}}. \tag{6}$$

If $(x,y) \in \Lambda_C^{\mathrm{reg}}$ we say y is a regular conormal vector to $x \in C$. In [CFM$^+$21, Section 6.5] we introduce the variety Λ^{sreg} of *strongly regular* elements of Λ defined by the property: $(x,y) \in \Lambda$ is strongly regular if the H-orbit $\mathcal{O}_H(x,y) \subseteq \Lambda$ is open and dense in Λ_C, where C is the H-orbit of x in V. It follows that $\Lambda_x^{\mathrm{sreg}}$ is non-empty if and only if Λ_x is a prehomogeneous vector space for the $Z_H(x)$-action. In [CFM$^+$21, Proposition 6.5.1] we show

$$\Lambda^{\mathrm{sreg}} \subseteq \Lambda^{\mathrm{reg}}.$$

For this proof, we find it convenient to introduce, for any H-orbit C in V, the variety

$$\Lambda'_C := \{(x,y) \in \Lambda_C \mid y \in C^*\} \tag{7}$$

situated between the regular part of the conormal variety and the conormal variety itself, above C:

$$\Lambda_C^{\mathrm{reg}} \subseteq \Lambda'_C \subseteq \Lambda_C.$$

Now, for a contradiction, suppose π_{KS} is of Arthur type, with Arthur parameter $\psi_{\mathrm{KS}} : W''_F \to \mathrm{GL}_{16}(\mathbb{C})$ where $W''_F := W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$; then $\phi_{\mathrm{KS}}(w,x) = \psi_{\mathrm{KS}}(w,x,d_w)$ where $d_w = \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})$. Note that $\exp x_\psi = \psi(1,e,1) \in V$ where $e = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Now define $y_\psi \in V^*$ by $\exp y_\psi = \psi(1,1,f)$ where $f = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$; see [CFM$^+$21, Proposition 4.2.2]. By [CFM$^+$21, Proposition 6.6.2], $(x_\psi, y_\psi) \in \Lambda_{C_\psi}^{\mathrm{sreg}}$, which means the H-orbit of (x_ψ, y_ψ) is open and dense in Λ_{C_ψ}. By [CFM$^+$21, Proposition 6.5.1], $y_\psi \in C_{\mathrm{KS}}^*$, so $(x_\psi, y_\psi) \in \Lambda'$ and the H-orbit of (x_ψ, y_ψ) is open and dense in Λ_{C_ψ}. By Proposition 2.2.3, the H-orbit of any $(x,y) \in \Lambda'_{C_{\mathrm{KS}}}$ has codimension at least 2 in Λ_{C_ψ}, so the orbit is not open and dense in Λ_{C_ψ}. In otherwords, $\Lambda_{C_\psi}^{\mathrm{sreg}}$ is empty. This contradiction proves the theorem. □

Recall that $\Lambda_C \to C$ is a bundle and $\mathcal{O}_H(x,y)$ is a subbundle of Λ_C. Also recall the dense open subbundle $\Lambda'_C \to C$ from Equation (7). For any $x \in C$, the fibres of these bundles are

$$\Lambda_x = \{y \in V^* \mid [x,y] = 0\} \qquad \text{and} \qquad \Lambda'_x = \{y \in C^* \mid [x,y] = 0\}.$$

Then Λ'_x is a dense, locally closed subvariety of the vector space Λ_x.

Lemma 2.2.1. *If* $x \in C_{\mathrm{KS}}$ *then*

$$\Lambda'_x \cong \left\{ (a,b,c,d) \in \mathrm{Mat}_{2,2}(\mathbb{C})^{\times 4} \;\middle|\; \begin{array}{c} \det(ac) \neq 0, \\ \det(ac+bd) \neq 0 \end{array} \right\}.$$

In particular, $\dim \Lambda'_x = 16$.

Proof. For any $x \in C_{\mathrm{KS}}$, there exists an $h \in H$ such that $h \cdot x = x_{\mathrm{KS}}$. Then, using the H-stability of $\Lambda'_{C_{\mathrm{KS}}}$, the morphism $\Lambda'_x \to \Lambda'_{x_{\mathrm{KS}}}$ given by $y \mapsto h \cdot y$ defines an isomorphism. By a direct calculation of $[x_{\mathrm{KS}}, y] = 0$ for $y \in C^*_{\mathrm{KS}}$, we find that $\Lambda_{x_{\mathrm{KS}}}$ is precisely the set of $y \in V^*$ that take the form

$$y_1 = \begin{pmatrix} 0 & a \end{pmatrix}, \quad y_2 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}, \quad y_4 = \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{8}$$

where a, b, c, and d are 2 by 2 matricies. Since $\widehat{C}_{\mathrm{KS}} = C_{\mathrm{KS}}$, the rank conditions which define $\Lambda'_{x_{\mathrm{KS}}} \subset \Lambda_{x_{\mathrm{KS}}}$ imply $\det(ac) \neq 0$ and $\det(ac + bd) \neq 0$. This gives us the conditions that a, c and $ac + bd$ are invertible. □

Lemma 2.2.2. *Every $Z_H(x_{\mathrm{KS}})$-orbit in the 16-dimensional vector space $\Lambda_{x_{\mathrm{KS}}}$ has dimension between 10 and 14. Consequently, the vector space $\Lambda_{x_{\mathrm{KS}}}$ with $Z_H(x_{\mathrm{KS}})$-action is not a prehomogeneous vector space.*

Proof. Since $\Lambda'_{x_{\mathrm{KS}}}$ is dense in $\Lambda_{x_{\mathrm{KS}}}$, to prove the lemma it suffices to study the dimensions of orbits passing through $\Lambda'_{x_{\mathrm{KS}}}$. Let $y_{\mathrm{KS}}(a, b, c, d) \in \Lambda'_{x_{\mathrm{KS}}}$ be defined by Equation (8). A direct calculation shows that any $h = (h_4, h_3, h_2, h_1, h_0) \in Z_H(x_{\mathrm{KS}})$ takes the form

$$h_1 = \begin{pmatrix} h_0 & u \\ 0 & k_1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad h_3 = \begin{pmatrix} k_1 & v \\ 0 & h_4 \end{pmatrix}, \tag{9}$$

for unique $h_0, h_4, k_1, k_2 \in \mathrm{GL}_2(\mathbb{C})$ and $u, v \in \mathrm{Mat}_{2,2}(\mathbb{C})$. Observe that we have $\dim Z_H(x_{\mathrm{KS}}) = 6 \times 4 = 24$. With reference to the notation above, the action of h on $\Lambda'_{x_{\mathrm{KS}}}$ is given by

$$h \cdot y_{\mathrm{KS}}(a, b, c, d) = y_{\mathrm{KS}}(h_0 a k_1^{-1}, h_0 b k_2^{-1}, k_1 c h_4^{-1}, k_2 d h_4^{-1}). \tag{10}$$

From this we see that the conjugacy class of $(ac)^{-1}(bd)$ is an invariant of this action.

We now find $Z_H(x_{\mathrm{KS}}, y_{\mathrm{KS}}(a, b, c, d))$ by solving the equations

$$a = h_0 a k_1^{-1}, \quad b = h_0 b k_2^{-1}, \quad c = k_1 c h_4^{-1}, \quad d = k_2 d h_4^{-1},$$

for $(h_0, h_4, k_1, k_2, u, v)$ in terms of (a, b, c, d). Clearly, u and v are arbitrary and h_0 and k_1 may be expressed in terms of h_4 by

$$h_0 = (ac)h_4(ac)^{-1}, \quad k_1 = c h_4 c^{-1}.$$

- If $b = 0$ and $d = 0$, then h_4 and k_2 are arbitrary; in this case,

  $$\dim Z_H(x_{\mathrm{KS}}, y_{\mathrm{KS}}(a, b, c, d)) = 16$$

  and this is largest this centralizer can be.
- At the other extreme, if b and d are both invertible, then k_2 may also be expressed in terms of h_4 by $k_2 = d h_4 d^{-1}$, but now h_4 is constrained by

  $$(ac)^{-1}(bd)h_4 = h_4(ac)^{-1}(bd),$$

  so h_4 centralizes $(ac)^{-1}(bd)$. The dimension of the centralizer of $(ac)^{-1}(bd)$ is minimal when $(ac)^{-1}(bd)$ is regular (either regular semisimple or regular unipotent modulo the centre), in which case its centralizer is 2-dimensional. It follows that

  $$\dim Z_H(x_{\mathrm{KS}}, y_{\mathrm{KS}}(a, b, c, d)) = 10,$$

  and this is the smallest this centralizer can be.

It follows that, for any $(x,y) \in \Lambda'_{C_{\mathrm{KS}}}$,

$$10 \leq \dim Z_H(x,y) \leq 16$$

and that these extreme dimensions are attained. Since $\dim Z_H(x) = 24$, for any $x \in C_{\mathrm{KS}}$, it follows that

$$8 = 24 - 16 \leq \dim \mathcal{O}_{Z_H(x)}(y) \leq 24 - 10 = 14$$

with the lower bound is attained when (x,y) is H-conjugate to $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(a,b,c,d))$ where $(ac)^{-1}(bd)$ is invertible and regular semisimple or regular unipotent. □

Proposition 2.2.3. *For every $(x,y) \in \Lambda'_{C_{\mathrm{KS}}}$, the H-orbit $\mathcal{O}_H(x,y)$ of (x,y) in $\Lambda_{C_{\mathrm{KS}}}$ has codimension at least 2 and at most 8, with these values attained.*

Proof. Recall that $\Lambda_{C_{\mathrm{KS}}} \to C_{\mathrm{KS}}$ is a bundle and $\mathcal{O}_H(x,y)$ is a subbundle of $\Lambda_{C_{\mathrm{KS}}}$. Restricting to $\mathcal{O}_H(x,y)$, the fibre of $\mathcal{O}_H(x,y) \to C_{\mathrm{KS}}$ above $x \in C_{\mathrm{KS}}$ is $\mathcal{O}_{Z_H(x)}(y)$. Since $\Lambda_{C_{\mathrm{KS}}}$ and $\mathcal{O}_H(x,y)$ are bundles over C_{KS}, the codimension of $\mathcal{O}_H(x,y)$ in $\Lambda_{C_{\mathrm{KS}}}$ is equal to

$$\begin{aligned}\dim \Lambda_{C_{\mathrm{KS}}} - \dim \mathcal{O}_H(x,y) &= (\dim C_{\mathrm{KS}} + \dim \Lambda'_x) - (\dim C_{\mathrm{KS}} + \dim \mathcal{O}_{Z_H(x)}(y)) \\ &= \dim \Lambda'_x - \dim \mathcal{O}_{Z_H(x)}(y).\end{aligned}$$

Thus, the codimension of $\mathcal{O}_H(x,y)$ in $\Lambda_{C_{\mathrm{KS}}}$ coincides with the codimension of $\mathcal{O}_{Z_H(x)}(y)$ in Λ'_x. Using $\dim \Lambda'_x = 16$ from Lemma 2.2.1, we conclude that

$$2 = 16 - 14 \leq \operatorname{codim} \mathcal{O}_{Z_H(x)}(y) \leq 16 - 8 = 8$$

with both extremes attained by Lemma 2.2.2. □

2.3. **The generic conormal bundle.** For a moment, let G be any connected reductive algebraic group over F and let $\lambda : W_F \to {}^LG$ be any infinitesimal parameter in the sense of [CFM+21, Section 4.1]. The H-variety Λ_C^{gen} appearing in the codomain of the functor

$$\mathsf{Evs}_C : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(\Lambda_C^{\mathrm{gen}})$$

is defined in [CFM+21, Section 7.9]. It is an open, dense sub-bundle of Λ_C situated between $\Lambda_C^{\mathrm{sreg}}$ and Λ_C^{reg}:

$$\Lambda_C^{\mathrm{sreg}} \subseteq \Lambda_C^{\mathrm{gen}} \subseteq \Lambda_C^{\mathrm{reg}}.$$

This sub-bundle is defined so that $\mathsf{Ev}_C\, \mathcal{P}|_{\Lambda_C^{\mathrm{gen}}}[\dim C^* - \dim V - 1]$ is an equivariant local system, for every equivariant perverse sheaf $\mathcal{P} \in \mathsf{Per}_H(V)$. We denote the equivariant fundamental group of Λ_C^{gen} by A_ϕ^{ABV} for a Langlands parameter ϕ with $x_\phi \in C$. If there is an Arthur parameter ψ with Langlands parameter ϕ, we say that C is of Arthur type; in this case $\emptyset \neq \Lambda_C^{\mathrm{sreg}} = \Lambda_C^{\mathrm{gen}} \subsetneq \Lambda_C^{\mathrm{reg}}$ and $A_\phi^{\mathrm{ABV}} \cong A_\psi := \pi_0(Z_{\widehat{G}}(\psi))$.

Return to $G = \mathrm{GL}_{16}$ over F. In this section we find an explicit description of $\Lambda_{C_{\mathrm{KS}}}^{\mathrm{gen}}$ and we show in Proposition 2.3.2 that its equivariant fundamental group, denoted by $A_{\phi_{\mathrm{KS}}}^{\mathrm{ABV}}$, is non-trivial. This was an unexpected result, since if ϕ is a Langlands parameter of Arthur type for a general linear group, then $A_\phi^{\mathrm{ABV}} = A_\psi$ is trivial, where ψ is the Arthur parameter for ϕ. The proof that the open dense subbundle $\Lambda_{C_{\mathrm{KS}}}^{\mathrm{gen}}$ described in this Section satisfies the definition appearing in [CFM+21, Section 7.9] is a consequence of the calculations performed in Section 3.

We begin by improving our parametrization of H-orbits in $\Lambda'_{C_{\mathrm{KS}}}$, or more precisely, by making a more detailed study of the $Z_H(x_{\mathrm{KS}})$-orbits in the 16-dimensional variety $\Lambda'_{x_{\mathrm{KS}}}$. From Section 2.2, recall

$$\Lambda'_{x_{\mathrm{KS}}} = \left\{ y_{\mathrm{KS}}(a,b,c,d) \in \mathrm{Mat}_{2,2}(\mathbb{C})^{\times 4} \;\middle|\; \begin{array}{c} \det(ac) \neq 0, \\ \det(ac+bd) \neq 0 \end{array} \right\}.$$

As we remarked in the proof of Lemma 2.2.3, a direct calculation shows that the $\mathrm{GL}_2(\mathbb{C})$-conjugacy class of $(ac)^{-1}(bd) \in \mathrm{GL}_2(\mathbb{C})$ is a fundamental invariant of the $Z_H(x_{\mathrm{KS}})$-orbit of $y_{\mathrm{KS}}(a,b,c,d)$.

Let Z be the Steinberg quotient for $\mathrm{Mat}_{2,2}$; we write $[g] \in Z$ for the characteristic polynomial or the GL_2-conjugacy class of $g \in \mathrm{Mat}_{2,2}$, according to taste. Set $Z' = \{[g] \in Z \mid \det(1+g) \neq 0\}$. The map

$$\begin{array}{rcl} \Lambda'_{x_{\mathrm{KS}}} & \to & Z' \\ y_{\mathrm{KS}}(a,b,c,d) & \mapsto & [(ac)^{-1}(bd)] \end{array} \tag{11}$$

is a useful tool in the study of $\Lambda'_{x_{\mathrm{KS}}}$. We find the smooth locus of this map.

Lemma 2.3.1. *If $y \in \Lambda'_{x_{\mathrm{KS}}}$ and $2 \leq \operatorname{codim} \mathcal{O}_{Z_H(x_{\mathrm{KS}})}(y) \leq 4$ then there is an $h \in Z_H(x_{\mathrm{KS}})$ such that $h \cdot y = y_{\mathrm{KS}}(1,b,1,1)$ for $b \in \mathrm{GL}_2(\mathbb{C})$ such that $\det(1+b) \neq 0$. The $\mathrm{GL}_2(\mathbb{C})$-conjugacy class of b is an invariant of the $Z_H(x_{\mathrm{KS}})$-orbit of y.*

Proof. From the proof of Lemma 2.2.2, if $y \in \Lambda'_{x_{\mathrm{KS}}}$ then $y = y_{\mathrm{KS}}(a,b,c,d)$ for $a,c \in \mathrm{GL}_2(\mathbb{C})$ and $b,d \in \mathrm{Mat}_{2,2}(\mathbb{C})$ such that $\det(ac+bd) \neq 0$. Further, the proof can be extended to show that $2 \leq \operatorname{codim} \mathcal{O}_{Z_H(x_{\mathrm{KS}})}(y) \leq 4$ if and only if $b,d \in \mathrm{GL}_2(\mathbb{C})$. In this case, we set

$$k_1 := h_4 c^{-1}, \qquad k_2 := h_4 d^{-1}, \qquad h_0 := h_4 (ac)^{-1},$$

and let u and v be arbitrary; using Equations (9) and (10), this determines an $h \in Z_H(x_{\mathrm{KS}})$ such that

$$h \cdot y_{\mathrm{KS}}(a,b,c,d) = y_{\mathrm{KS}}(1,b',1,1),$$

where $b' := h_4(ac)^{-1}(bd)h_4^{-1}$. Henceforth we use the notation

$$y_{\mathrm{KS}}(b') := y_{\mathrm{KS}}(1,b',1,1), \tag{12}$$

for any $b' \in \mathrm{GL}_2(\mathbb{C})$. We have already seen that $q(y) = [b'] = [(ac)^{-1}(bd)]$ is an invariant of the $Z_H(x_{\mathrm{KS}})$-orbit of y. A direct calculation shows that if

$$h \cdot y_{\mathrm{KS}}(b) = y_{\mathrm{KS}}(b')$$

then $[b] = [b']$. □

We now extend $\Lambda'_{x_{\mathrm{KS}}} \to Z'$ to

$$q' : \Lambda'_{C_{\mathrm{KS}}} \to Z',$$

which we will use to give a concrete definition of $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ which agrees with [CFM+21, Section 7.9]. For every $(x,y) \in \Lambda'_{C_{\mathrm{KS}}}$, recall from Section 2.1 that $x = (x_4,x_3,x_2,x_1)$ and $y = (y_1,y_2,y_3,y_4)$ where $x_i : E_{\lambda_{i-1}} \to E_{\lambda_i}$ and $y_i : E_{\lambda_i} \to E_{\lambda_{i_1}}$ for $1 \leq i \leq 4$. The condition $y \in C^*_{\mathrm{KS}}$ implies $\operatorname{rank} y_3 = 2$, so $\dim\ker y_2 = 2$. Likewise, $x \in C_{\mathrm{KS}}$ implies $\operatorname{rank} x_2 = \operatorname{rank} x_3 = \operatorname{rank} x_2x_3 = 2$; note that

$$E_{\lambda_2} = \operatorname{im} x_2 \oplus \ker x_3.$$

Also note that $\dim \operatorname{im} y_2 = 2$. Let a, b, c, d be the linear transformations of 2-dimensional vector spaces defined by

$$\begin{array}{rcl} a & := & y_2|_{\operatorname{im} x_2} : \operatorname{im} x_2 \to \operatorname{im} y_2 \\ b & := & y_2|_{\ker x_3} : \ker x_3 \to \operatorname{im} y_2 \\ c & := & y_3 : E_{\lambda_3}/\ker y_3 \to E_{\lambda_2} \to E_{\lambda_2}/\ker x_3 \\ d & := & y_3 : E_{\lambda_3}/\ker y_3 \to E_{\lambda_2} \to E_{\lambda_2}/\operatorname{im} x_2, \end{array}$$

as pictured below,

(13)
$$\begin{array}{ccccccc} & & \operatorname{im} x_2 & \xleftarrow{\cong} & E_{\lambda_2}/\ker x_3 & & \\ & \swarrow^{a} & \searrow & \nearrow & & \nwarrow^{c} & \\ \operatorname{im} y_2 & \xleftarrow{y_2'} & & E_{\lambda_2} & & \xleftarrow{y_3'} & E_{\lambda_3}/\ker y_3 \\ & \nwarrow_{b} & \nearrow & \searrow & & \swarrow_{d} & \\ & & \ker x_3 & \xleftarrow[\cong]{} & E_{\lambda_2}/\operatorname{im} x_2, & & \end{array}$$

where the maps y_2' and y_3' are induced from y_2 and y_3. This notation is compatible with Lemma 2.2.2, which is to say, if $x = x_{\mathrm{KS}}$ then $y = y_{\mathrm{KS}}(a, b, c, d)$. Returning to the general case of $(x, y) \in \Lambda'_{C_{\mathrm{KS}}}$, we know $\det(ac) \neq 0$. A direct calculation shows that the $\mathrm{GL}_2(\mathbb{C})$-conjugacy class of the endomorphism

$$(ac)^{-1}(bd) : E_{\lambda_3}/\ker y_3 \to E_{\lambda_3}/\ker y_3$$

is an invariant of the H-conjugacy class of $(x, y) \in \Lambda'_{C_{\mathrm{KS}}}$. In this way we define

(14)
$$\begin{array}{rcl} q' : \Lambda'_{C_{\mathrm{KS}}} & \to & Z' \\ (x, y) & \mapsto & [(ac)^{-1}(bd)]. \end{array}$$

Let Z_{reg} be the open dense subvariety of Z over which the Steinberg quotient is smooth and recall that the fibre of the Steinberg quotient over Z_{reg} is $\mathrm{GL}_2(\mathbb{C})_{\mathrm{rss}}$, the subvariety of regular semisimple elements of $\mathrm{GL}_2(\mathbb{C})$. Then $Z_{\mathrm{reg}} = T_{\mathrm{reg}}//W$, the moduli space of regular semisimple conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$, where T is the maximal torus in $\mathrm{GL}_2(\mathbb{C})$. Set $T' := \{t \in T \mid \det(1+t) \neq 0\}$ and recall $Z' := \{[g] \in Z \mid \det(1+g) \neq 0\}$. Set $T'_{\mathrm{reg}} = T' \cap T_{\mathrm{reg}}$ and $Z'_{\mathrm{reg}} = Z' \cap Z_{\mathrm{reg}}$. Let $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ be the dense, open subvariety of $\Lambda_{C_{\mathrm{KS}}}$ defined by the pullback $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to Z'_{\mathrm{reg}}$ of q' along $Z'_{\mathrm{reg}} \hookrightarrow Z'$:

$$\begin{array}{ccc} \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} & \xrightarrow{q} & Z'_{\mathrm{reg}} \\ \downarrow & & \downarrow \\ \Lambda'_{C_{\mathrm{KS}}} & \xrightarrow{q'} & Z'. \end{array}$$

Proposition 2.3.2. *If $(x, y) \in \Lambda^{gen}_{C_{\mathrm{KS}}}$ then codim $\mathcal{O}_H(x, y) = 2$. The bundle $q : \Lambda^{gen}_{C_{\mathrm{KS}}} \to Z'_{reg}$ is smooth. The equivariant fundamental group $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ of $\Lambda^{gen}_{C_{\mathrm{KS}}}$ is non-trivial.*

Proof. The first point is baked into the definition of $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ using Proposition 2.2.3. For the second, observe that q is locally trivializable and

$$q^{-1}([t]) = \mathcal{O}_H(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t)) \cong H/K$$

where $K = Z_H(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t))$ is the group of $h = (h_0, h_1, h_2, h_3, h_4) \in H$ such that $h_0 \in T(\mathbb{C})$ and

$$h_1 = \begin{pmatrix} h_0 & u \\ 0 & h_0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} h_0 & 0 \\ 0 & h_0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} h_0 & v \\ 0 & h_0 \end{pmatrix}, \quad h_4 = h_0;$$

this group K does not depend on t. Finally, consider the pull-back $z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ of z along q:

$$\begin{array}{ccc} \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} & \xrightarrow{q'} & T'_{\mathrm{reg}} \\ {\scriptstyle z'}\downarrow & \swarrow & \downarrow{\scriptstyle z} \\ \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} & \xrightarrow{q} & Z'_{\mathrm{reg}}. \end{array}$$

Then $z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ is an H-equivariant double cover. Since the equivariant fundamental group of a connected H-variety S is the fundamental group for the site of H-equivariant etale covers of S, it follows that $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ is not trivial. □

2.4. A curious ABV-packet.

Corollary 2.4.1. *For any p-adic field F, the ABV-packet for ϕ_{KS} contains exactly two representations:*

$$\Pi^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}(\mathrm{GL}_{16}(F)) := \{\pi_{\mathrm{KS}}, \pi_\psi\}.$$

Proof. Using [CFM+21, Definition 1], this corollary is a direct consequence of Theorem 3.1, which we will prove in Section 3, relying on tools explained in Appendix A. We note that general linear groups have no pure inner forms. □

3. The main geometric result

Recall the functor $\mathsf{Evs}_C : \mathsf{Per}_H(V) \to \mathsf{Loc}_H(\Lambda^{\mathrm{gen}}_C)$ from [CFM+21, Section 7.9].

Theorem 3.1.

$$\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_C) = \begin{cases} \mathbb{1}_{\Lambda^{gen}_{C_{\mathrm{KS}}}} & C = C_{\mathrm{KS}}, \\ \mathcal{L}_{\Lambda^{gen}_{C_{\mathrm{KS}}}} & C = C_\psi. \\ 0, & \textit{otherwise}, \end{cases}$$

where $\mathcal{L}_{\Lambda^{gen}_{C_{\mathrm{KS}}}}$ is the rank-1 local system defined by the double cover $z' : \widetilde{\Lambda}^{gen}_{C_{\mathrm{KS}}} \to \Lambda^{gen}_{C_{\mathrm{KS}}}$ appearing in the proof of Proposition 2.3.2.

The proof of Theorem 3.1 will occupy all of Section 3. Some of the supporting computations for the proof are explained in Appendix A.

Corollary 3.0.1. *The rank of $\Lambda_{C_{\mathrm{KS}}}$ in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ is 1 and the rank of Λ_{C_ψ} in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ is 1.*

Proof. By [CFM+21, Proposition 7.6.2], rank $\mathsf{Evs}_{C_\psi} \mathcal{IC}(\mathbb{1}_{C_\psi}) = 1$. By Theorem 3.1, rank $\mathsf{Ev}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_\psi}) = 1$. By [CFM+21, Section 7.11], this proves the corollary. □

Remark 3.2. By [KS97, Theorem 7.2.1], the rank of $\Lambda_{C_{\mathrm{KS}}}$ in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ is at least 1 and the rank of Λ_{C_ψ} in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ is at least 1. Thus, Theorem 3.1 is stronger than [KS97, Theorem 7.2.1]. The remark following [KS97, Theorem 7.2.1] promises – but does not prove – that the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C_\psi})$ is exactly $\Lambda_{C_{\mathrm{KS}}}$ and Λ_{C_ψ}. This is consistent with Theorem 3.1.

For use in the arguments below, we define $y_{\mathrm{KS}} : T'_{\mathrm{reg}} \to \Lambda^{\mathrm{gen}}_{x_{\mathrm{KS}}}$ by

$$y_{\mathrm{KS}}(t_1, t_2) := y_{\mathrm{KS}} \begin{pmatrix} t_1 & 1 \\ 0 & t_2 \end{pmatrix}. \tag{15}$$

Then $q(y_{\mathrm{KS}}(t_1, t_2)) = z(t_1, t_2)$ for $\mathrm{diag}(t_1, t_2) \in T'_{\mathrm{reg}}$. The results of Section 2.3 show that $y_{\mathrm{KS}}(T'_{\mathrm{reg}})$ is a parametrized slice of $\Lambda^{\mathrm{gen}}_{x_{\mathrm{KS}}}$, intersecting each $Z_H(x_{\mathrm{KS}})$-orbit exactly twice. We use this parametrized slice of $\Lambda^{\mathrm{gen}}_{x_{\mathrm{KS}}}$ because it makes the singularities of the cover $\rho_\psi : \widetilde{C}_\psi \to \overline{C}_\psi$ above $x_{\mathrm{KS}} \in C_{\mathrm{KS}} \subset \overline{C}_\psi$ easily accessible in the affine chart for $\widetilde{C}_\psi$ that intersects $\rho_\psi^{-1}(x_{\mathrm{KS}})$ appearing in Section 3.7. We return to the consequences of this choice in Section 3.8.

3.1. $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$. We start the proof of Theorem 3.1 with

$$\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}}) = \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}. \tag{16}$$

The local system $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$ is studied in [CFM+21, Section 7.10], where it is shown that it is a rank-1 local system corresponding to a character of $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ of order at most 2. Equation (16) is equivalent to the claim that this character is trivial. Pick $(x, y) = (x_{\mathrm{KS}}, y_{\mathrm{KS}}(t)) \in \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ so that we may identify $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ with the equivariant fundamental group of $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ at (x, y) and find the character as a representation of the one-dimensional vector space $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})_{(x,y)}$. Use [CFM+21, Theorem 7.7.5] to see that the character of $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ associated to $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$ is trivial if and only if the Hessian determinant of f at (x, y) is a square in the local coordinate ring of the singular locus at this point. To that end, we employ the following method, using Macaulay2 code documented in Appendix A.3.

(1) Use the Jacobian of the ideal defining the closure of $C_{\mathrm{KS}} \times C^*_{\mathrm{KS}}$ to identify a system of variables, from the polynomial ring over which $C_{\mathrm{KS}} \times C^*_{\mathrm{KS}}$ is defined, that may be used as local coordinates for the (completed) local ring at the point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t))$.
(2) Use implicit partial differentiation to solve for the partial derivatives of the other variables with respect to the chosen system of local coordinates.
(3) Use partial differentiation with the chain rule to compute the Hessian of f as a function in the completed local ring. Recall the Hessian is a symmetric matrix of functions from the completed local ring.
(4) Verify that the rank of the Hessian evaluated at (x, y) is $e_{C_{\mathrm{KS}}} = \dim C_{\mathrm{KS}} - \operatorname{codim} C^*_{\mathrm{KS}} = 32 - (48 - 32) = 16$.
(5) Find a 16×16 minor of the Hessian whose rank when evaluated at (x, y) is 16.
(6) Find an ordering of these 16 variables that makes it clear the determinant of this minor is the square of a function in the local ring.

Using this method, we found that the Hessian determinant is locally the perfect square of a function and hence $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$ is the constant local system.

Remark 3.3. These calculations verify the claim about points of $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ even if (x, y) is not generic because the rank of the Hessian is 16 and the calculations give us the same conclusion on an open neighbourhood of (x, y) in $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$, which must include generic vectors.

3.2. **Reduction of the problem.** For each orbit C in V, there exists a unique orbit C^* for V^* with $\overline{\Lambda_C} \cong \overline{\Lambda_{C^*}}$; see, for example, [Pja75, Corollary 2]. The map $C \mapsto C^*$ defines a bijection between the set of H-orbits in V and the set of H-orbits in V^*. Another bijection between H-orbits in V^* and H-orbits in V, called transposition, is $t : V^* \to V$ defined by $y = (y_1, y_2, y_3, y_4) \mapsto {}^t y := ({}^t y_4, {}^t y_3, {}^t y_2, {}^t y_1)$, where ${}^t y_i$ is the transpose of the matrix y_i. Now set

$$\widehat{C} := {}^t C^*.$$

Then $C \mapsto \widehat{C}$ is an involution on the set of H-orbits in V. The involution $C \mapsto \widehat{C}$ on H-orbits in V is the Zelevinsky involution on multisegments; we refer the reader to [KZ96] for more in this direction. The orbits C_{KS} and C_ψ are self-dual:

$$\widehat{C}_{\text{KS}} = C_{\text{KS}} \qquad \text{and} \qquad \widehat{C}_\psi = C_\psi.$$

Proposition 3.2.1. *If* $\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_{C'}) \neq 0$ *then* $\mathsf{Evs}_{\widehat{C}}\, \mathcal{IC}(\mathbb{1}_{\widehat{C}'}) \neq 0$ *and* $C \leq C'$ *and* $\widehat{C} \leq \widehat{C}'$.

Proof. By combining [CFM+21, Proposition 7.4.1] and [Kas90, Proposition 8.6.4] we have that $\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_{C'})$ is non-zero if and only if Λ_C is contained in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C'})$. Combining [EM97, Proposition 7.2] and [Kas90, Theorem 5.5.5] it follows that Λ_C is contained in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{C'})$ if and only if $\Lambda_{\widehat{C}}$ is contained in the characteristic cycles of $\mathcal{IC}(\mathbb{1}_{\widehat{C}'})$. Thus $\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_{C'}) \neq 0$ if and only if $\mathsf{Evs}_{\widehat{C}}\, \mathcal{IC}(\mathbb{1}_{\widehat{C}'}) \neq 0$. By the definition of Evs_C it is clear that $\mathsf{Evs}_C\, \mathcal{IC}(\mathbb{1}_{C'}) \neq 0$ implies $C \leq C'$ and correspondingly that $\mathsf{Evs}_{\widehat{C}}\, \mathcal{IC}(\mathbb{1}_{\widehat{C}'}) \neq 0$ implies $\widehat{C} \leq \widehat{C}'$. □

Remark 3.4. Proposition 3.2.1 is equivalent to the following statement:

$$a^* \left(\left(\mathsf{R}\Phi_f \left(\mathcal{IC}(\mathbb{1}_C) \boxtimes \mathbb{1}^!_{C^*} \right) \right) |_{\Lambda^{\text{gen}}_C} \right) = 0 \iff \left(\mathsf{R}\Phi_f \left(\mathcal{IC}(\mathbb{1}_{C^*}) \boxtimes \mathbb{1}^!_{C} \right) \right) |_{\Lambda^{\text{gen}}_{C^*}} = 0,$$

where $a : T^*(V) \to T^*(V^*)$ is defined by $a(x, y) = (y, -x)$ using $(V^*)^* \cong V$.

Lemma 3.2.2. *Besides* C_{KS}*, there are exactly six* H*-orbits* C *that satisfy the conditions*

$$C_{\text{KS}} \leq C \quad \textit{and} \quad C_{\text{KS}} \leq \widehat{C}.$$

These six orbits are listed in Table 3.1.

Proof. The proof is given by a direct calculation using Macaulay2 as explained in Appendix A.1. □

The proof of Theorem 3.1 is now reduced to showing $\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_C) = 0$ for the six orbits C from Lemma 3.2.2 and $\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_\psi}) = \mathcal{L}_{\Lambda^{\text{gen}}_{C_{\text{KS}}}}$. We have already seen that the multisegments for C_ψ and C_{KS} are self-dual, hence $\widehat{C}_\psi = C_\psi$ and $\widehat{C}_{\text{KS}} = C_{\text{KS}}$. A simple calculation, illustrated in Appendix A.1, shows that $C_R = \widehat{C}_L$ and $C_r = \widehat{C}_l$. It now follows from Proposition 3.2.1 that

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_L}) = 0 \qquad \iff \qquad \mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_R}) = 0$$

and

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_l}) = 0 \qquad \iff \qquad \mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_r}) = 0.$$

Therefore, the proof of Theorem 3.1 reduces to the following four statements:

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_r}) = 0 \tag{17}$$

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_m}) = 0 \tag{18}$$

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_R}) = 0 \tag{19}$$

$$\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_\psi}) = \mathcal{L}_{\Lambda^{\text{gen}}_{C_{\text{KS}}}}. \tag{20}$$

Table 3.1. The six H-orbits $C \subset V$ that satisfy the relations $C_{\mathrm{KS}} < C$ and $C_{\mathrm{KS}} < \widehat{C}$.

$$C_L:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&4&&2&&2&\\ &&2&&2&&0&&\\ &&&0&&0&&&\\ &&&&0&&&& \end{array} \qquad C_\psi:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&3&&3&&2&\\ &&1&&2&&1&&\\ &&&1&&1&&&\\ &&&&0&&&& \end{array} \qquad C_R:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&2&&4&&2&\\ &&0&&2&&2&&\\ &&&0&&0&&&\\ &&&&0&&&& \end{array}$$

$$C_l:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&3&&2&&2&\\ &&1&&2&&0&&\\ &&&0&&0&&&\\ &&&&0&&&& \end{array} \qquad C_m:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&3&&3&&2&\\ &&1&&2&&1&&\\ &&&0&&0&&&\\ &&&&0&&&& \end{array} \qquad C_r:\quad \begin{array}{ccccccccc} 2&&4&&4&&4&&2\\ \hline &2&&2&&3&&2&\\ &&0&&2&&1&&\\ &&&0&&0&&&\\ &&&&0&&&& \end{array}$$

3.3. **Overview of the calculations.** We now explain the strategy of the proofs of Equations (17), (18), (19) and (20).

We have seen that the rank triangle for every H-orbit C records the equations that define C as a variety in V:

$$C = \{x \in V \mid \operatorname{rank}(x_i \cdots x_j) = r_{ij}, 1 \le j \le i \le 4\},$$

where r_{ij} refer to the terms appearing in the rank triangle for C. The closure of C in V is then given by

$$\overline{C} = \{x \in V \mid \operatorname{rank}(x_i \cdots x_j) \le r_{ij}, 1 \le j \le i \le 4\}.$$

The rank triangle can also be used to construct a smooth H-variety $\widetilde{C}$ and an H-equivariant proper morphism

$$\rho : \widetilde{C} \to \overline{C}.$$

We show how this is done in each of the cases C_r, C_m, C_R, C_ψ in Sections 3.4 through 3.7.

Now let C be any of the orbits C_r, C_m, C_R or C_ψ. Then, by construction, $\rho : \widetilde{C} \to \overline{C}$ is an isomorphism over C. It follows from the decomposition theorem [BBD82, Théorème 6.2.5], [dCM09] that the push forward $\rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}]$ is a semisimple complex containing $\mathcal{IC}(\mathbb{1}_C)$: so

$$\rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}] = \mathcal{IC}(\mathbb{1}_C) \oplus \bigoplus_{C',i} m_i(C', C) \mathcal{IC}(\mathbb{1}_{C'})[d_{C',C,i}],$$

for non-negative integers $m_i(C', C)$ and integers $d_{C',C,i}$, where C' ranges over all orbits $C' < C$ and i ranges over $\mathbb{Z}$.

Remark 3.5. In fact, each ρ is semi-small, so the shift integers $d_{C,C',i}$ are all 0, and we can calculate all strata C' that are relevant to the cover, in the sense of [dCM09, Definition 4.2.3], namely those C' for which $m_i(C', C) \neq 0$; we only actually need to verify this for a small number of orbits in one case, see Remark 3.10. The code which does these checks is documented in Appendix A.2. We expect that the procedure we use to construct ρ will always yield a semi-small cover.

By [CFM+21, Proposition 7.4.1],

$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}] = \mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_C) \oplus \bigoplus_{C',i} m_i(C',C)\, \mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C'})[d_{C',C,i}], \tag{21}$$

where the sum is taken over the H-orbits C' such that $C_{\mathrm{KS}} \leq C'$ and $C_{\mathrm{KS}} \leq \widehat{C'}$ and $C' < C$; by Lemma 3.2.2 there are at most 7 such C'. Our strategy to calculate $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_C)$ is to use Equation (21) inductively, starting with small orbits C, and in each case calculate $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}]$.

Using [CFM+21, Lemma 7.5.2], the left-hand side of Equation (21) is given by

$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}] = \rho''_! \left(\left(\mathsf{R}\Phi_{\tilde{f}} \mathbb{1}_{\widetilde{C} \times C^*_{\mathrm{KS}}} \right) |_{\tilde{\mathcal{O}}} \right) [\dim \widetilde{C} - \operatorname{codim} C^*_{\mathrm{KS}}],$$

where $\tilde{f} : \widetilde{C} \times C^*_{\mathrm{KS}} \to \mathbb{A}^1$ is defined by $\tilde{f}(\tilde{x}, y) = (\, \rho(\tilde{x}) \,|\, y\,)$, $(\ |\ ) : V \times V^* \to \mathbb{A}^1$ is the usual pairing, $\rho' = \rho'$, $\tilde{\mathcal{O}} := (\rho')^{-1}(\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}})$ and $\rho'' : \tilde{\mathcal{O}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ is the pullback of ρ' along $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \overline{C} \times C^*_{\mathrm{KS}}$; see the following diagram:

$$\begin{array}{ccccccc}
U & \longleftarrow & U \times C^*_{\mathrm{KS}} & & & & \\
\downarrow & & \downarrow & & & & \\
\widetilde{C} & \longleftarrow & \widetilde{C} \times C^*_{\mathrm{KS}} & \longleftarrow & \tilde{\mathcal{O}} & \longleftarrow & \operatorname{sing} \tilde{f}^{-1}(0) \cap \tilde{\mathcal{O}} \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho''} & \swarrow{\scriptstyle \rho'''} & \\
\overline{C} & \longleftarrow & \overline{C} \times C^*_{\mathrm{KS}} & \longleftarrow & \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} & & \\
& & \downarrow{\scriptstyle (\ |\ )} & & \downarrow & & \\
& & \mathbb{A}^1 & \longleftarrow & \{0\} & &
\end{array}$$

Both $\tilde{f}$ and ρ'' are H-equivariant. By [CFM+21, Proposition 7.8.1 and Lemma 7.8.2], we know that

$$\mathsf{R}\Phi_{\tilde{f}}[-1] \mathbb{1}_{\widetilde{C} \times C^*_{\mathrm{KS}}}[\dim C + \dim C^*_{\mathrm{KS}}]$$

is a perverse sheaf supported on the singular locus of $\tilde{f}^{-1}(0)$ in $\tilde{\mathcal{O}}$. We are interested in the restriction of this perverse sheaf to $\tilde{\mathcal{O}}$ which, when shifted by $-\dim V$, is a finite rank local system with support $(\operatorname{sing} \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$. Set

$$\mathcal{M} := \left(\mathsf{R}\Phi_{\tilde{f}}[-1] \mathbb{1}_{\widetilde{C} \times C^*_{\mathrm{KS}}} \right) |_{(\operatorname{sing} \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}} [\dim C - \operatorname{codim} C^*_{\mathrm{KS}}]. \tag{22}$$

Then

$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}] = \rho'''_! \mathcal{M}$$

where ρ''' is the restriction of ρ'' to $(\operatorname{sing} \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$. In each of the four cases appearing in Equations (17) through (20), we use the Jacobian condition for smoothness to study $(\operatorname{sing} \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$. We use the following general strategy.

(1) We compute, by hand, the fibre of ρ above x_{KS}. In all cases, we find that the fibre is a single point or projective.
(2) We choose an affine chart U for $\widetilde{C}$ that intersects $\rho^{-1}(x_{\mathrm{KS}})$. We form the Jacobian for the system of equations that describe $(U \times C^*_{\mathrm{KS}}) \cap \tilde{f}^{-1}(0)$. We then perform Steps 4-7 for each of these Jacobians. In the case that $\rho^{-1}(x_{\mathrm{KS}})$ is projective we will need to vary U over an atlas for $\widetilde{C}$.
(3) We compute the generic rank of the Jacobian from Step 2.

(4) We evaluate the Jacobian at the generic element $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$ and further add the conditions on the variables in the Jacobian that correspond to $U \cap \rho^{-1}(x_{\mathrm{KS}})$. The Jacobian is now a matrix M over some polynomial ring $\mathbb{Q}[t_1, t_2, X_1, \ldots, X_s]$, where the X_i are variables used to describe the fibre from Step 2; in the case that the fibre is a single point, there are no X_i.
(5) Next, we perform elementary row and column operations on the matrix M. In each case, we are left with a block diagonal matrix whose blocks consist of an identity matrix I_m of some size m, and a matrix B over $\mathbb{Q}[t_1, t_2, X_1, \ldots, X_s]$.
(6) The conditions that describe when the rank of M is strictly less than the generic rank from Step 4 are now encoded in the matrix B. This gives simple equations for $\mathrm{sing}(\tilde{f}^{-1}(0) \cap (U \times C^*_{\mathrm{KS}})) \cap (\rho')^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$.
(7) Return to Step 2 until $\widetilde{C} \cap \rho^{-1}(x_{\mathrm{KS}})$ has been covered by affine charts U for $\widetilde{C}$. This determines $(\mathrm{sing}\, \tilde{f}^{-1}(0)) \cap (\rho')^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$. Finally, letting H act on this set we find $(\mathrm{sing}\, \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$.

In the cases where C is either C_r, C_m or C_R, the intersection $(\mathrm{sing}\, \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$ is empty, so $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_! \mathbb{1}_{\widetilde{C}}[\dim \widetilde{C}] = \rho'''_! \mathcal{M} = 0$. This gives Equations (17), (18) and (19). In the only remaining case, when $C = C_\psi$, we find that the intersection $(\mathrm{sing}\, \tilde{f}^{-1}(0)) \cap \tilde{\mathcal{O}}$ is non-empty and the map ρ'''_ψ is $2:1$. We find the rank of the local system $\mathcal{M}$ and the corresponding representation of the fundamental group $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ by computing the Hessian of $\tilde{f}$ near a singular point of $\tilde{f}$ in $\tilde{\mathcal{O}}$. After verifying that the rank of this Hessian is the codimension of the singular locus, we conclude that the rank of $\mathcal{M}$ is 1. Further, the determinant of this Hessian determinant then identifies the cover which trivializes $\mathcal{M}$. The strategy to compute the Hessian locally near a point in the singular locus of $\tilde{f}$ is the same strategy employed in Section 3.1 and documented in detail in Appendix A.3.

Remark 3.6. All of the calculations we are describing are performed without finding a Gröbner basis for the ideal defining $\tilde{\mathcal{O}}$ as computing this would have been intractable.

3.4. $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_r})$**.** In this section we prove Equation (17); see Appendix A.2.1 for a detailed description of the code used for the computations in this section.

We begin by defining $\rho_r : \widetilde{C}_r \to \overline{C}_r$. Recall the rank triangle for C_r from Table 3.1. As we saw in Section 1.5, the top row describes the infinitesimal parameter of the representations by listing the multiplicites of the eigenvalues of $\lambda(\mathrm{Fr})$; these eigenvalues are, from right to left in the rank triangle, $\lambda_0 = q^{-2}$, $\lambda_1 = q^{-1}$, $\lambda_2 = q^0$, $\lambda_3 = q^1$, $\lambda_4 = q^2$ and the multiplicities of these eigenvalues are, in the same order, 2, 4, 4, 4, 2. For each eigenvalue λ_i, consider the complete flag variety $\mathcal{F}(E_{\lambda_i})$ for $\mathrm{GL}(E_{\lambda_i})$, which is to say, consider the projective variety of complete flags in E_{λ_i}; points on this variety are given by a chain of subspaces of E_{λ_i}

$$0 \subset E^1_{\lambda_i} \subset E^2_{\lambda_i} \subset \cdots \subset E_{\lambda_i},$$

where $\dim E^j_{\lambda_i} = j$. The action of H on the flag variety is defined by letting $h \in H$ send this chain to

$$0 \subset h_i(E^1_{\lambda_i}) \subset h_i(E^2_{\lambda_i}) \subset \cdots \subset h_i(E_{\lambda_i}),$$

We make one complete chain for each eigenspace and then use the rank-triangle to introduce relations on these flags, thus defining a subvariety of $\prod_{i=0}^{4} \mathcal{F}(E_{\lambda_i})$, as follows.

(1) Looking at the second row of the rank triangle for C_r we see the ranks $(2, 2, 3, 2)$. This means, for any $x = (x_4, x_3, x_2, x_1) \in C_r$, we have $\operatorname{rank} x_1 = 2$, $\operatorname{rank} x_2 = 3$, $\operatorname{rank} x_3 = 2$ and $\operatorname{rank} x_4 = 2$.
   (a) Since $x_1 : E_{\lambda_0} \to E_{\lambda_1}$ and $\operatorname{rank} x_1 = 2$, we declare that the image of x_1 should be contained in $E^2_{\lambda_1}$, which is to say $x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1}$.
   (b) Since $x_2 : E_{\lambda_1} \to E_{\lambda_2}$ and $\operatorname{rank} x_2 = 3$, we declare that the image of x_2 should be contained in $E^3_{\lambda_2}$, so $x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2}$.
   (c) Since $x_3 : E_{\lambda_2} \to E_{\lambda_3}$ and $\operatorname{rank} x_3 = 2$, we declare that the image of x_3 should be contained in $E^2_{\lambda_3}$, so $x_3(E_{\lambda_2}) \subseteq E^2_{\lambda_3}$.
   (d) Since $x_4 : E_{\lambda_3} \to E_{\lambda_4}$ and $\operatorname{rank} x_4 = 2$, we declare that the image of x_4 should be contained in $E^2_{\lambda_4}$. This is no condition at all since $E^2_{\lambda_4} = E_{\lambda_4}$.

(2) Looking at the third row of the rank triangle for C_r we see the ranks $(0, 2, 1)$. This means $\operatorname{rank} x_2x_1 \leq 1$, $\operatorname{rank} x_3x_2 \leq 2$ and $\operatorname{rank} x_4x_3 \leq 0$.
   (a) Since $x_2x_1 : E_{\lambda_0} \to E_{\lambda_2}$ and $\operatorname{rank} x_2x_1 \leq 1$, we declare $x_2(E^2_{\lambda_1}) \subseteq E^1_{\lambda_2}$. Note that this is compatible with the condition $x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_2}$ from the rank triangle and condition $x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1}$ from above.
   (b) Since $x_3x_2 : E_{\lambda_1} \to E_{\lambda_3}$ and $\operatorname{rank} x_3x_2 \leq 2$, we declare $x_3(E^3_{\lambda_2}) \subseteq E^2_{\lambda_3}$. Note that this is compatible with $x_3x_2(E_{\lambda_1}) \subseteq E^2_{\lambda_3}$ and $x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2}$.
   (c) Since $x_4x_3 : E_{\lambda_2} \to E_{\lambda_4}$ and $\operatorname{rank} x_4x_3 \leq 0$, we declare $x_4(E^2_{\lambda_3}) = 0$. Note that this is compatible with $x_4x_3(E_{\lambda_2}) = 0$ and $x_3(E_{\lambda_2}) \subseteq E^2_{\lambda_3}$.

(3) Looking at the fourth row of the rank triangle for C_r we see the ranks $(0, 0)$. This means $\operatorname{rank} x_3x_2x_1 \leq 0$ and $\operatorname{rank} x_4x_3x_2 \leq 0$, or equivalently, $x_3x_2x_1 = 0$ and $x_4x_3x_2 = 0$.
   (a) Since $x_3x_2x_1 : E_{\lambda_0} \to E_{\lambda_3}$ and $x_3x_2x_1 = 0$, we declare $x_3(E^1_{\lambda_2}) = 0$. Note that this is compatible with $x_3x_2x_1(E_{\lambda_0}) = 0$ and $x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_2}$.
   (b) Since $x_4x_3x_2 : E_{\lambda_1} \to E_{\lambda_4}$ and $x_4x_3x_2 = 0$, we declare $x_4(E^2_{\lambda_3}) = 0$, which is not a new condition.

(4) Looking at the bottom entry in the rank triangle for C_r we see the rank (0). This means $\operatorname{rank} x_4x_3x_2x_1 \leq 0$ or equivalently, $x_4x_3x_2x_1 = 0$. Since $x_3x_2x_1 = 0$, this gives no new condition.

The diagram appearing in Table 3.2 defines a subvariety of $\overline{C}_r \times \prod_{i=0}^{4} \mathcal{F}(E_{\lambda_i})$ by the relations:

$$
\begin{array}{lll}
x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1} & x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2} & x_3(E_{\lambda_2}) \subseteq E^2_{\lambda_3} \\
x_2(E^2_{\lambda_1}) \subseteq E^1_{\lambda_2} & x_3(E^3_{\lambda_2}) \subseteq E^2_{\lambda_3} & \\
x_3(E^1_{\lambda_2}) = 0 & x_4(E^2_{\lambda_3}) = 0. &
\end{array}
$$

We remove the subspaces not appearing in the relations above; in this case, remove $E^1_{\lambda_0}, E^1_{\lambda_1}, E^3_{\lambda_1}, E^2_{\lambda_2}, E^1_{\lambda_3}, E^3_{\lambda_3}$ and $E^1_{\lambda_4}$. Let $\widetilde{C}_r$ be the subvariety in the product of $\overline{C}_r$ and this partial flag variety defined by the equations above and summarized in

TABLE 3.2. Equations defining $\widetilde{C}_r$.

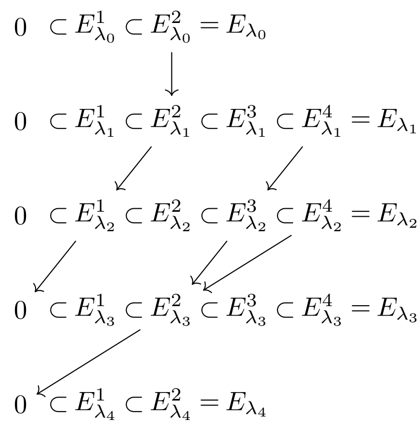


Table 3.2. Now define $\rho_r : \widetilde{C_r} \to \overline{C_r}$ by projection, so $\rho_r(x, E) = x$ where E is the partial flag $(E_{\lambda_0}, E^2_{\lambda_1}, E_{\lambda_1}, E^1_{\lambda_2}, E^3_{\lambda_2}, E_{\lambda_2}, E^2_{\lambda_3})$. Then ρ_r is H-equivariant.

Remark 3.7. By construction, ρ_r induces an isomorphism over C_r. One can calculate the dimensions of the fibres for each orbit $C < C_r$ to verify the map is semi-small; see Appendix A.2.1. The $C < C_r$ for which $m_i(C, C_r) \neq 0$ are:

$$\begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 2 & & 1 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 2 & & 2 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}.$$

Having defined the proper projection $\rho_r : \widetilde{C}_r \to \overline{C}_r$, we return to Equation (21). From Section 3.1, we know that $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}}) = \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}$, and since ρ_r is an isomorphism over C_r, Equation (21) now takes the form

$$\mathsf{Evs}_{C_{\mathrm{KS}}} \rho_{r!} \mathbb{1}_{\widetilde{C}_r}[\dim \widetilde{C}_r] = \mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_r}) \oplus \bigoplus_i m_i(C_{\mathrm{KS}}, C_r) \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}[d_{C_{\mathrm{KS}}, C_r, i}]. \tag{23}$$

Below, we calculate the left hand side and find that $\mathsf{Evs}_{C_{\mathrm{KS}}} \rho_{r!} \mathbb{1}_{\widetilde{C}_r}[\dim \widetilde{C}_r] = 0$. It then follows that $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_r}) = 0$.

We now begin the calculation showing that the left-hand side of Equation (23) is 0. As in Section 3.3, we define $\tilde{f}_r : \widetilde{C}_r \times C^*_{\mathrm{KS}} \to \mathbb{A}^1$ by $\tilde{f}_r(x, E, y) = (x|y)$, where $(-\,|\,-) : V \times V^* \to \mathbb{A}^1$ is the dot-product pairing.

By a direct computation, we find that the fibre $\rho_r^{-1}(x_{\mathrm{KS}})$ of ρ_r above x_{KS} is:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & b \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

thus, $\rho_r^{-1}(x_{\mathrm{KS}}) \cong \mathbb{P}^1$; we use the global coordinate $[a : b]$ for $\rho_r^{-1}(x_{\mathrm{KS}})$.

Introduce local coordinates for $\widetilde{C}_r$ for an affine chart U that contains the affine part $a \neq 0$ of $\rho_r^{-1}(x_{\text{KS}})$ under the isomorphism $\rho_r^{-1}(x_{\text{KS}}) \cong \{[a:b]\}$ above:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ b_3 \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}.$$

Form the Jacobian for the equations that describe $U \times C^*_{\text{KS}}$ and $\tilde{f}_r^{-1}(0)$. The generic rank of this Jacobian matrix is

$$(\text{number of variables}) - \dim \tilde{f}_r^{-1}(0) = 110 - (35 + 32 - 1) = 44.$$

We evaluate the Jacobian at the generic point $(x_{\text{KS}}, y_{\text{KS}}(t_1, t_2))$, with $\text{diag}(t_1, t_2) \in T'_{\text{reg}}$, and further add the conditions that describe $U \cap \rho_r^{-1}(x_{\text{KS}})$. Now the Jacobian becomes a matrix M_r over $\mathbb{Q}[t_1, t_2, c_3]$. Performing elementary row and column operations on M_r results in a 45×45 block diagonal matrix whose blocks consist of a 43×43 identity matrix and the matrix

$$B_r := \begin{bmatrix} t_1 c_3 + c_3^2 & -t_1 - c_3 \\ t_2 c_3^2 & -t_2 c_3 \end{bmatrix}.$$

The rank of M_r will drop from 44 to 43 if and only if $B_r = 0$. If $B_r = 0$ then $-t_1 - c_3 = 0$ and $-t_2 c_3 = 0$, so $t_1 = -c_3$ and hence $t_1 t_2 = -t_2 c_3 = 0$; but $t_1 t_2 \neq 0$ since $\text{diag}(t_1, t_2) \in T'_{\text{reg}}$, so $B_r \neq 0$ and the rank of M_r is 44. This shows that

$$(\text{sing}\, \tilde{f}_r^{-1}(0)) \cap (U \times C^*_{\text{KS}}) \cap (\rho'_r)^{-1}(x_{\text{KS}}, y_{\text{KS}}(t_1, t_2)) = \emptyset.$$

Since the fibre of ρ_r above x_{KS} is isomorphic to $\mathbb{P}^1$, we must also check for singularities in an affine chart for $\widetilde{C}_r$ that contains the affine part $b \neq 0$ of $\rho_r^{-1}(x_{\text{KS}})$. Smoothness is verified using the same approach; Appendix A.2.1 shows how this calculation is made.

We conclude that $\tilde{f}_r^{-1}(0)$ is smooth on $(\rho')^{-1}(x_{\text{KS}}, y_{\text{KS}}(t_1 t_2))$, and so, letting H act we get $\text{sing}(\tilde{f}_r^{-1}(0)) \cap \tilde{\mathcal{O}}_r = \emptyset$. It follows that $\mathsf{Evs}_{C_{\text{KS}}}\, \rho_{r!} \mathbb{1}_{\widetilde{C}_r} = 0$. As explained above, Equation (17) follows immediately.

3.5. $\mathsf{Evs}_{C_{\text{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_m})$.

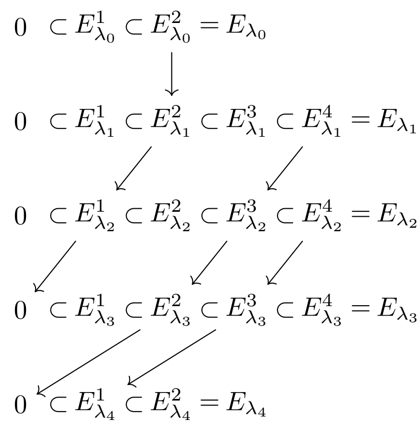
We now provide the proof of Equation (18); see Appendix A.2.2 for a detailed description of the code used for the computations in this section.

As in Section 3.4, we use the rank triangle for C_m to construct a proper cover $\rho_m : \widetilde{C}_m \to \overline{C}_m$. The diagram appearing in Table 3.3 defines a subvariety of $\overline{C}_m \times \prod_{i=1}^4 \mathcal{F}(E_{\lambda_i})$ by the relations:

$$\begin{array}{lll} x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1} & x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2} & x_3(E_{\lambda_2}) \subseteq E^3_{\lambda_3} \\ x_2(E^2_{\lambda_1}) \subseteq E^1_{\lambda_2} & x_3(E^3_{\lambda_2}) \subseteq E^2_{\lambda_3} & x_4(E^3_{\lambda_3}) \subseteq E^1_{\lambda_4} \\ x_3(E^1_{\lambda_2}) = 0 & x_4(E^2_{\lambda_3}) = 0. & \end{array}$$

We remove the subspaces not appearing in the above relations; in this case, remove $E^1_{\lambda_0}, E^1_{\lambda_1}, E^3_{\lambda_1}, E^2_{\lambda_2}$ and $E^1_{\lambda_3}$. Let $\widetilde{C}_m$ be the subvariety in the product of $\overline{C}_m$ and this partial flag variety defined by the equations above and summarized in Table 3.3. Define $\rho_m : \widetilde{C}_m \to \overline{C}_m$ by $\rho_m(x, E) = x$ where E is the partial flag $(E_{\lambda_0}, E^2_{\lambda_1}, E_{\lambda_1}, E^1_{\lambda_2}, E^3_{\lambda_2}, E_{\lambda_2}, E^2_{\lambda_3}, E^3_{\lambda_3}, E^1_{\lambda_4})$.

TABLE 3.3. Equations defining $\widetilde{C}_m$.



Remark 3.8. As in the previous case, by construction, ρ_m is an isomorpism over C_m. We may again verify the map is semi-small, see Appendix A.2.2. The $C < C_m$ for which $m_i(C, C_m) \neq 0$ are:

$$\begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 1 & & 2 & & 3 & & 2 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 1 & & 3 & & 2 & & 1 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 1 & & 3 & & 2 & & 2 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 1 & & 3 & & 3 & & 2 & \\ & & 0 & & 2 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}$$

$$\begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 2 & & 1 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 2 & & 2 & \\ & & 0 & & 1 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array} \qquad \begin{array}{ccccccccc} 2 & & 4 & & 4 & & 4 & & 2 \\ \hline & 2 & & 2 & & 3 & & 2 & \\ & & 0 & & 2 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}$$

By Lemma 3.2.2, and using that the cover is an isomorphism over C_m, Equation (21) now takes the form

(24)
$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_{m!} \mathbb{1}_{\widetilde{C}_m}[\dim \widetilde{C}_m] = \mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_m}) \oplus \bigoplus_i m_i(C_{\mathrm{KS}}, C_m) \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}[d_{C_{\mathrm{KS}},C_m,i}].$$

Below, we calculate the left hand side and find $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_{m!} \mathbb{1}_{\widetilde{C}_m}[\dim \widetilde{C}_m] = 0$.

By a direct computation, we find that the fibre $\rho_m^{-1}(x_{\mathrm{KS}})$ of ρ_m above x_{KS} is:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E^1_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}, \qquad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & b \end{bmatrix},$$

$$E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{bmatrix}, \qquad E^1_{\lambda_4} = \begin{bmatrix} c \\ d \end{bmatrix};$$

thus, $\rho_m^{-1}(x_{\text{KS}}) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Introduce local coordinates for $\widetilde{C}_R$ for an affine chart U that contains the affine part $a \neq 0$ and $c \neq 0$ of $\rho_m^{-1}(x_{\text{KS}})$ under the isomorphism $\rho_m^{-1}(x_{\text{KS}}) \cong \{[a:b]\} \times \{[c:d]\}$ above:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \qquad E^1_{\lambda_2} = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ b_3 \end{bmatrix}, \qquad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

$$E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}, \qquad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_1 & e_2 & e_3 \end{bmatrix}, \qquad E^1_{\lambda_4} = \begin{bmatrix} 1 \\ g \end{bmatrix}.$$

Next, we form the Jacobian for the equations that describe $U \times C^*_{\text{KS}}$ and $\tilde{f}_m^{-1}(0)$. The generic rank of this Jacobian matrix is

$$(\text{number of variables}) - \dim \tilde{f}_m^{-1}(0) = 114 - (37 + 32 - 1) = 46.$$

We evaluate the Jacobian at the generic point $(x_{\text{KS}}, y_{\text{KS}}(t_1,t_2))$, with $\operatorname{diag}(t_1,t_2) \in T'_{\text{reg}}$, and further add the conditions that describe $U \cap \rho_m^{-1}(x_{\text{KS}})$. We are left with a matrix M_m over $\mathbb{Q}[t_1,t_2,c_3,g]$. Performing elementary row and column operations on M_m results in a 48×47 block diagonal matrix whose blocks consists of a 45×45 identity matrix and the matrix

$$B_m := \begin{bmatrix} c_3 - g & -c_3 + g \\ -t_1 - c_3 & t_1 + c_3 \\ -t_2 c_3 & t_2 c_3 \end{bmatrix}.$$

The rank of M_m will drop from 46 to 45 if and only if $B_m = 0$. If $B_m = 0$ we get the relations $t_1 + c_3 = 0$ and $t_2 c_3 = 0$, so $t_1 = -c_3$ and hence $t_1 t_2 = -t_2 c_3 = 0$; but, $t_1 t_2 \neq 0$ since $\operatorname{diag}(t_1,t_2) \in T'_{\text{reg}}$, so $B_m \neq 0$ and the rank of M_m is 46. This shows that

$$(\operatorname{sing} \tilde{f}_m^{-1}(0)) \cap (U \times C^*_{\text{KS}}) \cap (\rho'_m)^{-1}(x_{\text{KS}}, y_{\text{KS}}(t_1,t_2)) = \emptyset.$$

Since $\rho_m^{-1}(x_{\text{KS}}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ we must also check for singularities in a collection of affine charts for $\widetilde{C}_m$ that, together with U, cover $\widetilde{C}_r \cap \rho_m^{-1}(x_{\text{KS}})$. Smoothness is verified using the same method above, but with a different affine choice for the $E^k_{\lambda_i}$ in $\widetilde{C}_m$. There are 3 such affine choices that need to be made; Appendix A.2.2 shows how these extra calculations were made. We conclude that $\tilde{f}_m^{-1}(0)$ is smooth on $\tilde{\mathcal{O}}_m$; that is, $\operatorname{sing}(\tilde{f}_m^{-1}(0)) \cap \tilde{\mathcal{O}}_m = \emptyset$. From this, Equation (18) follows immediately: $\mathsf{Evs}_{C_{\text{KS}}} \mathcal{IC}(\mathbb{1}_{C_m}) = 0$.

3.6. $\mathsf{Evs}_{C_{\text{KS}}} \mathcal{IC}(\mathbb{1}_{C_R})$. In this section we prove Equation (19); see Appendix A.2.3 for a detailed description of the code used for the computations in this section.

Recall the rank triangle for C_R from Table 3.1. As in Section 3.4, we use this rank triangle to construct a proper cover $\rho_R : \widetilde{C}_R \to \overline{C}_R$ which is semi-small, and realizes $\widetilde{C}_R$ as a smooth variety. The diagram appearing in Table 3.4 defines a

TABLE 3.4. Equations defining $\widetilde{C}_R$.

$$
\begin{array}{l}
0 \subset E^1_{\lambda_0} \subset E^2_{\lambda_0} = E_{\lambda_0} \\
\qquad\qquad \downarrow \\
0 \subset E^1_{\lambda_1} \subset E^2_{\lambda_1} \subset E^3_{\lambda_1} \subset E^4_{\lambda_1} = E_{\lambda_1} \\
\qquad\qquad \downarrow \\
0 \subset E^1_{\lambda_2} \subset E^2_{\lambda_2} \subset E^3_{\lambda_2} \subset E^4_{\lambda_2} = E_{\lambda_2} \\
\quad \swarrow \qquad\qquad \swarrow \\
0 \subset E^1_{\lambda_3} \subset E^2_{\lambda_3} \subset E^3_{\lambda_3} \subset E^4_{\lambda_3} = E_{\lambda_3} \\
\quad \swarrow \\
0 \subset E^1_{\lambda_4} \subset E^2_{\lambda_4} = E_{\lambda_4}
\end{array}
$$

subvariety of $\overline{C}_R \times \prod_{i=0}^4 \mathcal{F}(E_{\lambda_i})$ by the relations:

$$
\begin{aligned}
&x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1} && x_3(E_{\lambda_2}) \subseteq E^2_{\lambda_3} \\
&x_2(E^2_{\lambda_1}) \subseteq E^2_{\lambda_2} && x_4(E^2_{\lambda_3}) = 0 \\
&x_3(E^2_{\lambda_2}) = 0.
\end{aligned}
$$

We remove the subspaces not appearing in the relations above; in this case, remove $E^1_{\lambda_0}, E^1_{\lambda_1}, E^3_{\lambda_1}, E^1_{\lambda_2}, E^3_{\lambda_2}, E^1_{\lambda_3}, E^3_{\lambda_3}$ and $E^1_{\lambda_4}$. Let $\widetilde{C}_R$ be the subvariety in the product of $\overline{C}_R$ and this partial flag variety defined by the equations above and summarized in Table 3.4. For elements in $\widetilde{C}_R$ we use the notation (x, E), where $x \in \overline{C}_R$ and E is a point in the partial flag variety described above.

Remark 3.9. As in the previous case, by construction, ρ_R is an isomorpism over C_R. In this case, the map is small and hence $m_i(C, C_R) = 0$ for all $C < C_R$; see Appendix A.2.3.

Once again, Equation (21) takes the form
(25)

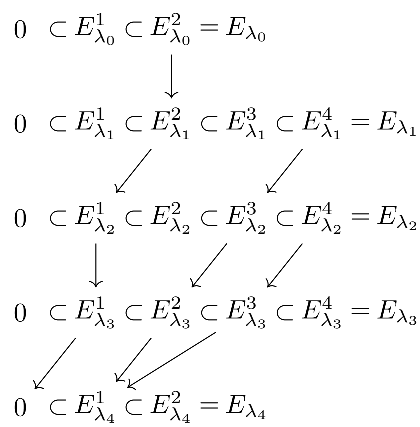
$$
\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_{R!} \mathbb{1}_{\widetilde{C}_R}[\dim \widetilde{C}_R] = \mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_R}) \oplus \bigoplus_i m_i(C_{\mathrm{KS}}, C_R) \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}[d_{C_{\mathrm{KS}}, C_R, i}].
$$

By a direct computation, we find that the fibre $\rho_R^{-1}(x_{\mathrm{KS}})$ of ρ_R above x_{KS} is:

$$
E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^2_{\lambda_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
$$

And so, $\rho_R : \widetilde{C}_R \to \overline{C}_R$ is an isomorphism over C_{KS}. Now choose local coordinates by introducing the following variables for an affine chart U in $\widetilde{C}_R$ that contains

TABLE 3.5. Equations defining $\widetilde{C}_\psi$.



$\rho_R^{-1}(x_{\mathrm{KS}})$, by specifying flags $E^k_{\lambda_i}$ as follows:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad E^2_{\lambda_2} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$$

We compute the Jacobian for the system of equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_R^{-1}(0)$. The generic rank of this Jacobian is

$$(\text{number of variables}) - \dim \tilde{f}_R^{-1}(0) = 108 - (36 + 32 - 1) = 41.$$

We evaluate the Jacobian at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$, with $\mathrm{diag}(t_1, t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that correspond to $\rho_R(x_{\mathrm{KS}}, E) = x_{\mathrm{KS}}$. We are left with a matrix M_R over $\mathbb{Q}[t_1, t_2]$. Performing elementary row and column operations on M_R results in a 41×41 identity matrix. We conclude that M_R has rank 41. In this case, it is not necessary to consider more than one affine $U \subset \widetilde{C}_R$, so $\tilde{f}_R^{-1}(0)$ is smooth on $(\rho'_R)^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$, and so $(\mathrm{sing}\, \tilde{f}_R^{-1}(0)) \cap \tilde{\mathcal{O}}_R = \emptyset$. From this, Equation (19) follows immediately: $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_R}) = 0$.

3.7. $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_\psi})$**.** In this section we prove Equation (20); see Appendix A.2.4 for a detailed description of the code used for the computations in this section.

Recall the rank triangle for C_ψ from Table 3.1. We use this rank triangle to define $\rho_\psi : \widetilde{C}_\psi \to \overline{C}_\psi$ as follows.

(1) Looking at the second row of the rank triangle for C_ψ we see the ranks $(2, 3, 3, 2)$. This means, for any $x = (x_4, x_3, x_2, x_1) \in C_\psi$, we have $\mathrm{rank}\, x_1 = 2$, $\mathrm{rank}\, x_2 = 3$, $\mathrm{rank}\, x_3 = 3$ and $\mathrm{rank}\, x_4 = 2$.
   (a) Since $x_1 : E_{\lambda_0} \to E_{\lambda_1}$ and $\mathrm{rank}\, x_1 = 2$, we declare $x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1}$.
   (b) Since $x_2 : E_{\lambda_1} \to E_{\lambda_2}$ and $\mathrm{rank}\, x_2 = 3$, we declare $x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2}$.
   (c) Since $x_3 : E_{\lambda_2} \to E_{\lambda_3}$ and $\mathrm{rank}\, x_3 = 3$, we declare $x_3(E_{\lambda_2}) \subseteq E^3_{\lambda_3}$.
   (d) Since $x_4 : E_{\lambda_3} \to E_{\lambda_4}$ and $\mathrm{rank}\, x_4 = 2$, we declare $x_4(E_{\lambda_3}) \subseteq E^2_{\lambda_4}$; of course, this is no condition at all since $E^2_{\lambda_4} = E_{\lambda_4}$.

(2) Looking at the third row of the rank triangle for C_ψ we see the ranks $(1,2,1)$. This means $\operatorname{rank} x_2x_1 \leq 1$, $\operatorname{rank} x_3x_2 \leq 2$ and $\operatorname{rank} x_4x_3 \leq 1$.
   (a) Since $x_2x_1 : E_{\lambda_0} \to E_{\lambda_2}$ and $\operatorname{rank} x_2x_1 \leq 1$, we declare $x_2(E^2_{\lambda_1}) \subseteq E^1_{\lambda_2}$. This is compatible with $x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_2}$ and $x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1}$.
   (b) Since $x_3x_2 : E_{\lambda_1} \to E_{\lambda_3}$ and $\operatorname{rank} x_3x_2 \leq 2$, we declare $x_3(E^3_{\lambda_2}) \subseteq E^2_{\lambda_3}$. This is compatible with $x_3x_2(E_{\lambda_1}) \subseteq E^2_{\lambda_3}$ and $x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2}$.
   (c) Since $x_4x_3 : E_{\lambda_2} \to E_{\lambda_4}$ and $\operatorname{rank} x_4x_3 \leq 1$, we declare $x_4(E^3_{\lambda_3}) \subseteq E^1_{\lambda_4}$. This is compatible with $x_4x_3(E_{\lambda_2}) \subseteq E^1_{\lambda_4}$ and $x_3(E_{\lambda_2}) \subseteq E^3_{\lambda_3}$.

(3) Looking at the fourth row of the rank triangle for C_ψ we see the ranks $(1,1)$. This means $\operatorname{rank} x_3x_2x_1 \leq 1$ and $\operatorname{rank} x_4x_3x_2 \leq 1$.
   (a) Since $x_3x_2x_1 : E_{\lambda_0} \to E_{\lambda_3}$ and $\operatorname{rank} x_3x_2x_1 \leq 1$, we declare $x_3(E^1_{\lambda_2}) \subseteq E^1_{\lambda_3}$. This is compatible with $x_3x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_3}$ and $x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_2}$.
   (b) Since $x_4x_3x_2 : E_{\lambda_1} \to E_{\lambda_4}$ and $\operatorname{rank} x_4x_3x_2 \leq 1$, we declare $x_4(E^2_{\lambda_3}) \subseteq E^1_{\lambda_4}$. This is compatible with $x_4x_3x_2(E_{\lambda_1}) \subseteq E^1_{\lambda_4}$ and $x_3x_2(E_{\lambda_1}) \subseteq E^2_{\lambda_3}$.

(4) Looking at the bottom entry in the rank triangle for C_ψ we see the rank (0). This means $\operatorname{rank} x_4x_3x_2x_1 \leq 0$ or equivalently, $x_4x_3x_2x_1 = 0$. We declare $x_4(E^1_{\lambda_3}) = 0$, which is compatible with $x_3x_2x_1(E_{\lambda_0}) \subseteq E^1_{\lambda_3}$ and $x_4x_3x_2x_1 = 0$.

Now consider the subvariety of $\overline{C}_\psi \times \prod_{i=0}^4 \mathcal{F}(E_{\lambda_i})$ defined by the relations:

$$
\begin{array}{lll}
x_1(E_{\lambda_0}) \subseteq E^2_{\lambda_1} & x_2(E_{\lambda_1}) \subseteq E^3_{\lambda_2} & x_3(E_{\lambda_2}) \subseteq E^3_{\lambda_3} \\
x_2(E^2_{\lambda_1}) \subseteq E^1_{\lambda_2} & x_3(E^3_{\lambda_2}) \subseteq E^2_{\lambda_3} & x_4(E^3_{\lambda_3}) \subseteq E^1_{\lambda_4} \\
x_3(E^1_{\lambda_2}) \subseteq E^1_{\lambda_3} & x_4(E^2_{\lambda_3}) \subseteq E^1_{\lambda_4} & \\
x_4(E^1_{\lambda_3}) = 0. & &
\end{array}
$$

We remove the subspaces not appearing in the relations above; in this case, remove $E^1_{\lambda_0}$, $E^1_{\lambda_1}$, $E^3_{\lambda_1}$ and $E^2_{\lambda_2}$. Let $\widetilde{C}_\psi$ be the subvariety in the product of $\overline{C}_\psi$ and this partial flag variety defined by the equations above and summarized in Table 3.5. We define $\rho_\psi(x,E) = x$ where E is a partial flag.

By a direct computation, we find that the fibre $\rho_\psi^{-1}(x_{\mathrm{KS}})$ of ρ_ψ above x_{KS} is

$$
E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{bmatrix},
$$

$$
E^1_{\lambda_3} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e \\ 0 & 0 & h \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} e \\ h \end{bmatrix},
$$

subject to the conditions $av - bu = 0$ and $cz - dw = 0$; this is the product of $\mathbb{P}^1 \cong \{[e:h] \mid (e,h) \neq (0,0)\}$ with the variety

$$
F := \{([a:b],[c:d],[u:v:w:z]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \mid av - bu = 0,\ cz - dw = 0\},
$$

which is the blow up of $\mathbb{P}^3$ along the two copies of $\mathbb{P}^1$ given by $(u,v) = (0,0)$ and $(w,z) = (0,0)$.

Remark 3.10. Again, by construction, ρ_ψ is an isomorpism over C_ψ. For the 15 orbits with $C_{\mathrm{KS}} < C < C_\psi$ we must verify that the cover is small in the sense of decomposition theorem. This check is performed using code as documented in Appendix A.2.4. A cohomology calculation on the fibre above then gives that $m_0(C_{\mathrm{KS}}, C_\psi) = 1$ and $m_i(C_{\mathrm{KS}}, C_\psi) = 0$ for $i \neq 0$. By checking all other orbits we verify the map is again semi-small. We find that there are no other $C < C_\psi$ for which $m_i(C, C_\psi) \neq 0$.

Using that $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}}) = \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}$, Equation (21) takes the form

$$\mathsf{Evs}_{C_{\mathrm{KS}}} \rho_{\psi !} \mathbb{1}_{\widetilde{C}_\psi}[\dim \widetilde{C}_\psi] = \mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_\psi}) \oplus \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}. \tag{26}$$

We calculate the left-hand side of Equation (26). Introduce local coordinates for $\widetilde{C}_\psi$ for an affine chart U that contains the affine part $a \neq 0, c \neq 0, e \neq 0$, and $u \neq 0$ of $\rho_\psi^{-1}(x_{\mathrm{KS}})$ under the isomorphism $\rho_\psi^{-1}(x_{\mathrm{KS}}) \cong \mathbb{P}^1 \times F$ above:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

$$E^1_{\lambda_3} = \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_1 & f_2 & f_3 \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} 1 \\ g \end{bmatrix}.$$

Next, we form the Jacobian for the equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_\psi^{-1}(0)$. The generic rank of this Jacobian matrix is

$$(\text{number of variables}) - \dim \tilde{f}_\psi^{-1}(0) = 117 - (40 + 32 - 1) = 46.$$

We evaluate the Jacobian at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1, t_2))$, with $\mathrm{diag}(t_1, t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that describe $U \cap \rho_\psi^{-1}(x_{\mathrm{KS}})$. We are left with a matrix M_ψ over $\mathbb{Q}[t_1, t_2, b_1, b_2, c_3, f_3]$. Performing row and column operations on M_ψ results in a 52×51 diagonal block matrix whose blocks consist of a 45×45 identity matrix and a 7×6 matrix B_ψ, displayed in the Section A.2.4. The rank of M_ψ will drop from its generic rank of 46 to 45 if and only if $B_\psi = 0$. If $B_\psi = 0$, then we get the system of equations:

$$\begin{gathered} -b_1 + f_3 = 0, \qquad -c_3 + f_3 = 0, \\ t_1 b_2 + b_2 c_3 + 1 = 0, \qquad t_2 b_2 c_3 + b_1 = 0. \end{gathered}$$

There are exactly two solutions to the above system: if $c_3 = 0$, then

$$b_1 = f_3 = c_3 = 0, b_2 = -t_1^{-1}$$

is a solution; and if $c_3 \neq 0$, then

$$b_1 = f_3 = c_3 = t_2 - t_1, b_2 = -t_2^{-1}$$

is a solution. This calculation shows that, in $(U \times C^*_{\mathrm{KS}}) \cap (\rho'_\psi)^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, there are exactly two singularities of $\tilde{f}_\psi^{-1}(0)$: one of these singularities is

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} t_1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E^1_{\lambda_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

while the other is

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} t_2 \\ t_2(t_2-t_1) \\ -1 \\ -(t_2-t_1) \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & t_2-t_1 \end{bmatrix},$$

$$E^1_{\lambda_3} = \begin{bmatrix} 1 \\ t_2-t_1 \\ 0 \\ 0 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & t_2-t_1 \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} 1 \\ t_2-t_1 \end{bmatrix}.$$

Using the same approach, we check for singularities in a further 15 affine charts for $\widetilde{C}_\psi$ that cover $\widetilde{C}_\psi \cap \rho_\psi^{-1}(x_{\mathrm{KS}})$, corresponding to different choices for the subspaces $E^k_{\lambda_i}$ in $\widetilde{C}_\psi$. In all the other cases, no new singularities appear. We conclude that $|(\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap (\rho'_\psi)^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))| = 2$ for every $\operatorname{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$. Now, letting H act on $(\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap (\rho'_\psi)^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, it follows that

$$\rho'''_\psi : (\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$$

is a double cover.

Having found the singular locus of $\tilde{f}_\psi^{-1}(0)$ in $\tilde{\mathcal{O}}_\psi$, we next compute the rank of the Hessian of $\tilde{f}_\psi$ at a point in the singular locus and verify the determinant of a maximal minor of the Hessian is a square. This first calculation verifies the rank of the local system $\mathcal{M}_\psi$, defined by Equation (22), is 1 and the later identifies the character of $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ associated to the local system. We make these calculations by adapting the Hessian calculations described in Section 3.1, which in turn are adapted from [CFM+21, Theorem 7.7.5] and documented in Appendix A.3. We use the Jacobian of the ideal defining $\tilde{\mathcal{O}}_\psi$ to identify a system of variables, from the polynomial ring over which $\tilde{\mathcal{O}}_\psi$ is defined, that may be used as local coordinates for the (completed) local ring at the chosen point in the singular locus. The rest of the computation follows the method of Section 3.1. We find that the rank of the Hessian at the chosen point is $\dim C_\psi - \operatorname{codim} C^*_{\mathrm{KS}} = 40 - (48-32) = 24$. This shows that the rank of the Hessian is the codimension of the singular locus, and from this we conclude that the rank of $\mathcal{M}_\psi$ is 1. Though this was automatic in the cases under consideration in [CFM+21, Theorem 7.7.5], it should not be seen as automatic for this case as there are groups for which this will not occur. Next, we verify that the Hessian determinant is the perfect square of a function so that we may conclude the representation associated to $\mathcal{M}_\psi$ is the trivial representation. We note that these determinants are often not perfect squares, this occurs in several of the examples considered in [CFM+21]. We explain how to use our code to reproduce these calculations in Appendix A.3. Unfortunately, these calculations are not particularly amenable to being displayed here.

With reference to Equation (22), it follows from these calculations that

$$\mathcal{M}_\psi = \mathbb{1}_{(\operatorname{sing} \tilde{f}_\psi^{-1}(0))\cap\tilde{\mathcal{O}}_\psi},$$

and therefore

$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \rho_{\psi!} \mathbb{1}_{\widetilde{C}_\psi}[\dim \widetilde{C}_\psi] = \rho'''_{\psi}{}_! \mathbb{1}_{(\operatorname{sing} \tilde{f}_\psi^{-1}(0))\cap\tilde{\mathcal{O}}_\psi} = \mathbb{1}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}} \oplus \mathcal{L}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}, \tag{27}$$

where $\operatorname{rank} \mathcal{L}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}} = 1$ and $\mathcal{L}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}$ corresponds to a quadratic character of $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$. Comparing (27) with (26) gives

$$\mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_\psi}) \oplus \mathbb{1}_{\Lambda^{gen}_{C_{\mathrm{KS}}}} = \mathbb{1}_{\Lambda^{gen}_{C_{\mathrm{KS}}}} \oplus \mathcal{L}_{\Lambda^{gen}_{C_{\mathrm{KS}}}}. \tag{28}$$

It follows that $\mathsf{Evs}_{C_{\mathrm{KS}}}\, \mathcal{IC}(\mathbb{1}_{C_\psi}) = \mathcal{L}_{\Lambda^{gen}_{C_{\mathrm{KS}}}}$, where this local system is defined by the non-trivial character of the automorphism group of the double cover

$$\rho'''_\psi : (\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}.$$

3.8. **A study of quadratic covers of the generic conormal bundle.** Two H-equivariant double covers of $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ appear in this paper: the first is

$$z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} := \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \times_{Z'_{\mathrm{reg}}} T'_{\mathrm{reg}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$$

appearing in Proposition 2.3.2 and the second is

$$\rho'''_\psi : (\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$$

appearing in Section 3.7. In this section we show that these two quadratic covers are isomorphic. It will then follow that the local system $\mathcal{L}_{\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}}$ appearing in Section 3.7 is the local system defined by the non-trivial character of the automorphism group of the double cover $z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$, thus concluding the proof of Theorem 3.1

From Section 2.3 recall the cover $q : \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to Z'_{\mathrm{reg}}$ given by $q(x,y) = [(ac)^{-1}(bd)]$, where $a,b,c,d \in \mathrm{GL}_2(\mathbb{C})$ are defined by Equation (13). Define

$$\begin{array}{rcl} \dot{q} : \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} & \to & \mathrm{GL}(E_{\lambda_3}/\ker y_3) \\ (x,y) & \mapsto & (ac)^{-1}(bd). \end{array}$$

Using this, consider the varieties

$$\Lambda''_{C_{\mathrm{KS}}} := \left\{ (x,y,E^3_{\lambda_3}) \in \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \times \mathrm{Gr}^3(E_{\lambda_3}) \;\middle|\; \begin{array}{c} \ker y_3 \subset E^3_{\lambda_3} \\ \dot{q}(x,y)(E^3_{\lambda_3}/\ker y_3) \subset E^3_{\lambda_3}/\ker y_3 \end{array} \right\}$$

and

$$\Lambda'''_{C_{\mathrm{KS}}} := \left\{ (x,y,E^3_{\lambda_3},\lambda) \in \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \times \mathrm{Gr}^3(E_{\lambda_3}) \times \mathbb{G}_{\mathrm{m}} \;\middle|\; \begin{array}{c} \ker y_3 \subset E^3_{\lambda_3} \\ (\dot{q}(x,y) - \lambda)(E^3_{\lambda_3}/\ker y_3) = 0 \end{array} \right\}.$$

Note that in the definition of $\Lambda''_{C_{\mathrm{KS}}}$ the condition $\dot{q}(x,y)(E^3_{\lambda_3}/\ker y_3) \subset E^3_{\lambda_3}/\ker y_3$ encodes that $E^3_{\lambda_3}/\ker y_3$ is an eigenspace for $\dot{q}(x,y)$, whereas in the definition of $\Lambda'''_{C_{\mathrm{KS}}}$ the condition $(\dot{q}(x,y) - \lambda)(E^3_{\lambda_3}/\ker y_3) = 0$ encodes that $E^3_{\lambda_3}/\ker y_3$ is an eigenspace for $\dot{q}(x,y)$ with eigenvalue λ. It is thus immediate that the natural map $\Lambda'''_{C_{\mathrm{KS}}} \to \Lambda''_{C_{\mathrm{KS}}}$ is an isomorphism.

We now claim that the natural map $(\operatorname{sing} \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \times \mathrm{Gr}^3(E_{\lambda_3})$ defined by projection to $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ and the three-dimensional vector space in the partial flag induces an isomorphism to $\Lambda''_{C_{\mathrm{KS}}}$. To see this we first note that the map is equivariant. We next note that by direct inspection the map induces an isomorphism over $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$. Because $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ is the H orbit of $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$ it follows that the induced map

$$(\mathrm{sing}\, \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda''_{C_{\mathrm{KS}}}$$

is indeed an isomorphism. Now define

$$\begin{array}{ccc} \Lambda'''_{C_{\mathrm{KS}}} & \to & \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \\ (x,y,E^3_{\lambda_3},\lambda) & \mapsto & (x,y,\mathrm{diag}(\lambda, \det \dot{q}(x,y)/\lambda)). \end{array}$$

This is an H-equivariant isomorphism and clearly commutes with the H-equivariant projections $\Lambda''_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ and $z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$.

Combining the maps above, it follows that

$$z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} := \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}} \times_{Z'_{\mathrm{reg}}} T'_{\mathrm{reg}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$$

is isomorphic to

$$\rho'''_\psi : (\mathrm{sing}\, \tilde{f}_\psi^{-1}(0)) \cap \tilde{\mathcal{O}}_\psi \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$$

as covers $\Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$. This concludes the proof of Theorem 3.1.

3.9. **Speculation on endoscopy for the Kashiwara-Saito representation.** Motivated by the theory of endoscopy, we note that the algebraic group

$$\mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}} := Z_H(\{(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t)) \mid t \in T'_{\mathrm{reg}}\}) \tag{29}$$

determines non-trivial group homomorphisms

$$\mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}} \to \pi_0(\mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}) \to A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}, \tag{30}$$

defined as follows. First note that $\mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ has two connected components: the identity component is the group of $h = (h_0,h_1,h_2,h_3,h_4) \in H$ for which $h_0 \in T$ and

$$h_1 = \begin{pmatrix} h_0 & u \\ 0 & h_0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} h_0 & 0 \\ 0 & h_0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} h_0 & v \\ 0 & h_0 \end{pmatrix}, \quad h_4 = h_0;$$

the other component is the coset represented by $s_{\mathrm{KS}} := (s_0,s_1,s_2,s_3,s_4) \in Z_H(x_{\mathrm{KS}})$ for which $s_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and

$$s_1 = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \end{pmatrix}, \quad s_4 = s_0.$$

As an element of $\mathrm{GL}_{16}(\mathbb{C})$, s_{KS} is the block matrix with 8 copies of s_0 down the diagonal. Now define $\pi_0(\mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}) \to A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ as follows. The image of the component of s_{KS} is the non-trivial automorphism of the cover $z' : \widetilde{\Lambda}^{\mathrm{gen}}_{C_{\mathrm{KS}}} \to \Lambda^{\mathrm{gen}}_{C_{\mathrm{KS}}}$ defined by

$$s_{\mathrm{KS}} : (x_{\mathrm{KS}}, y_{\mathrm{KS}}(t), t) \mapsto (x_{\mathrm{KS}}, y_{\mathrm{KS}}(t), s_0 t s_0^{-1}).$$

Now, building on [CFM+21, Definition 3] and [CFZb, Definition 4], consider

$$\Theta_{\phi_{\mathrm{KS}},s} := \sum_{\pi \in \Pi^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}(G(F))} (-1)^{\dim(\phi_{\mathrm{KS}})-\dim(\phi_\pi)}\, \mathrm{trace}\langle s,\pi\rangle\ \Theta_\pi \tag{31}$$

where we identify $s \in \mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ with its image in $A^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ using Equation (30). By Theorem 3.1 and the work above, if $s \in \mathcal{S}^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ is in the component of s_{KS} then the value of the character $\langle -,\pi\rangle$ at s is -1, so

$$\Theta_{\phi,s} = \Theta_{\pi_{\mathrm{KS}}} - \Theta_{\pi_\psi}, \tag{32}$$

while if $s \in (\mathcal{S}^{\text{ABV}}_{\phi_{\text{KS}}})^\circ$ then

$$\Theta_{\phi,s} = \Theta_{\pi_{\text{KS}}} + \Theta_{\pi_\psi}. \tag{33}$$

We ask if ϕ_{KS} factors as a Langlands parameter through $\widehat{G}' := Z_{\text{GL}_{16}(\mathbb{C})}(s)$ and, further, if for every f in the Hecke algebra $C_c^\infty(\text{GL}_{16}(F))$,

$$\Theta_{\phi_{\text{KS}},s}(f) = \Theta_{\phi'_{\text{KS}}}(f')$$

where f' is the Langlands-Shelstad transfer to $C_c^\infty(G'(F))$ of f?

Appendix A. Macaulay2 instructions

The purpose of this appendix is to present the Macaulay2 code used in the proof of the geometric main result, Theorem 3.1. This code is available from the Voganish-Project GitHub repository which includes:

`KSrepresentations.m2`,
`ComputeDuals.m2`,
`NetworkFlow.m2`,
`ComputeCover.m2`,
`VanishingCycles.m2`,
`VoganV.m2`,
`PSNF.m2` (`partialSmithNormalForm`).

Detailed descriptions of each function used in this appendix can be found in these files. Our code for `ComputeDuals.m2` uses `NetworkFlow.m2` which, as the title suggests, relies on ideas from the theory of network flows; in particular, it does not use the Moeglin-Waldspurger algorithm [MW86] for the corresponding duality on multisegments. We remark that V is an example of a Vogan variety; see [CFM+21, Section 4.2] for the definition. There are several functions in the files listed above, some of which we do not describe in this appendix, that can be applied to arbitrary Vogan varieties.

A.1. **General computations.** In this section, we introduce functions that perform the following tasks: list all rank triangles/orbits of a Vogan variety; display an orbit in terms of its rank conditions; compute the dual of an orbit; compute the dimension of an orbit and its closure; and determine if an orbit is contained in the closure of another. We also provide a proof of Lemma 3.2.2. The code in this section is particularly relevant to Sections 1.5, 2.1, and 3.2. Note that all functions used in this section can be applied to an arbitrary Vogan variety.

We obtain a complete list of rank triangles/H-orbits for V using the eigenspace dimensions $(2, 4, 4, 4, 2)$ of $\lambda_{\text{KS}}(\text{Fr})$:

```
i1 :  L := getAllStrataR({2,4,4,4,2});
```

We store an H-orbit of V as follows:

```
i2 :  CKS := new RankConditions from ({2,4,4,4,2},{{2,2,2,2},{0,2,0},{0,0},{0}});

i3 :  Cpsi := new RankConditions from ({2,4,4,4,2},{{2,3,3,2},{1,2,1},{1,1},{0}});

i4 :  CL := new RankConditions from ({2,4,4,4,2},{{2,4,2,2},{2,2,0},{0,0},{0}});
```

We compute the duals of our H-orbits:

```
i5 :  computeDual CKS
```

```
o5 = 2  4  4  4  2
       2  2  2  2
         0  2  0
           0  0
             0

i6 :  computeDual Cpsi

o6 = 2  4  4  4  2
       2  3  3  2
         1  2  1
           1  1
             0

i7 :  CR := computeDual CL

o7 = 2  4  4  4  2
       2  2  4  2
         0  2  2
           0  0
             0
```

We form an ideal consisting of polynomial equations that describes the closure of an H-orbit:

```
i8:  getEquations CKS;
```

Recall that the dimension of an H-orbit coincides with the dimension of its closure. Thus, the dimension of an H-orbit can be obtained from the ideal that describes its closure:

```
i9:  dim o8

o9 = 32

i10:  dim getEquations Cpsi

o10 = 40

i11:  dim getEquations CR

o11 = 36
```

Given any two orbits C' and C, with $C' \neq C$, we can check if C' is in the closure of C:

```
i12:  CKS < Cpsi;

o12 = true
```

Here, we use the order defined in Section 1.5 by $C' < C$ if and only if $C' \subsetneq \overline{C}$. We compute the list of H-orbits C of V that satisfy $C_{\mathrm{KS}} < C$ and $C_{\mathrm{KS}} < \widehat{C}$, thus providing a proof of Lemma 3.2.2:

```
i12:  orbits6 := {};

i13:  for C in L do (if CKS < C and CKS < (computeDual C) then
                             orbits6 = append(orbits6, C))

i14:  orbits6

o15 = {2  4  4  4  2, 2  4  4  4  2, 2  4  4  4  2, 2  4  4  4  2,
         2  2  3  2     2  2  4  2     2  3  2  2     2  3  3  2
           0  2  1        0  2  2        1  2  0        1  2  1
             0  0           0  0           0  0           0  0
               0              0              0              0
      ----------------------------------------------------------------
        2  4  4  4  2, 2  4  4  4  2}
          2  3  3  2     2  4  2  2
            1  2  1        2  2  0
              1  1           0  0
                0              0
```

This computation, together with Proposition 3.2.1, allows us to reduce Theorem 3.1 to Equations (17), (18), (19) and (20).

A.2. **Microlocal vanishing cycle calculations.** In this section, we illustrate how to reproduce the Macaulay2 computations used in the proof of Equations (17), (18), (19) and (20). We introduce functions that perform the following tasks: verify that our covers are semi-small; generate the relevant Jacobians needed in the proofs; substitute particular points into these Jacobians; and perform elementary row and column operations on matrices over polynomial rings. The code in this section is relevant to Sections 3.3 - 3.7.

All functions in this section, with the exception of `partialSmithNormalForm` (`PSNF.m2`), are specific to V. Throughout this appendix t_1 refers to `y_{2,0,2}` in the code, and t_2 to `y_{2,1,3}`.

A.2.1. $\mathsf{Evs}_{C_{\mathrm{KS}}}\,\mathcal{IC}(\mathbb{1}_{C_r})$. The computations in this section are specific to Section 3.4. The equations that describe the affine chart $U \subseteq \widetilde{C}_r$ are the generators of the ideal `coverCr()`. One can optionally verify that the cover is semi-small using `smallCr()`. The output lists the relevant strata for the cover. We compute the Jacobian for the equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_r^{-1}(0)$:

```
i1 :  JacCr();
```

Next, we evaluate the Jacobian `o1` at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, with $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that describe $U \cap \rho_r^{-1}(x_{\mathrm{KS}})$:

```
i2:  subJacCr(o1);
```

The resulting matrix `o2` is over $\mathbb{Q}[t_1,t_2,c_3]$.We perform elementary row and column operations on `o2`:

```
i3:  partialSmithNormalForm(o2);
```

The output `o3` is a 45×45 block diagonal matrix whose blocks consist of a 43×43 identity matrix and the matrix

$$\begin{bmatrix} t_1c_3 + c_3^2 & -t_1 - c_3 \\ t_2c_3^2 & -t_2c_3 \end{bmatrix}.$$

Recall that the fibre of ρ_r above C_{KS} is $\mathbb{P}^1 \cong \{[a:b]\}$. The above calculations are for verifying smoothness in the affine chart U that contains the points $\{[1:b]\}$. We must also check smoothness in an affine chart for $\widetilde{C}_r$ that contains the points $\{[a:1]\}$. To do this, we use the affine chart U' for $\widetilde{C}_r$, which is different from the one used in `JacCr()`, corresponding to:

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 1 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}.$$

We then continue with the same approach illustrated above. Compute the Jacobian for the equations that describe $U' \times C^*_{\mathrm{KS}}$ and $\tilde{f}_r^{-1}(0)$:

```
i4:  JacCr2();
```

We evaluate the Jacobian `o4` at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, with $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that describe $U' \cap \rho_r^{-1}(x_{\mathrm{KS}})$:

```
i5:  subJacCr(o4);
```

The resulting matrix `o5` is over $\mathbb{Q}[t_1,t_2]$.We perform elementary row and column operations on `o5`:

```
i6:  partialSmithNormalForm(o5);
```

The output `o6` is a 45×45 block diagonal matrix whose blocks consist of a 43×43 identity matrix and the matrix

$$\begin{bmatrix} -t_1c_3 - 1 & t_1c_3^2 + c_3 \\ -t_2 & t_2c_3 \end{bmatrix}.$$

The rank of `o6` is 43 if and only if $-t_1c_3 - 1 = 0$ and $-t_2 = 0$. But, $t_2 \neq 0$ since $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$. And so the rank of `o6` is 44. Since the rank of `o6` coincides with the generic rank of `o4`, we are smooth.

A.2.2. $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_m})$. The computations in this section are specific to Section 3.5. The equations that describe the affine chart $U \subseteq \widetilde{C}_m$ are the generators of the ideal `coverCm()`. One can optionally verify that the cover is semi-small using `smallCm()`. The output lists the relevant strata for the cover. We compute the Jacobian for the equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_m^{-1}(0)$:

```
i1 :  JacCm();
```

Next, we evaluate the Jacobian `o1` at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, with $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that describe $U \cap \rho_m^{-1}(x_{\mathrm{KS}})$:

```
i2:  subJacCm(o1);
```

The resulting matrix `o2` is over $\mathbb{Q}[t_1,t_2,c_3,g]$. We perform elementary row and column operations on `o2`:

```
i3:  partialSmithNormalForm(o2);
```

The output `o3` is a 48×47 block diagonal matrix whose blocks consist of a 45×45 identity matrix and the matrix

$$\begin{bmatrix} c_3 - g & -c_3 + g \\ -t_1 - c_3 & t_1 + c_3 \\ -t_2c_3 & t_2c_3 \end{bmatrix}.$$

Let $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \rho_m^{-1}(x_{\mathrm{KS}})$ be the isomorphism $p([a:b],[c:d]) = (E^2_{\lambda_1}, E^1_{\lambda_2}, E^3_{\lambda_2}, E^2_{\lambda_3}, E^1_{\lambda_4})$ where

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & b \end{bmatrix},$$

$$E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

The above calculations are for verifying smoothness in the affine chart U that contains the points $p([1:b],[1:d])$. We use the same method as above to verify smoothness in a collection of affine charts for $\widetilde{C}_m$ that, together with U, cover $\widetilde{C}_m \cap \rho_m^{-1}(x_{\mathrm{KS}})$. For each of these charts, we list below the relevant points of the fibre that it contains, together with the corresponding Jacobian and analogous function to `subJacCm()` needed for the computation:

(2) $p([1:b],[c:1])$ use `JacCm2()` and `subJaCm`;
(3) $p([a:1],[1:d])$ use `JacCm3()` and `subJaCm`;
(4) $p([a:1],[c:1])$ use `JacCm4()` and `subJaCm`.

A.2.3. $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_R})$. The computations in this section are specific to Section 3.6. The equations that describe the affine chart $U \subseteq \widetilde{C}_R$ are the generators of the ideal `coverCR()`. One can optionally verify that the cover is small using `smallCR()`. The output lists the relevant strata for the cover. We compute the Jacobian for the equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_R^{-1}(0)$:

```
i1 :  JacCR();
```

Next, we evaluate the Jacobian `o1` at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, with $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that correspond to $\rho_R(x_{\mathrm{KS}}, E) = x_{\mathrm{KS}}$:

```
i2:  subJacCR(o1);
```

The resulting matrix `o2` is over $\mathbb{Q}[t_1,t_2]$. We perform elementary row and column operations on `o2`:

```
i3 :  partialSmithNormalForm(o2);
```

The output `o3` is a 41×41 identity matrix.

A.2.4. $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_\psi})$. The computations in this section are specific to Section 3.7. The equations that describe the affine chart $U \subseteq \widetilde{C}_\psi$ are the generators of the ideal returned by `coverCpsi()`. In this case we must verify that our cover $\rho_\psi : \widetilde{C}_\psi \to \overline{C}_\psi$ is semi-small:

```
i1 :  smallCpsi()

o1 = ({2  4  4  4  2, 2  4  4  4  2}, {}, {})
         2  2  2  2     2  3  3  2
```

```
0  2  0        1  2  1
 0  0           1  1
  0              0
```

The second list in `o1` tells us that

$$2\dim(\rho_\psi^{-1}(x)) + \dim C_i \leq \dim \widetilde{C}_\psi, \quad x \in C_i$$

for all orbits $C_i \leq C_\psi$. Equality is obtained if C_i is one of the orbits in the first list of `o1`. Together, these two statements tell us that $\rho_\psi : \widetilde{C}_\psi \to \overline{C}_\psi$ is semi-small. Next, we compute the Jacobian for the equations that describe $U \times C^*_{\mathrm{KS}}$ and $\tilde{f}_\psi^{-1}(0)$:

```
i2 :  JacCpsi();
```

We evaluate the Jacobian `o2` at the generic point $(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$, with $\mathrm{diag}(t_1,t_2) \in T'_{\mathrm{reg}}$, and further add the conditions that describe $U \cap \rho_\psi^{-1}(x_{\mathrm{KS}})$:

```
i3:  subJacCpsi(o2);
```

The resulting matrix `o3` is over $\mathbb{Q}[t_1,t_2,b_1,b_2,c_3,f_3]$. We perform elementary row and column operations on `o3`:

```
i4:  partialSmithNormalForm(o3);
```

To make `o4` easier to read, simply compute `matrix o4`. The output `o4` is a 52×51 block diagonal matrix whose blocks consist of a 45×45 identity matrix and the matrix

$$B_\psi = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v & b_1 v & f_3 v & b_1 f_3 v & v & f_3 v \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad \text{with} \quad v = \begin{bmatrix} -b_1 + f_3 \\ -b_1 + f_3 \\ -c_3 + f_3 \\ t_1 b_2 + b_2 c_3 + 1 \\ t_2 b_2 c_3 + b_1 \\ t_1 b_2 + b_2 c_3 + 1 \\ t_2 b_2 c_3 + b_1 \end{bmatrix}.$$

Recall that the fibre of ρ_ψ above x_{KS} is the product of $\mathbb{P}^1 \cong \{[e:h]\}$ with the variety

$$F := \{([a:b],[c:d],[u:v:w:z]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \mid av - bu = 0,\ cz - dw = 0\},$$

which is the blow up of $\mathbb{P}^3$ along the two copies of $\mathbb{P}^1$ given by $(u,v) = (0,0)$ and $(w,z) = (0,0)$. For the calculations below, we find it convenient to use the map $p : (\mathbb{P}^1)^{\times 4} \to \rho_\psi^{-1}(x_{\mathrm{KS}})$ defined by

$$p([a:b],[c:d],[e:h],[\beta:\alpha]) = (E^2_{\lambda_1}, E^1_{\lambda_2}, E^3_{\lambda_2}, E^1_{\lambda_3}, E^2_{\lambda_3}, E^3_{\lambda_3}, E^1_{\lambda_4})$$

where

$$E^2_{\lambda_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^1_{\lambda_2} = \begin{bmatrix} \beta a \\ \beta b \\ \alpha c \\ \alpha d \end{bmatrix}, \quad E^3_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{bmatrix},$$

$$E^1_{\lambda_3} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \quad E^2_{\lambda_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^3_{\lambda_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e \\ 0 & 0 & h \end{bmatrix}, \quad E^1_{\lambda_4} = \begin{bmatrix} e \\ h \end{bmatrix}.$$

The calculations above are for finding singularities in the affine chart U that contains $p([1:b],[1:d],[1:h],[1:\alpha])$; these calculations show that the intersection of $\mathrm{sing}(\tilde{f}_\psi^{-1}(0))$

with $(U \times C^*_{\mathrm{KS}}) \cap (\rho'_\psi)^{-1}(x_{\mathrm{KS}}, y_{\mathrm{KS}}(t_1,t_2))$ is

$$\{p([1:0],[1:0],[1:0],[1:-1/t_1]), p([1:t_2-t_1],[1:t_2-t_1],[1:t_2-t_1],[1:-1/t_2])\}.$$

Notice that $[1:0]$ corresponds to the eigenspace of $\left(\begin{smallmatrix} t_1 & 1 \\ 0 & t_2 \end{smallmatrix}\right)$ for the eigenvalue t_1 and $[1:t_2-t_1]$ corresponds to the eigenspace for the eigenvalue t_2.

We use the same method as above to check for singularities in a collection of affine charts that, together with U, cover $\widetilde{C}_\psi \cap \rho_\psi^{-1}(x_{\mathrm{KS}})$. In all cases, we find no new singularities. For each of these charts, we list below the relevant points of the fibre that it contains, together with the corresponding Jacobian and analogous function to `subJacCpsi()` needed for the computation:

(2) $p([1:b],[1:d],[1:h],[\beta:1])$ use `JacCpsi2()` and `subJacCpsi2()`;
(3) $p([1:b],[1:d],[e:1],[1:\alpha])$ use `JacCpsi3()` and `subJacCpsi3()`;
(4) $p([1:b],[1:d],[e:1],[\beta:1])$ use `JacCpsi4()` and `subJacCpsi4()`;
(5) $p([1:b],[c:1],[1:h],[1:\alpha])$ use `JacCpsi5()` and `subJacCpsi5()`;
(6) $p([1:b],[c:1],[1:h],[\beta:1])$ use `JacCpsi6()` and `subJacCpsi6()`;
(7) $p([1:b],[c:1],[e:1],[1:\alpha])$ use `JacCpsi7()` and `subJacCpsi7()`;
(8) $p([1:b],[c:1],[e:1],[\beta:1])$ use `JacCpsi8()` and `subJacCpsi8()`;
(9) $p([a:1],[1:d],[1:h],[1:\alpha])$ use `JacCpsi9()` and `subJacCpsi9()`;
(10) $p([a:1],[1:d],[1:h],[\beta:1])$ use `JacCpsi10()` and `subJacCpsi10()`;
(11) $p([a:1],[1:d],[e:1],[1:\alpha])$ use `JacCpsi11()` and `subJacCpsi11()`;
(12) $p([a:1],[1:d],[e:1],[\beta:1])$ use `JacCpsi12()` and `subJacCpsi12()`;
(13) $p([a:1],[c:1],[1:h],[1:\alpha])$ use `JacCpsi13()` and `subJacCpsi13()`;
(14) $p([a:1],[c:1],[1:h],[\beta:1])$ use `JacCpsi14()` and `subJacCpsi14()`;
(15) $p([a:1],[c:1],[h:1],[1:\alpha])$ use `JacCpsi15()` and `subJacCpsi15()`;
(16) $p([a:1],[c:1],[e:1],[\beta:1])$ use `JacCpsi16()` and `subJacCpsi16()`.

A.3. **Local Hessian Calculations.** In this section we document how to reproduce our calculation to find the Hessians and Hessian determinants. The code which performs our Hessian calculations can all be found in `KSHess.m2`. We first focus on the case relevant for Section 3.7. First use the function

```
i1 :  eqcpsi = EqCpsi();
```

to compute the defining ideals and functions $\tilde{f}_\psi$ for the cover $\tilde{\mathcal{O}}_\psi$ for C_ψ on which we wish to compute the Hessian. Next, the function

```
i2 :  locvar = getLocalCoordList( eqcpsi#1, subXCpsi );
```

computes a system of local coordinates which generate the completed local ring in an open neighbourhood of the point described by `subXCpsi`. Then the function

```
i3 :  impvarpart = getImpVarPartials( eqcpsi#1 , locvar, subXCpsi );
```

computes the partial derivatives of the implicit functions in terms of the chosen local coordinates. The resulting formulas are valid in an open neighbourhood of the point described by `subXCpsi`. Now the function

```
i4 :  hess = getHessian(eqcpsi#0, locvar, impvarpart);
```

computes the hessian of $\tilde{f}_\psi$ in terms of the chosen local coordinates using the precomputed formulas for the partial derivatives for the implicit functions. At this point we may use

```
i5 :  rnk = rank( subXCpsi( hess ) )

o5 = 24
```

to obtain the rank of the Hessian at the point described by `subXCpsi`. This rank will be locally constant on an open neighbourhood of `subXCpsi` on the singular locus of $\tilde{f}_\psi$. We obtain a rank of 24. Now the function

```
i6 :  subhess = getSubHessian( hess, subXCpsi );
```

returns a minor of the Hessian with non-zero determinant which has maximal rank on an open neighbourhood of `subXCpsi` on the singular locus of $\tilde{f}_\psi$. Finally the function

```
i7 :  iso=findIso(subhess#0, subXCpsi);
```

optimistically searches for a large isotropic subspace. The subspace it finds is isotropic on an open neighbourhood of `subXCpsi` on the singular locus of $\tilde{f}_\psi$. If `numcolumns(iso#0)` is half of `rnk` then the Hessian determinant is a square on an open neighbourhood of `subXCpsi` on the singular locus of $\tilde{f}_\psi$. We find a 12 dimensional isotropic subspace

To perform the calculations for Section 3.1 relevant to verifying that the character of the local system $\mathsf{Evs}_{C_{\mathrm{KS}}} \mathcal{IC}(\mathbb{1}_{C_{\mathrm{KS}}})$ is trivial simply use `EqCk();` and `subXCks` in the above. In this case the Hessian has rank 16 and the isotropic subspace has dimension 8.

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