

LIFTS OF HILBERT MODULAR FORMS AND APPLICATION TO MODULARITY OF ABELIAN VARIETIES

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To Benedict H. Gross on his recent retirement

ABSTRACT. In this paper, we prove the existence of certain lifts of Hilbert cusp forms to general odd spin groups. We then use those lifts to provide evidence for a conjecture of Gross on the modularity of abelian varieties *not* of GL_2 -type.

1. Introduction

Let F/E be a cyclic extension of totally real number fields of degree g . Let π be a cuspidal Hilbert automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$, with trivial central character, which is not a base change from any proper sub-extension F'/E of F . In this paper, we show that the automorphic induction Π' of π from F to E , which *a priori* lives on $\mathrm{GL}_{2g}(\mathbf{A}_E)$, descends to an automorphic representation Π of $\mathrm{GSpin}_{2g+1}(\mathbf{A}_E)$. In fact, our result is a little stronger (see Theorem 3.3).

As an application of our construction, we provide some evidence for the following conjecture of Gross on the modularity of abelian varieties (see Conjecture 5.1 for a slightly more general statement).

Conjecture 1.1 (Gross-Langlands). ¹ *Let B be an abelian variety of dimension g and conductor N defined over $\mathbf{Q}$ such that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$. Then, there exists a globally generic cuspidal automorphic representation Π on $\mathrm{GSpin}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$ of weight 2 and paramodular level structure N , with field of rationality $\mathbf{Q}$, such that*

$$L(B, s) = L(\Pi, s).$$

Gross' conjecture deals more generally with *discrete* symplectic motives of pure weight 1, with Conjecture 1.1 being a special case, see [Gro16, Conjectures 3 and 4, and Proposition 6]. It proposes a candidate admissible representation $\Pi = \bigotimes'_v \Pi_v$, whose local components Π_v are described using the local Langlands correspondence for $\mathrm{SO}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$ [JS04] after some appropriate normalisation. It also predicts the level of Π in terms of the conductor of B . The main challenge is to prove that this Π is indeed automorphic and cuspidal. (We note that, although the field of rationality of Π is not explicitly discussed in [Gro16], its existence could be inferred from the theory of newforms developed in §§6-8. Also, the paramodular level structure is only defined for the group SO_{2g+1} , but one can extend the theory to GSpin_{2g+1} .)

Conjecture 1.1 is a generalisation of the genus 2 case also known as the paramodularity conjecture [BK14, Conjectures 1.1 and 1.4]; the best theoretical evidence

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¹Although there was a folklore conjecture before it, to the best of our knowledge, this precise and verifiable formulation is due to Gross. It is analogous to Weil's refinement of the Shimura-Taniyama conjecture for elliptic curves over $\mathbf{Q}$.

for the conjecture is only available in that case. Let B be an abelian surface defined over $\mathbf{Q}$, with $\text{End}_{\mathbf{Q}}(B) = \mathbf{Z}$. Let N be the conductor of B , and $\ell \geq 5$ a prime not dividing N . Consider the Galois representation

$$\rho_{B,\ell} : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{GSp}_4(\mathbf{Z}_{\ell}),$$

arising from the ℓ -adic Tate module of B . Under some technical conditions, Tilouine [Til06] and Pilloni [Pil12] prove an overconvergent “ $R = T$ ” from which they deduce that there exists an overconvergent Siegel modular form f of genus 2 and weight 2 such that $\rho_{B,\ell} \simeq \rho_{f,\ell}$. In this case, one is left with proving that the form f is indeed classical. That crucial step has now been settled in [Pil17]. In [CG16], they also prove a modularity lifting theorem for Siegel modular forms of genus 2 and non-regular weights. However, as it stands, this result doesn’t apply yet to abelian surfaces. All of the results above impose a “big image” condition. This makes it extremely hard to check residual modularity in practice since $\text{GSp}_4(\mathbf{F}_{\ell})$ is rather big for $\ell \geq 5$.

The infinite component Π_{∞}^h of the automorphic representation Π^h , associated to the form f described above, is a limit of holomorphic discrete series. But, although the form f itself is not cohomological, its construction relies crucially on the fact that the symmetric space associated to $\text{SO}(3, 2)$ carries a complex structure. For example, in [Til06, Pil12, Pil17], the form f arises from ℓ -adic families of overconvergent Siegel modular forms. This requires working with the geometry of Siegel modular threefolds; this is equally true for the approach used in [CG16].

Conjecture 1.1 predicts an automorphic representation Π which is globally generic. Its infinite component Π_{∞} is a limit of generic discrete series. Unfortunately, the symmetric space associated to the split group GSpin_{2g+1} no longer carries a complex structure when $g \geq 3$ (see [Gro16]). Moreover, as was pointed out to us by Tilouine [Til15], Π_{∞} becomes more and more degenerate as the dimension g of B increases. This means that, for $g \geq 3$, none of the approaches mentioned above readily applies, and that any substantial progress towards Conjecture 1.1 will require some novel ideas.

Our goal in this paper is one that is more modest. We want to provide some evidence for Conjecture 1.1 by means of functoriality. Our approach here is a generalisation of [DK16, BDPŠ15]. It consists in showing that Conjecture 1.1, or rather its converse, is compatible with Galois descent of isogeny classes, which corresponds to lifts on the automorphic side. More precisely, we prove the following:

Theorem 1.2. *Let $F/\mathbf{Q}$ be a totally real cyclic extension of degree g , and write $\text{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$. Let $\mathfrak{N}$ be an integral ideal such that $\mathfrak{N}^{\sigma} = \mathfrak{N}$, and π a cuspidal Hilbert automorphic representation on $\text{GL}_2(\mathbf{A}_F)$ of weight 2 and level $\mathfrak{N}$, with trivial central character, which is not a base change from any proper subfield F' of F . Let L be the field of rationality of π . Assume that the Hecke orbit of π is fixed by $\text{Gal}(F/\mathbf{Q})$. Further assume that there is an abelian variety A/F associated to π by the Eichler-Shimura construction. Then, we have the followings:*

- (i) *There exists a subfield K of L such that L/K is cyclic of degree g ;*
- (ii) *The automorphic representation π lifts to a globally generic cuspidal automorphic representation Π of $\text{GSpin}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$, whose field of rationality is K ;*
- (iii) *There exists an abelian variety B of dimension g defined over $\mathbf{Q}$ such that $B \times_{\mathbf{Q}} F \sim A$, $\text{End}_{\mathbf{Q}}(B) \otimes \mathbf{Q} = K$, and*

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s),$$

where $[\Pi]$ denotes the Hecke orbit of Π .

Combining Theorem 1.2 with the Serre Conjecture [Ser87, KW09], we obtain the following:

Theorem 1.3. *Let B be an abelian variety of prime dimension g over $\mathbf{Q}$ such that $\text{End}_{\mathbf{Q}}(B) = \mathbf{Z}$. Suppose that there exist totally real number fields F and L of degree g such that*

1. F is cyclic;
2. $\text{End}_F(A) \otimes \mathbf{Q} = L$, where $A = B \times_{\mathbf{Q}} F$;
3. *There is a rational prime $\ell \geq 3$ and a (totally) ramified prime $\lambda \mid \ell$ in L such that the residual Galois representation $\bar{\rho}_{A,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}_2(\mathbf{F}_{\ell})$ is surjective.*

Then, B is automorphic. More specifically, there exists a cuspidal Hilbert automorphic representation π on $\text{GL}_2(\mathbf{A}_F)$, which lifts to a globally generic cuspidal automorphic representation Π on $\text{GSpin}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$, such that

$$L(B, s) = L(\Pi, s).$$

The proof of Theorem 1.3 will show that Conditions (1) and (2) force L to be cyclic as a result of B acquiring extra endomorphism after base change to F . Since $L/\mathbf{Q}$ is of prime degree, any prime ℓ that ramifies in L will be totally ramified. So, the technical requirement that we impose on $\bar{\rho}_{A,\lambda}$ is rather mild. It allows us to use some of the most up to date modularity results available in the literature, namely, the Serre Conjecture over $\mathbf{Q}$ [Ser87, KW09] and results in [KT16, Tho17] to show that there is an abundance of abelian varieties which verify Conjecture 1.1. We illustrate this by giving some examples. But, we also refer to [DK16, BDP§15] for additional examples of abelian surfaces. We believe that, by combining all modularity lifting results that are currently available, one should be able to remove Condition (3) from Theorem 1.3. However, this would require a case by case detailed analysis that we do not want to perform here at the risk of overshadowing the primary goal of this paper, which is to provide some evidence for Conjecture 1.1. We remark, in closing, that one potential application of our results (see also Corollary 5.9), and their generalisation, is to the hypergeometric abelian varieties described in [Dar00] in connection with the Generalised Fermat Equations, see work of Billery-Chen-Dieulefait-Freitas [BCDF17] on this problem.

The content of the paper is as follows. In Section 2, we recall some background material on automorphic representations on GL_n and their associated Galois representations. In Section 3, we prove the existence of our lift. Section 4 contains preliminary results on Galois descent of isogeny classes of abelian varieties; they are results that could be interesting on their own. Section 5 is dedicated to showing that our lift is compatible with Langlands functoriality; more precisely, we show that Conjecture 1.1 is compatible with Galois descent of isogeny classes. Finally, we provide some illustrating examples in Sections 6 and 7.

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2. Automorphic representations for GL_n

In this section, we recall some results on automorphic representations for GL_n that we need. (We refer to [BLGGT14a, BLGGT14b, BLGGT12] for more details.) We let F be a totally real number field, and denote by J_F the set of all real embeddings of F . For every $\underline{a} = (a_\tau)_{\tau \in J_F}$, with $a_\tau = (a_{\tau,1}, \dots, a_{\tau,n}) \in \mathbf{Z}_+^n$, such that $a_{\tau,1} \geq \dots \geq a_{\tau,n}$, we let $\Xi_{\underline{a}}$ be the irreducible representation of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_n)$ given by

$$\Xi_{\underline{a}} = \bigotimes_{\tau \in J_F} \Xi_{a_\tau},$$

where Ξ_{a_τ} is the irreducible representation of GL_n of highest weight a_τ . We are interested in the $\underline{a}$ such that $\Xi_{\underline{a}}$ is an algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_n)$. In that case, there exists $n_0 \in \mathbf{Z}_+$ such that, for all $1 \leq i \leq n$,

$$a_{\tau,i} + a_{\tau,n+1-i} = n_0.$$

Let π be an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. We say that π has weight $\underline{a}$ if π_∞ has the same infinitesimal character as $\Xi_{\underline{a}}^\vee$, the dual of $\Xi_{\underline{a}}$. We will be interested in automorphic representations π which satisfy the following conditions:

1. π is *regular algebraic* of weight $\underline{a}$, i.e. the weight $\Xi_{\underline{a}}$ is an algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_n)$;
2. π is *essentially self dual*, i.e. $\pi^\vee \simeq \pi \otimes \chi$ for some Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbf{C}^\times$ such that $\chi_\tau(-1) = (-1)^n$ for all $\tau \in J_F$;
3. π is *cuspidal*.

An automorphic representation π which satisfies the conditions above will be called *RAESDC* (regular, algebraic, essentially self-dual and cuspidal). We will also use the same terminology for the pair (π, χ) .

Remark 2.1. Since strong multiplicity one holds for GL_n , Condition (2) is equivalent to saying that for almost all unramified places v we have $\pi_v \simeq \pi_v^\vee$.

2.1. Field of rationality and Hecke orbits. Let π be an RAESDC automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, and write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is the archmedian component of π , and π_f is finite part. Let V be the underlying complex vector space of π_f . For $\tau \in \mathrm{Aut}(\mathbf{C})$, let $V^\tau = V \otimes_{\mathbf{C}, \tau} \mathbf{C}$. Then V^τ admits a natural admissible representation of $\mathrm{GL}_n(\mathbf{A}_F^\tau)$, which we denote by π_f^τ . So, we get an action of $\mathrm{Aut}(\mathbf{C})$ on equivalence classes of admissible representations of $\mathrm{GL}_n(\mathbf{A}_F^\tau)$. Let $H(\pi_f)$ be the stabiliser of the class of π_f in $\mathrm{Aut}(\mathbf{C})$, so

$$H(\pi_f) = \{ \tau \in \mathrm{Aut}(\mathbf{C}) : \pi_f^\tau \simeq \pi_f \}.$$

The *field of rationality* of π denoted L_π is defined by

$$L_\pi := \mathbf{C}^{H(\pi_f)}.$$

In general, L_π is not a number field. But in our step up, this will always be the case thanks to [Clo90, Proposition 3.1] and [Clo14, Theorem 1.11].

Theorem 2.2 (Clozel). *Let π be an RAESDC automorphic representation on $\mathrm{GL}_n(\mathbf{A}_F)$. Then, the field of rationality L_π is a totally real or CM number field.*

Let π be an RAESDC automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Then the admissible representations π_f of $\mathrm{GL}_n(\mathbf{A}_F^\tau)$ has a model $\pi_{f,0}$ over L_π . This model is

unique up to scaling, and is stable under $\mathrm{GL}_n(\mathbf{A}_F^f)$ (see *loc. cit.*). We define the *Hecke orbit* of π to be the set

$$[\pi] := \{\pi^\tau = \pi_\infty \otimes \pi_f^\tau : \tau \in \mathrm{Hom}(L_\pi, \mathbf{C})\}.$$

We note that, for $\tau \in \mathrm{Hom}(L_\pi, \mathbf{C})$, the field of rationality of π^τ is $L_{\pi^\tau} := \tau(L_\pi)$, and that $\pi_{f,0}^\tau = \pi_{f,0} \otimes_{L_\pi, \tau} L_{\pi^\tau}$ is a model for π_f^τ .

2.2. Galois representations. The following result is the culmination of the work of lots of people (see [BLGGT14b, Theorem 2.1.1] and reference therein.)

Theorem 2.3. *Let (π, χ) be an RAESDC automorphic representation of weight $\underline{a}$ on $\mathrm{GL}_n(\mathbf{A}_F)$. Let λ be a prime of L_π with residue characteristic ℓ , and fix an isomorphism $\iota : \overline{L}_{\pi, \lambda} \simeq \mathbf{C}$. Then, there exists a Galois representation*

$$r_\lambda(\pi) : G_F \rightarrow \mathrm{GL}_n(\overline{L}_{\pi, \lambda}),$$

such that

- (i) $r_\lambda(\pi)^\vee \simeq r_\lambda(\pi) r_\lambda(\chi)$;
- (ii) $\mathrm{WD}(r_\lambda(\pi)|_{G_{F_v}})^{\mathrm{F-ss}} \simeq \mathrm{rec}_{F_v}(\pi_v \otimes |\det|^{(1-n)/2})$;
- (iii) For each $v \mid \ell$, $r_\lambda(\pi)|_{G_{F_v}}$ is de Rham, and for each embedding $\tau : F \hookrightarrow \overline{L}_{\pi, \lambda}$, the Hodge-Tate weights are given by

$$\mathrm{HT}_\tau(r_\lambda(\pi)) = \{a_{\tau, n}, a_{\tau, n-1} + 1, \dots, a_{\tau, 1} + n - 1\}.$$

These conditions determine $r_\lambda(\pi)$ uniquely up to isomorphism.

The requirement that π be essentially self-dual in Theorem 2.3 has been removed in recent work of Harris-Lan-Taylor-Thorne [HLTT16, Theorem A] (also see [Sch15, Theorem 5.1.4]), but we will not need this here. However, we will be dealing with automorphic representations that are essentially self-dual and cuspidal but *not* regular (AESDC), and for which Theorem 2.3 will still be true. So, for the remainder of this section, we will assume that the Langlands correspondence is known for all our automorphic representations.

2.3. Base change for GL_n . We recall that, if F/E is a Galois extension, then there is a natural action of $\mathrm{Gal}(F/E)$ on the set of automorphic representations on $\mathrm{GL}_n(\mathbf{A}_F)$ given by

$$\sigma\pi := \pi \circ \sigma.$$

If π has weight $\underline{a}$, then $\sigma\pi$ has weight $\sigma \cdot \underline{a} := (a_{\sigma\tau})$, $\sigma \in \mathrm{Gal}(F/E)$, with

$$\Xi_{\sigma \cdot \underline{a}} = \bigotimes_{\tau \in J_F} \Xi_{a_{\sigma\tau}}.$$

Solvable base change for GL_n for cyclic extensions of prime degree is due to Arthur-Clozel [AC89]. Their results were extended to general cyclic extensions by Henniart [Hen12].

Theorem 2.4 (Cyclic base change). *Let F/E be a cyclic extension of totally real fields. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$ of weight $\underline{a}$. Then there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

- (i) For all finite places v of F , $\mathrm{rec}_{F_v}(\Pi_v) = \mathrm{rec}_{E_w}(\pi_w)|_{W_{F_v}}$, where $v \mid w$. In particular, we have $r_\lambda(\Pi) \simeq r_\lambda(\pi)|_{G_F}$.
- (ii) $\Pi \simeq \Pi \circ \sigma$, where $\mathrm{Gal}(F/E) = \langle \sigma \rangle$.

The automorphic representation Π is uniquely determined by those two conditions. It is called the *base change of π from E to F* , and denoted by $\mathrm{BC}_{F/E}(\pi)$. It satisfies the additional property:

- (iii) $\mathrm{BC}_{F/E}(\pi) \simeq \mathrm{BC}_{F/E}(\pi')$ if and only if $\pi \simeq \pi' \otimes (\delta_{F/E}^i \circ \mathrm{Art}_E \circ \det)$ for some i , where $\mathrm{Gal}(F/E)^\vee = \langle \delta_{F/E} \rangle$ is the dual of $\mathrm{Gal}(F/E)$.
- (iv) The weight $\mathrm{BC}_{F/E}(\underline{a})$ of Π is given by $\mathrm{BC}_{F/E}(\underline{a})_\tau = a_{\tau'}$ where $\tau' = \tau|_E$.

2.4. Automorphic induction. In *loc. cit.*, Arthur-Clozel also proved automorphic induction for cyclic extensions F/E of prime degree. This result was also extended to general cyclic extensions in [Hen12].

Theorem 2.5 (Automorphic induction). *Let F/E be a cyclic extension of totally number fields of degree g , and π a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbf{A}_F)$, induced from cuspidal, such that π is not a base change from any sub-extension F'/E of F . Then, there exists a cuspidal automorphic representation Π of $\mathrm{GL}_{mg}(\mathbf{A}_E)$, induced from cuspidal, which satisfies the following properties:*

- (i) $\mathrm{BC}_{F/E}(\Pi) = \pi \boxplus \pi^\sigma \boxplus \cdots \boxplus \pi^{\sigma^{g-1}}$;
- (ii) $\Pi \simeq \Pi \otimes (\delta_{F/E} \circ \mathrm{Art}_E \circ \det)$.

The automorphic representation Π , which is uniquely determined by Conditions (i) and (ii) up to isomorphism, is called the automorphic induction of π , and denoted by $\mathrm{AI}_{E/F}(\pi)$. It satisfies the additional property:

- (iii) *For all finite places w of E , we have $\mathrm{rec}_w(\mathrm{AI}_{E/F}(\pi)_w) = \mathrm{rec}_w(\mathrm{AI}_{E_w/F_v}(\pi_v))$, where $w \mid v$.*

Remark 2.6. We have stated Theorem 2.5 in such a way that the characterising properties of automorphic induction mirror those of base change. Not only is this very useful from a practical point of view, but also these functorial properties guide the proof in [Hen12], where automorphic induction is constructed by combining base change with Galois descent.

2.5. Field of rationality of automorphic induction. It will be important for us to know the field of rationality of the automorphic representations we obtain from induction. So, we will need the following lemma.

Lemma 2.7. *Let π be a RAESDC automorphic representation of $\mathrm{GL}_m(\mathbf{A}_F)$ such that π is not a base change from any proper sub-extension F'/E of F . Let L_π and L_Π be the fields of rationality of π and $\Pi = \mathrm{AI}_{E/F}(\pi)$, respectively. Then, L_Π is a subfield of L_π . If further $[\pi] = [\sigma\pi]$ (i.e. the Hecke orbit of π is preserved by $\mathrm{Gal}(F/E)$), then L_π is a cyclic extension of L_Π of degree g .*

Proof. By Theorem 2.2, L_π is a number field, which is either totally real or CM, and by definition of base change, we have $L_\pi = L_{\sigma\pi}$. From Theorem 2.5 (1), we see that L_Π is the smallest field over which the cuspidal support of Π is defined. Since all the $\mathrm{Gal}(F/E)$ -conjugate of π are defined over L_π , we conclude that L_Π must be a subfield of L_π . This gives the first assertion. To prove the second one, observe that if $[\pi] = [\sigma\pi]$, then there exists $\tau \in \mathrm{Hom}(L_\pi, \mathbf{C})$ such that $\sigma\pi = \pi^\tau$. By uniqueness of models, this implies that $\sigma\pi_{f,0} = \pi_{f,0}^\tau$. Hence, $L_\pi = L_{\pi^\tau}$ and $\tau \in \mathrm{Aut}(L_\pi)$. The map $\sigma \mapsto \tau$ must be an injection. Otherwise, this would contradict the fact that the stabiliser of π in $\mathrm{Gal}(F/E)$ is trivial. Let H be the subgroup of $\mathrm{Aut}(L_\pi)$ generated by τ and $K = L_\pi^H$. Then, Π is clearly defined over K , and it is not hard to see that K is the smallest such field. So $L_\Pi = K$. $\square$

Remark 2.8. In general, it is a very delicate matter to determine the field of rationality of an automorphic representation Π whose infinite components Π_v , $v \mid \infty$, are limit of discrete series (see [BHR94, ST14] for example) as is the case in our applications to abelian varieties. However, since our representations are automorphic inductions of representations whose weights are regular, Lemma 2.7 ensures that their fields of rationality are still totally real or CM.

Lemma 2.9. *Let (π, χ) be an essentially self-dual cuspidal automorphic representation on $\mathrm{GL}_m(\mathbf{A}_F)$. Then $(\mathrm{AI}_{E/F}(\pi), \chi|_E)$ is an essentially self-dual cuspidal automorphic representation on $\mathrm{GL}_{mg}(\mathbf{A}_E)$.*

Proof. Let $\Pi = \text{AI}_{E/F}(\pi)$, so that $\Pi^\vee = \text{AI}_{E/F}(\pi^\vee)$. Then, by Theorem 2.5, Π is cuspidal. So, we only need to show that it is essentially self-dual. We recall that $\pi^\vee \simeq \pi \otimes \chi$, and for simplicity, we assume that the stabiliser of π in $\text{Gal}(F/E)$ is trivial. By Theorems 2.4 and 2.5, it follows that

$$\begin{aligned} \text{BC}_{F/E}(\Pi^\vee) &= \pi^\vee \boxplus (\pi^\vee)^\sigma \boxplus \cdots \boxplus (\pi^\vee)^{\sigma^{g-1}} \\ &= (\pi \otimes \chi) \boxplus (\pi \otimes \chi)^\sigma \boxplus \cdots \boxplus (\pi \otimes \chi)^{\sigma^{g-1}} \\ &= (\pi \boxplus \pi^\sigma \boxplus \cdots \boxplus \pi^{\sigma^{g-1}}) \otimes \chi|_E \circ \text{N}_{F/E} \\ &= \text{BC}_{F/E}(\Pi) \otimes \text{BC}_{F/E}(\chi|_E) \\ &= \text{BC}_{F/E}(\Pi \otimes \chi|_E). \end{aligned}$$

So, by uniqueness, we must have $\Pi^\vee = \Pi \otimes \chi|_E$. This implies that $(\Pi, \chi|_E)$ is essentially self-dual. $\square$

2.6. Automorphic descent from GL_{2g} to GSpin_{2g+1} . Let G/F be the algebraic group GSpin_{2g+1} or the split GSpin_{2g} or one of its non-split quasi-split forms GSpin_{2g}^* determined by some quadratic extension of F . (For a detailed description of these groups, we refer to [AS06, AS14].) The dual group of G is $\hat{G} = \text{GSp}_{2g}$ or GSO_{2g} , accordingly. There is a natural embedding $\hat{G} \hookrightarrow \text{GL}_{2g}$, which gives rise to a homomorphism

$$\iota : \hat{G}(\mathbf{C}) \hookrightarrow \text{GL}_{2g}(\mathbf{C}).$$

Let (Π', χ) be an AESDC automorphic representation of $\text{GL}_{2g}(\mathbf{A}_F)$. Let Π be a cuspidal automorphic representation of $G(\mathbf{A}_F)$. Then, for each place v of F , the local component Π_v of Π is determined by its local Langlands parameter

$$\phi_v : W_{F_v} \rightarrow \hat{G}(\mathbf{C}).$$

We say that (Π', χ) is the *weak functorial transfer* of Π to GL_{2g} if, for almost place v of F , the Langlands parameter of the local component Π'_v is given by $\iota \circ \phi_v$. We say that it is the *strong functorial transfer* if this is true for all v .

Let (Π', χ) be as above. Then, for every finite set of place S (containing the archimedians), we have the equality of incomplete L -series

$$L^S(s, \Pi' \times (\Pi' \otimes \chi^{-1})) = L^S(s, \text{Sym}^2(\Pi') \otimes \chi^{-1}) L^S(s, (\wedge^2 \Pi') \otimes \chi^{-1}).$$

For S sufficiently large, the left hand side has a simple pole at $s = 1$ (see [JS81b, JS81a]). Hence, only one of the L -series of the right hand side must have a pole. The following theorem characterises the image of automorphic descent to GSpin_{2g+1} . (We refer to [Sha11] for the notion of global genericity, and to their Conjecture 2.10, which predicts that every cuspidal automorphic representations of GSpin_{2g+1} that is everywhere local generic is globally generic. We also refer to [AS06, AS14, HS16] for a detailed study of the question of functorial transfer from GSpin_{2g+1} to GL_{2g} .)

Theorem 2.10. *Let F be a number field, and (Π', χ) an essentially self-dual cuspidal automorphic representation of $\text{GL}_{2g}(\mathbf{A}_F)$. Then, the followings are equivalent:*

- (a) *There exists a globally generic, cuspidal, automorphic representation Π of $\text{GSpin}_{2g+1}(\mathbf{A}_F)$, whose unique functorial transfer to $\text{GL}_{2g}(\mathbf{A}_F)$ is (Π', χ) ;*
- (b) *The L -series $L^S(s, \wedge^2 \Pi' \otimes \chi^{-1})$ has a pole at $s = 1$.*

Proof. Asgari-Shahidi [AS14, Theorem 4.26] prove that (a) implies (b), while Hundley-Sayag [HS16, Corollary 3.2.1] show the converse. $\square$

Remark 2.11. It is important to note that [AS14, Theorem 4.26] only gives that Π' is a weak transfer of Π . However, [AS14, Proposition 5.1] shows that the transfer is in fact strong.

3. Lifts of Hilbert automorphic representations

In this section, we prove the existence of lifts of Hilbert automorphic forms.

3.1. Automorphic representations on GL_2 . Let $k \geq 2$ and w be two integers with the same parity. We let $D_{k,w}$ be the *discrete* series representation of $\mathrm{GL}_2(\mathbf{R})$ defined in [Car86, §0.2]; it is the essentially square integrable sub-representation of the unitary induction $\mathrm{Ind}(\mu, \nu)$, where μ, ν are the characters of $\mathbf{R}^\times$ given by

$$\mu(t) = |t|^{\frac{k-1-w}{2}} \cdot \mathrm{sgn}(t), \quad \nu(t) = |t|^{\frac{-k+1-w}{2}}.$$

The central character of $D_{k,w}$ is the character $\mu\nu : t \mapsto t^{-w}$.

Let $\underline{k} = (k_\tau)_{\tau \in J_F} \in \mathbf{Z}_+^{J_F}$ be such that the k_τ have the same parity. Let $k_0 = \max\{k_\tau : \tau \in J_F\}$ and $w = k_0 - 2$. For each $\tau \in J_F$, define $m_\tau = \frac{k_0 - k_\tau}{2}$ and $n_\tau = k_\tau - 2$. We are interested in automorphic representations $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\pi_\tau \simeq D_{k_\tau, w}$. Such representations correspond to holomorphic Hilbert modular forms of weight $\underline{k}$. In the terminology of Section 2, we see that π has weight $\underline{a} = (a_\tau)_{\tau \in J_F}$, where $a_\tau = (a_{\tau,1}, a_{\tau,2}) = (k_\tau + m_\tau - 2, m_\tau)$. (Note that $a_{\tau,1} + a_{\tau,2} = w$.)

3.2. Galois representations. Let π be a cuspidal Hilbert automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $\underline{k}$, and L_π the field of rationality of π . Then, by work of Shimura [Shi78], L_π is the coefficient field of the newform $f \in \pi$, i.e. $L_\pi = \mathbf{Q}(\{a_p(f) : p \subset \mathcal{O}_F\})$ where $a_p(f)$ is the eigenvalue of the Hecke T_p at p acting on f . In this case, Theorem 2.3 is a bit more precise. In the form below, it follows from work of Carayol [Car86], Taylor [Tay89] and many other people.

Theorem 3.1. *Let ℓ be a rational primes, and $\lambda \mid \ell$ a prime in L_π . Then, there exists a Galois representation*

$$r_\lambda(\pi) : G_F \rightarrow \mathrm{GL}_2(\overline{L}_{\pi, \lambda}),$$

such that

1. For each finite place v such that π_v is unramified, and $v \nmid \ell$, then the characteristic polynomial of Frob_v is given by $X^2 - a_v X + q_v s_v$, where $q_v = \#k_v$, k_v the residue field of $\mathcal{O}_{F_v}$ at v , and a_v and s_v are the eigenvalues of the Hecke operators T_v and S_v respectively.
2. More generally, we have $\mathrm{WD}(r_\lambda(\pi)|_{G_{F_v}})^{\mathrm{F-ss}} \simeq \mathrm{rec}_{F_v}(\pi_v \otimes |\det|^{-1/2})$.
3. If $v \mid \ell$, then $r_\lambda(\pi)|_{G_{F_v}}$ is de Rham with Hodge-Tate weights

$$\mathrm{HT}_\tau(r_\lambda(\pi)) = \{m_\tau, k_\tau + m_\tau - 1\},$$

for each embedding $\tau : F \hookrightarrow \overline{L}_{\pi, \lambda} \simeq \mathbf{C}$. If π_v is unramified then $r_\lambda(\pi)|_{G_{F_v}}$ is crystalline.

4. If c_v is a complex conjugation, then $\det(r_\lambda(\pi)(c_v)) = -1$.

3.3. Local-global compatibility at archimedean places. Conditions (2) and (3) express local-global compatibility conditions at finite places. This also extends to archimedean places. Let $\zeta_{k_\tau, w}$ be the character

$$\begin{aligned} \zeta_{k_\tau, w} : \mathbf{C}^\times &\rightarrow \mathbf{C}^\times \\ z &\mapsto (z\bar{z})^{\frac{-w-k_\tau+1}{2}} z^{k_\tau-1}. \end{aligned}$$

This character determines the 2-dimensional Weil representation $\sigma_{k_\tau, w} = \mathrm{Ind}_{W_{\mathbf{C}}}^{W_{\mathbf{R}}} \zeta_{k_\tau, w}$. The local-global compatibility condition at infinite places, asserts that $\sigma_{k_\tau, w}$ corresponds to $D_{k_\tau, w}$.

3.4. Lifts of Hilbert automorphic representations. We are now ready to prove our main result on the existence of lifts. The method we used was suggested by F. Shahidi. It follows somewhat [RS07], with the advantage that we now have the results in [AS06, AS14] and [HS16] at our disposal in the form of Theorem 2.10.

Lemma 3.2. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$, with central character ω , which is not a base change from any proper sub-extension F'/E of F . Let $\Pi = \mathrm{AI}_{E/F}(\pi)$, and $t = \lfloor \frac{g-1}{2} \rfloor$. When g is even, let E_0/E be the sub-extension of degree $g/2$ of F , and Π_0 a cuspidal automorphic representation over $\mathrm{GL}_4(\mathbf{A}_{E_0})$ such that $\mathrm{BC}_{F/E_0}(\Pi_0) = \pi \times \pi^{\sigma^{g/2}}$. Then we have the following isobaric decompositions:*

$$\begin{aligned} \wedge^2 \Pi &= \mathrm{AI}_{E/F}(\omega) \cdot | \cdot |^{k_0-1} \boxplus_{i=1}^t \mathrm{AI}_{E/F}(\pi \times \pi^{\sigma^i}) \boxplus \begin{cases} 0 & \text{if } g \text{ is odd;} \\ \mathrm{AI}_{E/E_0}(\Pi_0) & \text{otherwise;} \end{cases} \\ \mathrm{Sym}^2 \Pi &= \mathrm{AI}_{E/F}(\mathrm{Sym}^2(\pi)) \boxplus_{i=1}^t \mathrm{AI}_{E/F}(\pi \times \pi^{\sigma^i}) \boxplus \begin{cases} 0 & \text{if } g \text{ is odd;} \\ \mathrm{AI}_{E/E_0}(\Pi_0) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Since F is cyclic, and π is not a base change from any proper sub-extension F'/E of F , Theorem 2.5 implies that $\Pi = \mathrm{AI}_{E/F}(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_{2g}(\mathbf{A}_E)$. Recall that $\pi^\vee = \pi \otimes \omega^{-1}$, and $\det(r_\lambda(\pi)) = r_\lambda(\omega) \epsilon^{k_0-1}$. So, by Lemma 2.9, it follows $\Pi^\vee = \Pi \otimes \omega_{|E}^{-1}$. By Theorem 2.5, we have

$$\begin{aligned} (\wedge^2 r_\lambda(\Pi)) \Big|_{G_F} &= \bigwedge^2 \left(r_\lambda(\pi) \oplus \cdots \oplus r_\lambda(\pi^{\sigma^{g-1}}) \right) \\ &= \left(r_\lambda(\omega) \oplus r_\lambda(\omega^\sigma) \oplus \cdots \oplus r_\lambda(\omega^{\sigma^{g-1}}) \right) \otimes \epsilon^{k_0-1} \\ &\quad \oplus \bigoplus_{0 \leq i < j \leq g-1} r_\lambda(\pi^{\sigma^i}) \otimes r_\lambda(\pi^{\sigma^j}) \\ &= \left(\mathrm{Ind}_E^F(r_\lambda(\omega)) \Big|_{G_F} \right) \otimes \epsilon^{k_0-1} \oplus \bigoplus_{0 \leq i < j \leq g-1} r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^j}). \end{aligned}$$

A similar calculation shows that

$$(\mathrm{Sym}^2 r_\lambda(\Pi)) \Big|_{G_F} = \left(\mathrm{Ind}_E^F(\mathrm{Sym}^2(r_\lambda(\pi))) \Big|_{G_F} \right) \oplus \bigoplus_{0 \leq i < j \leq g-1} r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^j}).$$

When g is odd, then we can rearrange the last sum as

$$\begin{aligned} \bigoplus_{0 \leq i < j \leq g-1} r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^j}) &= \bigoplus_{i=1}^t \left(r_\lambda(\pi \times \pi^{\sigma^i}) \oplus \cdots \oplus r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^{i+g-1}}) \right) \\ &= \bigoplus_{i=1}^t \left(\mathrm{Ind}_E^F(r_\lambda(\pi \times \pi^{\sigma^i})) \Big|_{G_F} \right). \end{aligned}$$

When g is even, then $\pi \times \pi^{\sigma^{g/2}}$ is a base change from the subfield E_0 of F of degree $g/2$. (It is cuspidal, see below.) Let Π_0 be such that $\mathrm{BC}_{F/E_0}(\Pi_0) = \pi \times \pi^{\sigma^{g/2}}$. Then, Π_0 is cuspidal by Theorem 2.4, and it is not hard to see that it cannot be a

base change from any proper subfield E'/E of E_0 . We have

$$\begin{aligned} \bigoplus_{0 \leq i < j \leq g-1} r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^j}) &= \bigoplus_{i=1}^t \left(r_\lambda(\pi \times \pi^{\sigma^i}) \oplus \cdots \oplus r_\lambda(\pi^{\sigma^i} \times \pi^{\sigma^{i+g-1}}) \right) \\ &\quad \oplus \left(r_\lambda(\Pi_0)|_{G_F} \oplus \cdots \oplus r_\lambda(\Pi_0^{\sigma^{g/2}})|_{G_F} \right) \\ &= \bigoplus_{i=1}^t \left(\text{Ind}_E^F(r_\lambda(\pi \times \pi^{\sigma^i}))|_{G_F} \right) \oplus \text{Ind}_E^{E_0}(r_\lambda(\Pi_0))|_{G_F}. \end{aligned}$$

By [JS81b, JS81a], $\pi^{\sigma^i} \times \pi^{\sigma^j}$ is cuspidal, for $0 \leq i < j \leq g-1$. So, we conclude the lemma by again applying Theorem 2.5. $\square$

Theorem 3.3. *Let F/E be a cyclic extension of totally real fields of degree g . Let π be a cuspidal Hilbert automorphic representation, with central character ω , which is not a base change from any sub-extension F'/E of F . Then, $\Pi' = \text{AI}_{E/F}(\pi)$ descends to a globally generic, cuspidal, automorphic representation Π of $\text{GSpin}_{2g+1}(\mathbf{A}_E)$ if and only if ω factors through the norm map $N_{F/E} : F \rightarrow E$.*

Proof. Let $r = \lfloor \frac{g-1}{2} \rfloor$ or $g/2$ according as to g is odd or even. Then, by Lemma 3.2, there exist cuspidal automorphic representations $\Pi_1, \dots, \Pi_r$ on $\text{GL}_{4g}(\mathbf{A}_E)$ such that

$$\begin{aligned} \wedge^2 \Pi' &= \text{AI}_{E/F}(\omega)| \cdot |^{k_0-1} \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r; \\ \text{Sym}^2(\Pi') &= \text{AI}_{E/F}(\text{Sym}^2(\pi)) \boxplus \Pi_1 \boxplus \cdots \boxplus \Pi_r. \end{aligned}$$

The product $L^S(s, \Pi_1 \otimes \omega|_E^{-1}) \cdots L^S(s, \Pi_r \otimes \omega|_E^{-1})$ is an entire function. Therefore, if $L^S(s, \text{Sym}^2(\Pi') \otimes \omega|_E^{-1})$ has a pole at $s = 1$ then $L^S(s, \text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1})$ has a pole at $s = 1$. Similarly, if $L^S(s, (\wedge^2 \Pi') \otimes \omega|_E^{-1})$ has a pole at $s = 1$, then so does $L^S(s, \text{AI}_{E/F}(\omega)| \cdot |^{k_0-1} \otimes \omega|_E^{-1})$. The latter is possible if and only if ω factors through the norm map $N_{F/E} : F \rightarrow E$, i.e. only if ω is a base change from E .

Conversely, assume that ω factors through the norm map $N_{F/E}$, and also that $L^S(s, (\wedge^2 \Pi') \otimes \omega|_E^{-1})$ does not have a pole at $s = 1$. By [Sha97, Theorem 1.1], the L -series $L^S(s, (\wedge^2 \Pi') \otimes \omega|_E^{-1})$ and $L^S(s, \text{Sym}^2(\Pi') \otimes \omega|_E^{-1})$ are always non-zero at $s = 1$. Hence $L^S(s, \Pi_1 \otimes \omega|_E^{-1}) \cdots L^S(s, \Pi_r \otimes \omega|_E^{-1})$ has a zero of order g at $s = 1$ since $L^S(s, \text{AI}_{E/F}(\omega)| \cdot |^{k_0-1} \otimes \omega|_E^{-1})$ has a pole of order g at $s = 1$. This means that $L^S(s, \text{Sym}^2(\Pi') \otimes \omega|_E^{-1})$ must have a simple pole at $s = 1$, or equivalently, that $L^S(s, \text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1})$ must have a pole of order $g+1$ at $s = 1$.

Since ω factors through the norm map, we have

$$\text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1} = \text{AI}_{E/F}(\text{Sym}^2(\pi) \otimes \omega^{-1}).$$

By Theorem 2.5, if $\text{Sym}^2(\pi) \otimes \omega^{-1}$ is cuspidal, then so is $\text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1}$. Therefore, if $L^S(s, \text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1})$ has a pole at $s = 1$ then the same is true for $L^S(s, \text{Sym}^2(\pi) \otimes \omega^{-1})$. However, by [Tak14, Theorem 5.1], $L^S(s, \text{Sym}^2(\Pi') \otimes \omega|_E^{-1})$ and $L^S(s, \text{Sym}^2(\pi) \otimes \omega^{-1})$ can have at most simple poles at $s = 1$. Therefore, the order of the pole of $L^S(s, \text{AI}_{E/F}(\text{Sym}^2(\pi)) \otimes \omega|_E^{-1})$ at $s = 1$ is at most g . So, $L^S(s, \text{Sym}^2(\Pi') \otimes \omega|_E^{-1})$ cannot have a pole at $s = 1$, which is a contradiction. So, we conclude that Π' satisfies the conditions of Theorem 2.10 if and only if ω is a base change from E . Therefore, Π' descends to a globally generic, cuspidal, automorphic representation Π of $\text{GSpin}_{2g+1}(\mathbf{A}_E)$ if and only if ω factors through the norm map $N_{F/E} : F \rightarrow E$. $\square$

Remark 3.4. Let F/E be a cyclic extension of totally real fields of degree g . Let π be a cuspidal Hilbert automorphic representation, with central character ω , which is not a base change from any sub-extension F'/E of F . Assume that ω is *not* a base change from E . Then, Theorem 3.3 also shows that π lifts to an automorphic representation Π on the even split GSpin_{2g} or one of its quasi-split non-split form GSpin_{2g}^* . The central character of Π' is $\omega|_E$, so we obtain a quadratic character $\mu : E^\times \backslash \mathbf{A}_E^\times \rightarrow \{\pm 1\}$ given by

$$\mu = \omega_{\Pi'} \omega|_E^{-g} = \omega|_E^{1-g}.$$

If μ is trivial, then π lifts to the split GSpin_{2g} . Otherwise, it lifts to the quasi-split non-split GSpin_{2g}^* determined by μ .

Corollary 3.5. *Let F/E be a quadratic extension of totally real fields, and π a cuspidal Hilbert automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$, which is not a base change from E . Let ω be the central character of π , and assume that it is of the form $\omega = N_{F/E} \circ \omega'$ for some character $\omega' : E^\times \backslash \mathbf{A}_E^\times \rightarrow \mathbf{C}^\times$. Then, there exists a holomorphic Hilbert-Siegel automorphic representation Π^h of $\mathrm{GSp}_4(\mathbf{A}_E)$ which lifts π in the sense that its functorial transfer to $\mathrm{GL}_4(\mathbf{A}_E)$ is $\Pi' = \mathrm{AI}_{E/F}(\pi)$.*

Proof. We use the same argument as in the proof of [Pil17, Proposition 15.4]. Let Π be the lift of π obtained from Theorem 3.3, and write

$$\Pi = \Pi_\infty^g \otimes \Pi_f = \left(\bigotimes_{\nu|\infty} \Pi_\nu^g \right) \otimes \Pi_f,$$

where Π_∞^g and Π_f are the infinite and finite parts of Π , respectively. This is a globally generic, cuspidal, automorphic representation of $\mathrm{GSpin}_5(\mathbf{A}_E) \simeq \mathrm{GSp}_4(\mathbf{A}_E)$. Therefore, the global A-packet of Π is generic in the sense of [Art04, The Classification Theorem]. Hence, it is stable and tempered. This means that every local component of this A-packet is an L-packet. For every $\nu \in J_E$, the L-packet of Π_ν^g is $\{\Pi_\nu^g, \Pi_\nu^h\}$, where Π_ν^h is the holomorphic (limit of) discrete series of the same weight as Π_ν^g (see [Sch15]). Therefore, by setting

$$\Pi^h = \Pi_\infty^h \otimes \Pi_f = \left(\bigotimes_{\nu|\infty} \Pi_\nu^h \right) \otimes \Pi_f,$$

we obtain an holomorphic automorphic representation Π^h of $\mathrm{GSp}_4(\mathbf{A}_E)$, which belongs to the same A-packet as Π . $\square$

Remark 3.6. For F real quadratic, Johnson-Leung-Roberts [JLR12] construct a lift to $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$ whose infinite component is holomorphic. This lift, also known as the twisted Yoshida lift has been studied extensively, see [Rob01, Theorem 8.2], [Vig86, Théorème 6.2] and [MT02, Section 7.3]. It is usually constructed using theta correspondence. Corollary 3.5 gives a different way of obtaining this lift.

3.5. The weight of the lift. Let $G = \mathrm{GSpin}_{2g+1}$ and $G' = \mathrm{GL}_{2g}$, and recall that $\widehat{G} = \mathrm{GSp}_{2g}$ and $\widehat{G}' = \mathrm{GL}_{2g}$. (See [AS14] or [MT02] for a nice description of these groups in terms of their root data.) Let (B, T) be a pair of a Borel subgroup B and a maximal torus T in G . Also let (B', T') be the pair consisting of the standard Borel and maximal torus of G' . We have the pairs $(\widehat{B}, \widehat{T})$ and $(\widehat{B}', \widehat{T}')$ for the dual groups $\widehat{G}$ and $\widehat{G}'$, respectively. Let $X^*(\widehat{T})$ and $X^*(\widehat{T}')$ (resp. $X_*(\widehat{T})$ and $X_*(\widehat{T}')$) be the corresponding character (resp. cocharacter) lattices. The natural inclusion of dual tori $\widehat{T} \hookrightarrow \widehat{T}'$ induces a homomorphism $X_*(\widehat{T}) \otimes \mathbf{Q} \hookrightarrow X_*(\widehat{T}') \otimes \mathbf{Q}$ of cocharacter lattices by composition. Let δ and δ' be the half-sum of the positive coroots in $X_*(\widehat{T})$ and $X_*(\widehat{T}')$, respectively.

Lemma 3.7. *Let Π be a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2g+1}(\mathbf{A}_E)$ of weight $\underline{\lambda}$, and Π' its functorial transfer to $\mathrm{GL}_{2g}(\mathbf{A}_E)$. Then, the weight $\underline{\lambda}'$ of Π' is the image of $\underline{\lambda}$ under the inclusion map $X_*(\hat{T}) \otimes \mathbf{Q} \hookrightarrow X_*(\hat{T}') \otimes \mathbf{Q}$.*

Proof. Let ϕ_ν and ϕ'_ν be the respective Langlands parameters of Π_ν and Π'_ν (see Section 2.6), so that we have:

$$\phi_\nu : W_{\mathbf{R}} \rightarrow \mathrm{GSp}_{2g}(\mathbf{C}), \text{ and } \phi'_\nu : W_{\mathbf{R}} \rightarrow \mathrm{GL}_{2g}(\mathbf{C}).$$

These Langlands parameters are determined by their Harish-Chandra parameters which are the orbits of $\eta_\nu := \lambda_\nu + \delta$ and $\eta'_\nu := \lambda'_\nu + \delta'$ under the respective Weyl groups. The fact that $\iota \circ \phi_\nu = \phi'_\nu$ means that the Weyl orbit of η'_ν is the image of the Weyl orbit of η_ν under the map $X_*(\hat{T}) \otimes \mathbf{Q} \hookrightarrow X_*(\hat{T}') \otimes \mathbf{Q}$. We recall that $X_*(\hat{T}) = X^*(T)$ and $X_*(\hat{T}') = X^*(T')$. Since G and G' are of type B_g and A_{2g-1} , respectively, one sees that δ maps to δ' . Hence λ_ν maps to λ'_ν for all $\nu \in J_E$. Therefore, $\underline{\lambda}$ maps to $\underline{\lambda}'$. $\square$

Remark 3.8. In [AS06, §6], there is an explicit recipe which relates the Langlands parameters of Π and Π' . Lemma 3.7 can be seen as a reinterpretation of this recipe in terms of weights. Similarly, for $G = \mathrm{GSp}_4$ and $G' = \mathrm{GL}_4$, and Π a holomorphic cuspidal Hilbert-Siegel automorphic representation on G , there is a recipe in [Sor10, §2], which gives the Langlands parameters of its functorial transfer Π' to G' in terms of the Harish-Chandra parameters of Π using theta correspondence. This recipe can also be recovered from Lemma 3.7 and the (accidental) isomorphism $\hat{G} = \mathrm{GSpin}_5 \simeq G$.

Corollary 3.9. *Let π be a cuspidal Hilbert automorphic representation of weight $\underline{k}$ which satisfies the conditions of Theorem 5.4. Let $\underline{k}_E = (k_{E,\nu})_{\nu \in J_E}$, where $k_{E,\nu} = (k_\tau)_{\tau \in S_\nu}$ and $S_\nu = \{\tau \in J_F : \tau|_E = \nu\}$. Up to labelling and reordering the set of places in S_ν , we can assume that $k_{E,\nu} = (k_{\nu,1}, \dots, k_{\nu,g})$ such that $k_{\nu,1} \geq k_{\nu,2} \geq \dots \geq k_{\nu,g} \geq 2$. Then, the weight $\underline{\lambda}$ of Π is the inverse image, under the inclusion map $X_*(\hat{T}) \otimes \mathbf{Q} \hookrightarrow X_*(\hat{T}') \otimes \mathbf{Q}$, of the weight $\underline{\lambda}' = (\lambda'_\nu)_{\nu \in J_E}$ of Π' given by*

$$\lambda'_\nu = (k_{\nu,1} + m_{\nu,1} - 2g, \dots, k_{\nu,g} + m_{\nu,g} - g - 1, m_{\nu,g} - g + 1, \dots, m_{\nu,1}).$$

Proof. For each prime λ of L_π , the Hodge-Tate weights of $r_\lambda(\Pi')$ are given by

$$\mathrm{HT}_\nu(r_\lambda(\Pi')) = \{m_{\nu,1}, \dots, m_{\nu,g}, k_{\nu,g} + m_{\nu,g} - 1, \dots, k_{\nu,1} + m_{\nu,1} - 1\}.$$

Therefore, the weight of Π' is given by $\underline{\lambda}' = (\lambda'_\nu)_{\nu \in J_E}$ where

$$\lambda'_\nu = (k_{\nu,1} + m_{\nu,1} - 2g, \dots, k_{\nu,g} + m_{\nu,g} - g - 1, m_{\nu,g} - g + 1, \dots, m_{\nu,1}).$$

We conclude by applying Lemma 3.7. $\square$

Remark 3.10. Corollary 3.9 implies that the weight of the lift Π in Theorem 3.3 is regular if and only if the k_τ 's are pairwise distinct, in which case $k_0 \geq 2g$. In particular, automorphic inductions of cuspidal Hilbert automorphic representation of parallel weight 2 are always limits of discrete series.

3.6. Level structure and new vectors theory for the lift. Let $\mathfrak{N}$ be the level of the Hilbert automorphic representation π in Theorem 3.3. Then, the level structure of Π is the paramodular group $K^{\mathrm{par}}(\mathfrak{N}')$ where $\mathfrak{N}' = \mathrm{N}_{F/E}(\mathfrak{N})\mathfrak{D}_{F/E}^g$ and $\mathfrak{D}_{F/E}$ is the different of the relative extension F/E . We refer to [Gro16] for the description of $K^{\mathrm{par}}(\mathfrak{N}')$ as well as the theory of new vectors. Although the discussion in there is concerned with SO_{2g+1} over $\mathbf{Q}$, the results extend to GSpin_{2g+1} over general number fields.

4. Descent of isogeny classes of abelian varieties

Let A be an abelian variety over F , with $L = \text{End}_F(A) \otimes \mathbf{Q}$. Let λ be a prime in L , and denote by $\rho_{A,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GSp}_{2g}(L_\lambda)$ the Galois representation into the λ -adic Tate module of A . If A is of GL_2 -type, in which case $\dim(A) = [L : \mathbf{Q}]$, then $g = 1$, and we have $\rho_{A,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}_2(L_\lambda)$. In that case, we let

$$a_{\mathfrak{p}} := \text{Tr}(\rho_{A,\lambda}(\text{Frob}_{\mathfrak{p}})).$$

Assume that F/E is a Galois extension, and let $\rho_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}(V_\lambda)$ be a λ -adic representation. There is a natural action of $\text{Gal}(F/E)$ on the set of such representations given by

$$(\sigma \cdot \rho_\lambda)(\text{Frob}_{\mathfrak{p}}) := \rho_\lambda(\text{Frob}_{\sigma(\mathfrak{p})}).$$

Theorem 4.1. *Let F/E and L/K be extensions of totally real number fields of degree g . Let B be an abelian variety of dimension $[L : \mathbf{Q}]$ defined over E . Suppose that the followings hold:*

1. F is a cyclic extension of E of degree g ;
2. $\text{End}_E(B) \otimes \mathbf{Q} = K$;
3. $\text{End}_F(A) \otimes \mathbf{Q} = L$, where $A = B \times_E F$, and B does not become of GL_2 -type over any proper sub-extension F'/E of F .

Then L/K is cyclic, and there exists a generator $\tau \in \text{Gal}(L/K)$ such that

$$a_{\sigma(\mathfrak{p})} = \tau(a_{\mathfrak{p}}), \text{ for all primes } \mathfrak{p}.$$

Proof. Let V_ℓ be the underlying $\mathbf{Q}_\ell$ -vector space to the ℓ -adic Tate modules of A and B ; and

$$\rho_{A,\ell} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}(V_\ell), \text{ and } \rho_{B,\ell} : \text{Gal}(\overline{\mathbf{Q}}/E) \rightarrow \text{GL}(V_\ell)$$

the corresponding Galois representations. By Faltings [Fal83, Sätzen 3 and 4], V_ℓ is a semi-simple $\text{Gal}(\overline{\mathbf{Q}}/F)$ -module, and we have

$$L \otimes \mathbf{Q}_\ell = \text{End}_F(A) \otimes \mathbf{Q}_\ell = \text{End}_{\mathbf{Q}_\ell[\text{Gal}(\overline{\mathbf{Q}}/F)]}(V_\ell).$$

Let $\lambda \mid \ell$ be a prime of L , and L_λ the completion of L at λ , and set $V_\lambda = V_\ell \otimes_{L \otimes \mathbf{Q}_\ell} L_\lambda$. Since A is of GL_2 -type, V_λ is a rank two L_λ -module. Let $\rho_{A,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}(V_\lambda)$ be the corresponding λ -adic Tate module. By [Rib04, Proposition 3.3], V_λ is absolute irreducible with $\text{End}_{\mathbf{Q}_\ell[\text{Gal}(\overline{\mathbf{Q}}/F)]}(V_\lambda) = L_\lambda$ and $\text{End}_{L_\lambda}(V_\lambda) = L_\lambda$.

Similarly, let $\lambda' \mid \ell$ be a prime of K , and $K_{\lambda'}$ the completion of K at λ' , and set $V_{\lambda'} = V_\ell \otimes_{K \otimes \mathbf{Q}_\ell} K_{\lambda'}$. Since V_ℓ has rank $2g$ over $K \otimes \mathbf{Q}_\ell$, $V_{\lambda'}$ is a $K_{\lambda'}$ -module of rank $2g$. A similar argument as in [Rib04, Proposition 3.3], shows that $V_{\lambda'}$ is absolute irreducible as a $K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]$ -module with $\text{End}_{\mathbf{Q}_\ell[\text{Gal}(\overline{\mathbf{Q}}/E)]}(V_{\lambda'}) = K_{\lambda'}$ and $\text{End}_{K_{\lambda'}}(V_{\lambda'}) = K_{\lambda'}$. As above, the corresponding λ' -adic Tate modules $\rho_{A,\lambda'} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}(V_{\lambda'})$ and $\rho_{B,\lambda'} : \text{Gal}(\overline{\mathbf{Q}}/E) \rightarrow \text{GL}(V_{\lambda'})$ can also be written as

$$\rho_{A,\lambda'} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GSp}_{2g}(K_{\lambda'}), \text{ and } \rho_{B,\lambda'} : \text{Gal}(\overline{\mathbf{Q}}/E) \rightarrow \text{GSp}_{2g}(K_{\lambda'}).$$

Let λ be a prime of L lying above λ' in K . Since $\rho_{A,\lambda'} = \rho_{B,\lambda'}|_{\text{Gal}(\overline{\mathbf{Q}}/F)}$ is reducible after extension of scalars to L_λ , and $\sigma \cdot \rho_{A,\lambda'} = \rho_{A,\lambda'}$, we have that

$$\rho_{A,\lambda'} \otimes_{K_{\lambda'}} L_\lambda = \rho_{A,\lambda} \oplus \sigma \cdot \rho_{A,\lambda} \oplus \cdots \oplus \sigma^{g-1} \cdot \rho_{A,\lambda}.$$

Since B doesn't become of GL_2 -type over any proper subfield F'/E of F , the summands are pairwise inequivalent. This means that the representation $\text{Ind}_E^F(\rho_{A,\lambda})$ is irreducible, and we have

$$\rho_{B,\lambda'} \otimes_{K_{\lambda'}} L_\lambda = \text{Ind}_E^F(\rho_{A,\lambda}).$$

So $\text{Ind}_E^F(\rho_{A,\lambda})$ is defined over K .

Let $\mathfrak{p}$ be a prime of F , and recall that $a_{\mathfrak{p}} = \text{Tr}(\rho_{A,\lambda}(\text{Frob}_{\mathfrak{p}})) \in L$. By construction, we have

$$(\sigma \cdot \rho_{A,\lambda})(\text{Frob}_{\mathfrak{p}}) = \rho_{A,\lambda}(\text{Frob}_{\sigma(\mathfrak{p})}),$$

and

$$\text{Tr}(\rho_{A,\lambda}(\text{Frob}_{\sigma(\mathfrak{p})})) = a_{\sigma(\mathfrak{p})} \in L.$$

We need to show that the map $(\tau : L \rightarrow L, a_{\mathfrak{p}} \mapsto a_{\sigma(\mathfrak{p})})$ is a homomorphism. To this end, let $S_K := \{\theta : L \hookrightarrow \overline{\mathbf{Q}} \mid \theta|_K = 1\}$. Then, since B becomes of GL_2 -type over F , we have

$$\text{charpoly}(\rho_{A,\lambda'}(\text{Frob}_{\mathfrak{p}})) = \prod_{\theta \in S_K} (x^2 - \theta(a_{\mathfrak{p}})x + N\mathfrak{p}) = \prod_{i=0}^{g-1} (x^2 - a_{\sigma^i(\mathfrak{p})}x + N\mathfrak{p}).$$

By Tate's conjecture on endomorphism rings of abelian varieties, the extension L/K is generated by the set $\{a_{\mathfrak{p}} : \mathfrak{p} \text{ prime}\}$ (see [Rib04, Proposition 3.5]). So, the map τ is indeed a homomorphism. Furthermore, since $\text{End}_E(B) \otimes \mathbf{Q} = K$, and B doesn't become of GL_2 -type over any proper subfield F'/E of F , there exists a totally split prime $\mathfrak{p} \subset \mathcal{O}_F$ such that $a_{\mathfrak{p}}$, and hence $a_{\sigma(\mathfrak{p})}$, is a primitive element. So, the map $(\phi : \text{Gal}(F/E) \rightarrow \text{Aut}(L), \sigma \mapsto \tau)$ is injective. By letting $H = \text{im}(\phi)$, we see that $K = L^H$ and $\text{Gal}(F/E) \simeq \text{Gal}(L/K)$. This concludes the proof of the theorem. $\square$

We conclude this section with the following converse statement to Theorem 4.1.

Theorem 4.2. *Let F/E and L/K be cyclic extensions of totally real number fields of degree g . Let A be an abelian variety of dimension $[L : \mathbf{Q}]$ defined over F . Suppose that the followings hold:*

1. $\text{End}_F(A) \otimes \mathbf{Q} = L$;
2. *There exists a generator $\tau \in \text{Gal}(L/K)$ such that*

$$a_{\sigma(\mathfrak{p})} = \tau(a_{\mathfrak{p}}), \text{ for all primes } \mathfrak{p}.$$

Then, there exists an abelian variety B defined over E , with $\text{End}_E(B) \otimes \mathbf{Q} = K$, such that $A \sim B \times_E F$.

Proof. Let $X = \text{Res}_{F/E}(A)$, ℓ a rational prime, and $V_{\ell}(X)$ the ℓ -adic Tate module of X . By [Fal83, Satz 4] and [Mil72, Section 1], we have

$$\begin{aligned} \text{End}_E(X) \otimes \mathbf{Q}_{\ell} &= \text{End}_{\mathbf{Q}_{\ell}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(V_{\ell}(X)) = \text{End}_{\mathbf{Q}_{\ell}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(\text{Ind}_E^F(V_{\ell}(A))) \\ &= \text{End}_{\mathbf{Q}_{\ell}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(\text{Ind}_E^F(\rho_{A,\ell})) = \prod_{\lambda'|\ell} \text{End}_{K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(\text{Ind}_E^F(\rho_{A,\lambda'})) \\ &= \prod_{\lambda'|\ell} \text{End}_{K_{\lambda'}}(\text{Ind}_E^F(\rho_{A,\lambda'})). \end{aligned}$$

Let λ', λ be primes in K and L , respectively, with $\lambda \mid \lambda'$. Then, we have

$$\rho_{A,\lambda'} \otimes_{K_{\lambda'}} L_{\lambda} = \rho_{A,\lambda} \oplus \sigma \cdot \rho_{A,\lambda} \oplus \cdots \oplus \sigma^{g-1} \cdot \rho_{A,\lambda}.$$

It follows that, we have

$$\text{End}_{L_{\lambda}}(\text{Ind}_E^F(\rho_{A,\lambda'} \otimes_{K_{\lambda'}} L_{\lambda})) = \text{End}_{L_{\lambda}} \left(\bigoplus_{i=0}^{g-1} \text{Ind}_E^F(\sigma^i \cdot \rho_{A,\lambda}) \right).$$

By construction, we have that

$$W_{\lambda} := \text{Ind}_E^F(\rho_{A,\lambda}) = \text{Ind}_E^F(\sigma \cdot \rho_{A,\lambda}) = \cdots = \text{Ind}_E^F(\sigma^{g-1} \cdot \rho_{A,\lambda}).$$

By Condition (2), W_{λ} is an irreducible $L_{\lambda}[\text{Gal}(\overline{\mathbf{Q}}/E)]$ -module, and the characteristic polynomial of every Frobenius element acting on it lies in $K[x] \subset K_{\lambda}[x]$. Therefore, there exists an irreducible $K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]$ -module $W_{\lambda'}$ such that $W_{\lambda} = W_{\lambda'} \otimes_{K_{\lambda'}} L_{\lambda}$, and $W_{\lambda'}$ satisfies one of the following two possibilities:

- (i) $\text{End}_{K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(W_{\lambda'}) = K_{\lambda'}$; or
- (ii) $\text{End}_{K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(W_{\lambda'}) = D'$ for the unique division algebra D' of dimension g^2 over $K_{\lambda'}$.

In case (ii), this would mean that there is a simple factor B of X , and a totally definite division algebra D of dimension g^2 over K such that $\text{End}_E(B) \otimes \mathbf{Q} \simeq D$, and $\text{End}_{K_{\lambda'}[\text{Gal}(\overline{\mathbf{Q}}/E)]}(W_{\lambda'}) = D_{\lambda'}$, where $D' = D_{\lambda'}$ is the completion of D at λ' . Since $\text{End}_F(X) \otimes \mathbf{Q} = M_g(L)$, such a division algebra would be split by L which is totally real. But this is impossible, so case (ii) cannot happen. Hence, W_{λ} descends to $K_{\lambda'}$, and we have

$$\text{End}_{L_{\lambda}}(\text{Ind}_E^F(\rho_{A,\lambda'} \otimes_{K_{\lambda'}} L_{\lambda})) = \text{End}_{L_{\lambda}}((W_{\lambda'} \otimes_{K_{\lambda'}} L_{\lambda})^g) = M_g(L_{\lambda}).$$

From this, we see that

$$\text{End}_{K_{\lambda'}}(\text{Ind}_E^F(\rho_{A,\lambda'})) = \text{End}_{K_{\lambda'}}(W_{\lambda'}^g) = M_g(K_{\lambda'}).$$

It follows that

$$\text{End}_E(X) \otimes \mathbf{Q}_{\ell} = \prod_{\lambda'|\ell} M_g(K_{\lambda'}).$$

Since this is true for every prime ℓ , we see that $\text{End}_E(X) \otimes \mathbf{Q} = M_g(K)$.

Let B be a simple factor of X . Then, one sees that $B \times_E F$ is necessarily simple, and that $B \times_E F \sim A$. By construction, we have $\text{End}_E(B) \otimes \mathbf{Q} = K$. $\square$

Remark 4.3. We expect Theorems 4.1 and 4.2 to generalise to arbitrary Galois extensions F/E . They show that, when B/E is an abelian variety such that $A = B \times_E F$ acquires extra endomorphisms, then there is a strong interaction between the group structure of $\text{Gal}(F/E)$ and that of $\text{Aut}_K(L)$ where $K = \text{End}_E(B) \otimes \mathbf{Q}$, and $L = \text{End}_F(A) \otimes \mathbf{Q}$. Theorem 5.4 below can be seen as a natural counterpart on the automorphic side which indicates the importance of this relation from a functorial point of view. The E -varieties, which have been extensively studied in [Rib04, Pyl04, Gui10], seem to represent an extreme case where this relation is simply absent as the action of the group $\text{Gal}(F/E)$ on L is trivial. (We refer to [Dem16] for more details on this.)

5. The Gross conjecture

The goal of this section is to provide some evidence for the following statement which is a slight generalisation of Conjecture 1.1 in the introduction. (Our statement should also be compared with [BK14, Conjecture 1.4]).

Conjecture 5.1 (Gross-Langlands). *Let E be a number field, and B be an abelian variety defined over E such that $\text{End}_E(B) \otimes \mathbf{Q} = K$ is a totally real field of degree d with $\dim(B) = dg$, for some $g \in \mathbf{Z}_{\geq 1}$. Assume that B is not of GL_2 -type, i.e., that $g \geq 2$ when $\dim(B) \geq 2$. Let $\text{cond}(B) = \mathfrak{N}^d$ be the conductor of B . Then there exists a globally generic cuspidal automorphic representation Π on $\text{GSpin}_{2g+1}(\mathbf{A}_E)$ of weight 2 and paramodular level structure $\mathfrak{N}$, with field of rationality K , such that*

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s),$$

where $[\Pi]$ denotes the Hecke orbit of Π .

Conjecture 5.1 was probably known to experts in some less imprecise form. To the best of our knowledge, however, Gross [Gro16] was the first to give a refinement which also predicts the level of Π in terms of the conductor of B . In [Gro16], after fixing an isomorphism $\iota : \overline{\mathbf{Q}}_{\ell} \simeq \mathbf{C}$, the Weil-Deligne representations arising from the ℓ -adic Tate module attached to B are normalised so that they take values in $\text{Sp}_{2g}(\mathbf{C})$. So, the conjectured automorphic representation Π lives on $\text{SO}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$.

Using the local Langlands correspondence for SO_{2g+1} (see [JS04]), Gross provides a precise recipe for the local component Π_v for the representation

$$\Pi = \bigotimes_v' \Pi_v.$$

The difficult is in proving that Π is automorphic and cuspidal.

Remark 5.2. Although the infinite components of the automorphic representation Π are limits of discrete series, the theory of newforms developed in [Gro16, §§6-8] allows for a more classical definition of the field of rationality L_Π of Π , and it should be equal to K . So, we can define the Hecke orbit of Π under that assumption. But we expect that one can also give a definition of L_Π that is more intrinsic.

Remark 5.3. The restriction to abelian varieties not of GL_2 -type in our formulation of Conjecture 5.1 is not essentially. We refer to the discussion following [Gro16, Proposition 6] to see how the conjecture can be restated in a way that encompasses abelian varieties of GL_2 -type.

5.1. Compatibility with Galois descent. In this subsection, we show that Conjecture 5.1 is compatible with lifts of Hilbert automorphic representations, which correspond to descent of isogeny classes on the geometric side.

Theorem 5.4. *Let F/E be a cyclic extension of totally real fields of degree g , and $\mathrm{Gal}(F/E) = \langle \sigma \rangle$. Let $\mathfrak{N}$ be an integral ideal such that $\mathfrak{N}^\sigma = \mathfrak{N}$. Let f be a Hilbert newform of parallel weight 2, level $\mathfrak{N}$ and trivial central character, with field of coefficients L , which is not a base change from any proper sub-extension F'/E of F . Assume that the Hecke orbit of f is preserved by $\mathrm{Gal}(F/E)$, and that there is an abelian variety A attached to f by the Eichler-Shimura construction. Then, the followings hold:*

- (i) *There exist a subfield K of L such that the extension L/K is cyclic Galois of degree g ; and a generator $\tau \in \mathrm{Gal}(L/K)$ such that, for all primes $\mathfrak{p}$,*

$$a_{\sigma(\mathfrak{p})}(f) = \tau(a_{\mathfrak{p}}(f)).$$

- (ii) *The isogeny class of A descends to E , i.e. there exists an abelian variety B defined over E such that $\mathrm{End}_E(B) \otimes \mathbf{Q} \simeq K$ and $A \sim B \times_E F$;*
- (iii) *The automorphic representation π on $\mathrm{GL}_2(\mathbf{A}_F)$ attached to f lifts to a globally generic cuspidal automorphic representation Π on $\mathrm{GSpin}_{2g+1}(\mathbf{A}_E)$, with field of rationality K , such that*

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

Proof. (i) Let π be the automorphic representation attached to f , and let $\sigma\pi = \pi \circ \sigma$. Then for $\mathfrak{p}$ prime, $a_{\sigma(\mathfrak{p})}(f)$ is the Hecke eigenvalue at $\mathfrak{p}$ of the newform $\sigma f \in \sigma\pi$. The newforms f and σf are in the same Hecke orbit if and only if $\sigma\pi$ and π are in the same Hecke orbit. This is equivalent to saying that there exists $\tau \in \mathrm{Gal}(L/K)$ non-trivial such that $\sigma\pi = \pi^\tau$, which translates into the stated identity. Furthermore, since f is not a base change from any intermediate field, τ must be a generator.

(ii) The identity (i) is the descent condition on the isogeny class of A given in Theorem 4.2. So, the isogeny class of A descends to that of an abelian variety B over E .

(iii) Since π is not a base change from any proper sub-extension F'/E of F , we deduce from Theorem 3.3 that π admits a lift Π to $\mathrm{GSpin}_{2g+1}(\mathbf{A}_E)$. By Lemma 2.7, the field of rationality of Π is K . By construction, we have

$$L(A, s) = \prod_{\pi' \in [\pi]} L(\pi', s).$$

So, it follows from the functorial properties of Theorem 2.5, and the local-global compatibility result stated in [AS14, Proposition 5.1], that

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

□

Remark 5.5. In Theorem 5.4, the condition that π is not a base change from any proper sub-extension F'/E of F can be relaxed as follows. Let $H = \text{Stab}_G([\pi])$, where $G = \text{Gal}(F/E)$, and $F' = F^H$, $g' = [F : F']$. Then, as before, we have an injection $\phi : H \hookrightarrow \text{Aut}(L)$. Letting $K' = L^{\phi(H)}$, Theorem 5.4 implies that π lifts to an automorphic representation Π on $\text{GSpin}_{2g'+1}(\mathbf{A}_{F'})$, and that A descends to an abelian variety B over F' such that $\text{End}_{F'}(B) \otimes \mathbf{Q} = K'$ and

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

5.2. Automorphy of abelian varieties acquiring extra endomorphisms. We now prove a converse statement to Theorem 5.4. We refer to [BDJ10] for the Serre conjecture for totally real number fields.

Theorem 5.6. *Let F/E and L/K be extensions of totally real number fields of degree g . Let B be an abelian variety of dimension $[L : \mathbf{Q}]$ defined over E . Suppose that the followings hold:*

1. F is a cyclic extension of E ;
2. $\text{End}_E(B) \otimes \mathbf{Q} = K$;
3. $\text{End}_F(A) \otimes \mathbf{Q} = L$, where $A = B \times_E F$, but B doesn't become of GL_2 -type over any proper subfield F'/E of F ;
4. There is a rational prime ℓ , and a prime $\lambda' \mid \ell$ in K which is totally ramified in L such that $\bar{\rho}_{A, \lambda}$ is surjective for the unique prime $\lambda \mid \lambda'$ in L . If $\ell = 2$, then further assume that $F(A[\lambda])$ is non-solvable.

If the Serre conjecture for totally real number fields is true for E , then B is automorphic. More specifically, there exists a cuspidal Hilbert automorphic representation π on $\text{GL}_2(\mathbf{A}_F)$, which lifts to a globally generic cuspidal automorphic representation Π on $\text{GSpin}_{2g+1}(\mathbf{A}_E)$ such that

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

Proof. Keeping the same notations as in Theorem 4.1, we have the commutative diagram

$$\begin{array}{ccc} \text{Gal}(\overline{\mathbf{Q}}/F) & \xrightarrow{\rho_{A, \lambda'}} & \text{GSp}_{2g}(K_{\lambda'}) \\ & \searrow \mathbf{1} \times \rho_{A, \lambda} & \uparrow \\ & & \text{GL}_2(L_{\lambda})^g \end{array}$$

where $\mathbf{1} \times \rho_{A, \lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}_2(L_{\lambda})^g$ is given by

$$(\mathbf{1} \times \rho_{A, \lambda})(\text{Frob}_{\mathfrak{p}}) := (\rho_{A, \lambda}(\text{Frob}_{\mathfrak{p}}), \dots, \rho_{A, \lambda}(\text{Frob}_{\sigma^{g-1}(\mathfrak{p})})).$$

We recall that, by Theorem 4.1, there exists a generator $\tau \in \text{Gal}(L/K)$ such that

$$a_{\sigma(\mathfrak{p})} = \tau(a_{\mathfrak{p}}),$$

for all primes $\mathfrak{p} \subset \mathcal{O}_F$. Let $\bar{\rho}_{A,\lambda}$ be the reduction of $\rho_{A,\lambda}$ modulo λ , and $\bar{\rho}_{A,\lambda}^{ss}$ its semi-simplification. Since λ is totally ramified in L , we see that

$$a_{\sigma(\mathfrak{p})} = a_{\mathfrak{p}} \pmod{\lambda},$$

for all primes $\mathfrak{p} \subset \mathcal{O}_F$. So, $\bar{\rho}_{A,\lambda}^{ss} : \text{Gal}(\bar{\mathbf{Q}}/F) \rightarrow \text{GL}_2(\mathbf{F}_\lambda)$ is a base change from E . By assumption $\bar{\rho}_{A,\lambda}$ is surjective. So it is absolutely irreducible, hence $\bar{\rho}_{A,\lambda} = \bar{\rho}_{A,\lambda}^{ss}$. Since we assume that the Serre conjecture for totally real number fields is true for E (see [BDJ10]), $\bar{\rho}_{A,\lambda}$ is modular. If $\ell \geq 3$, then $\rho_{A,\lambda}$ is modular by [KT16, Theorem 1.1]. If $\ell = 2$, then $\rho_{A,\lambda}$ is also modular by [Tho17, Theorem 6.1]. So, in either case, there exists a Hilbert automorphic representation π associated to A . The abelian varieties A and B , and the automorphic representation π satisfy the conditions of Theorem 5.4. So, there exists Π on $\text{GSpin}_{2g+1}(\mathbf{A}_E)$ such that

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

□

Remark 5.7. First, we note that the assumption that the Serre conjecture for totally real fields holds for E in Theorem 5.6 is more than what we need. It suffices that $\bar{\rho}_{A,\lambda}$ be modular.

Second, the conditions in Theorem 5.6 are clearly not optimal. Indeed, we assume that the residual representation $\bar{\rho}_{A,\lambda}$ is surjective. But, [KT16, Theorem 1.1] only requires the restriction $\bar{\rho}_{A,\lambda}|_{G_{F(\zeta_\ell)}}$ to be irreducible. One could also allow $\bar{\rho}_{A,\lambda}$ to be reducible and use [Tho15, Theorem 7.1] or [SW99, Theorem A]. In other words, Theorem 5.6 can be easily extended to a wider class of abelian varieties by combining some of the most recent modularity lifting results, see for example [Gee06, Gee09, Kis09, KT16, SW99, Tho15, Tho17].

Remark 5.8. In light of recent works of Pilloni [Pil17] and Calegari-Geraghty [CG16], one might also be able to extend our approach to other groups such as GSp_4 to produce further evidence to Conjecture 5.1 via functoriality.

Corollary 5.9. *Let $F/\mathbf{Q}$ and L/K be extensions of totally real number fields of degree g . Let B be an abelian variety of dimension $[L : \mathbf{Q}]$ defined over $\mathbf{Q}$. Suppose that the followings hold:*

1. F is cyclic;
2. $\text{End}_{\mathbf{Q}}(B) \otimes \mathbf{Q} = K$;
3. $\text{End}_F(A) \otimes \mathbf{Q} = L$, where $A = B \times_{\mathbf{Q}} F$, but B doesn't become of GL_2 -type over any proper subfield F'/E of F ;
4. There is a rational prime ℓ , and a prime $\lambda' \mid \ell$ in K which is totally ramified in L such that $\bar{\rho}_{A,\lambda}$ is surjective for the unique prime $\lambda \mid \lambda'$ in L . If $\ell = 2$, then further assume that $F(A[\lambda])$ is non-solvable.

Then B is automorphic, i.e., there exists a cuspidal Hilbert automorphic representation π on $\text{GL}_2(\mathbf{A}_F)$, which lifts to a globally generic cuspidal automorphic representation Π on $\text{GSpin}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$, such that

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

In particular, we have the following corollary for abelian varieties of prime dimension.

Corollary 5.10. *Let B be an abelian variety defined over $\mathbf{Q}$ such that $\text{End}_{\mathbf{Q}}(B) \otimes \mathbf{Q} = K$ is a totally real field. Suppose that there exist totally real extensions $F/\mathbf{Q}$ and L/K of prime degree g such that*

1. F is cyclic;
2. $[L : \mathbf{Q}] = \dim(B)$ and $\text{End}_F(A) \otimes \mathbf{Q} = L$, where $A = B \times_{\mathbf{Q}} F$;
3. There is a rational prime ℓ , and a prime $\lambda' \mid \ell$ in K which is ramified in L such that $\bar{\rho}_{A, \lambda}$ is surjective for the unique prime $\lambda \mid \lambda'$ in L . If $\ell = 2$, then further assume that $F(A[\lambda])$ is non-solvable.

Then, B is automorphic, i.e. there exists a cuspidal Hilbert automorphic representation π on $\text{GL}_2(\mathbf{A}_F)$, which lifts to a globally generic cuspidal automorphic representation Π on $\text{GSpin}_{2g+1}(\mathbf{A}_{\mathbf{Q}})$, such that

$$L(B, s) = \prod_{\Pi' \in [\Pi]} L(\Pi', s).$$

Proof of Corollary 5.9. Since $E = \mathbf{Q}$, the Serre conjecture is true [Ser87, KW09]. So we conclude by applying Theorem 5.6. $\square$

Remark 5.11. There are two separate aspects to Corollary 5.9 (and in fact Theorem 5.6). The first one is concerned with potential modularity of abelian varieties which acquire extra endomorphisms after base change, while the second is about functoriality. If one is only interested in potential modularity, then the assumption that the extension $F/\mathbf{Q}$ be cyclic is not strictly necessary. Indeed, there are results of Dieulefait on modularity of residual Galois representations that are non-solvable base change (see for example [Die12, §6], and also [Hid09]).

One easily deduces the following uniformisation result from Corollary 5.9.

Corollary 5.12. *Keeping the hypotheses of Corollary 5.9, assume that F has narrow class number one. Let $\text{cond}(A) = \mathfrak{N}^g$ be the conductor of the abelian variety A . If g is even, assume that there exists a prime $v \mid \mathfrak{N}$ such that π_v , the local component of the automorphic representation π at v , is a discrete series representation. Let D be the unique quaternion algebra D over F ramified exactly at*

- $g - 1$ of the real places if g is odd; and
- v and $g - 1$ of the real places of F if g is even.

Let $\mathcal{O}$ be an Eichler order of level $\mathfrak{N}$, and $X_0^D(\mathfrak{N})$ the Shimura curve attached to $\mathcal{O}$. Then, there is a morphism $\phi : \text{Jac}(X_0^D(\mathfrak{N})) \rightarrow B$ defined over $\mathbf{Q}$.

Proof. The conditions of Corollary 5.12 ensure that there is a non-constant morphism $\phi : \text{Jac}(X_0^D(\mathfrak{N})) \rightarrow A$. We want to show that ϕ descends to $\mathbf{Q}$. First, we note that since $\mathfrak{N} = \mathfrak{N}^\sigma$, and F has narrow class number one, the field of moduli of $X_0^D(\mathfrak{N})$ is $\mathbf{Q}$ (see [DN67]). Let $\text{Jac}(X_0^D(\mathfrak{N}))^{\text{new}}$ be the new part of $\text{Jac}(X_0^D(\mathfrak{N}))$, and consider the decomposition into isogeny classes over F :

$$\text{Jac}(X_0^D(\mathfrak{N}))^{\text{new}} \sim \prod_{[f]} A_f \sim \prod_{[f] \in \mathcal{E}} \left(\prod_{[h] \in G \cdot [f]} A_h \right) \sim \prod_{[f] \in \mathcal{E}} B_f,$$

where $S_2(\mathfrak{N})^{\text{new}}$ is the new subspace of cusp forms of level $\mathfrak{N}$ and weight 2 over F , and $\mathcal{E}$ a set of representatives of the Hecke orbits of the eigenforms in $S_2(\mathfrak{N})^{\text{new}}$ under the action of $G = \text{Gal}(F/\mathbf{Q})$. By Theorem 5.4 and Remark 5.5, the abelian variety B_f attached to the Galois orbit $G \cdot [f]$ descends to $\mathbf{Q}$. So, the decomposition of $\text{Jac}(X_0^D(\mathfrak{N}))^{\text{new}}$ into Galois orbits of its simple factors descends to $\mathbf{Q}$. The Hecke orbit of the newform attached to A is unique in its Galois orbit. So, in particular, the morphism $\phi : \text{Jac}(X_0^D(\mathfrak{N})) \rightarrow A$ descends to $\mathbf{Q}$. $\square$

Remark 5.13. Although the morphism $\phi : \text{Jac}(X_0^D(\mathfrak{N})) \rightarrow B$ is defined over $\mathbf{Q}$, the abelian variety B itself cannot be the quotient of the Jacobian of a Shimura curve arising from a quaternion algebra D defined over $\mathbf{Q}$ as B is not of GL_2 -type.

TABLE 1. Hecke eigenvalues for the newforms of level $\mathfrak{q}^2$ over the cubic subfield of $\mathbf{Q}(\zeta_7)^+$ with coefficient field $\mathbf{Q}(\zeta_7)^+$.

Np	p	$a_p(g)$	$a_p(g')$
7	$b^2 + 2b - 1$	2	2
8	2	3	-3
13	$-2b^2 - b + 2$	$3b + 2$	$3b + 2$
13	$b^2 + 2b - 2$	$3b^2 - 4$	$3b^2 - 4$
13	$b^2 - b - 3$	$-3b^2 - 3b + 5$	$-3b^2 - 3b + 5$
27	3	0	0
29	$b^2 - 2b - 3$	$3b^2 - 3b - 9$	$-3b^2 + 3b + 9$
29	$-b^2 - 3b + 1$	$-6b^2 - 3b + 6$	$6b^2 + 3b - 6$
29	$-2b^2 + b + 4$	$3b^2 + 6b - 6$	$-3b^2 - 6b + 6$
41	$b^2 - 2b - 4$	$-3b^2 - 6b$	$3b^2 + 6b$
41	$2b^2 + 3b - 4$	$6b^2 + 3b - 12$	$-6b^2 - 3b + 12$
41	$-3b^2 - b + 3$	$-3b^2 + 3b + 3$	$3b^2 - 3b - 3$

6. Examples

In this section, we illustrate Theorems 5.4 and 5.6 with some concrete examples.

6.1. An example over the real cubic field of smallest discriminant. Let $F = \mathbf{Q}(\zeta_7)^+ = \mathbf{Q}(b)$, where $b^3 + b^2 - 2b - 1 = 0$, be the totally real cubic subfield of the cyclotomic field generated by the 7-th root of unity. This is the totally real cubic field of smallest discriminant. It is Galois, we let σ be a generator of $\text{Gal}(F/\mathbf{Q})$. Let D be the quaternion algebra over F ramified at exactly 2 of the real places. Let $\mathcal{O}_D$ be a maximal order in D , and $\mathcal{O} \subset \mathcal{O}_D$ an Eichler order of level $\mathfrak{q}^2$ where $\mathfrak{q}$ is the inert prime above 3 in F . Let $X_0^D(\mathfrak{q}^2)$ be the Shimura curve attached to $\mathcal{O}$; it has genus 10. Let $\text{Jac}(X_0^D(\mathfrak{q}^2))$ be its Jacobian. The new part $\text{Jac}(X_0^D(\mathfrak{q}^2))^{\text{new}}$ is an 8-dimensional abelian variety.

Proposition 6.1. *Let $C/\mathbf{Q}$ the hyperelliptic curve given by*

$$C : y^2 = -7x^7 + 98x^6 - 1470x^5 + 4606x^4 - 1715x^3 - 4116x^2 - 4459x + 4802,$$

with discriminant $\text{disc}(C) = -2^{36} \cdot 3^{24} \cdot 7^{28}$. Let $B = \text{Jac}(C)$ be its Jacobian, and $A = B \times_{\mathbf{Q}} F$. Then, we have the followings:

- (i) $\text{End}_{\mathbf{Q}}(B) = \mathbf{Z}$, and $\text{End}_F(A) \otimes \mathbf{Q} \simeq F$;
- (ii) *There exists an automorphic representation Π on $\text{GSpin}_7(\mathbf{A}_{\mathbf{Q}})$ such that*

$$L(B, s) = L(\Pi, s);$$

- (iii) *There is a non-constant morphism $\phi : \text{Jac}(X_0^D(\mathfrak{q}^2)) \rightarrow B$ defined over $\mathbf{Q}$.*

Proof. We use the algorithm described in [CMSV17] to show that A has RM by a suborder of $\mathcal{O}_F$ of index 3. A **Magma** calculation shows that the prime-to-2-part of the conductor of A is $\mathfrak{q}^6$.

Let λ be the unique prime above 7. By computing the orders of Frobenii for the first few primes in F , one sees that $\bar{\rho}_{A, \lambda} : \text{Gal}(\bar{\mathbf{Q}}/F) \rightarrow \text{GL}_2(\mathbf{F}_7)$ is surjective. If B were of GL_2 -type, then A would be modular as a base change. Otherwise, we can apply Corollary 5.10. So, in either cases, A is modular.

Now, we will show that B cannot be of GL_2 -type. To this end, let C' be the quadratic twist of C by $d = 8$, and let $B' = \text{Jac}(C')$ and $A' = B' \times_{\mathbf{Q}} F$. Then A' is also modular. It is much easier to compute the conductor of A' , and we obtain that $\text{cond}(A') = (2^{18} \cdot 3^6) = \mathfrak{N}^3$. Since A' is the quadratic twist of A by $d = 8$, we get that $\text{cond}(A) = \mathfrak{q}^6 = \mathfrak{N}^3$. So A has good reduction at 2.

If B were of GL_2 -type, then the Hilbert newform corresponding to A would be a base change. The new subspace of Hilbert cusp forms of level $\mathfrak{N}$ and weight 2 is 8-dimensional (as explained above). It has two Hecke constituents of dimension 3. In Table 1, we have listed their Hecke eigenvalues for the first few primes. One can see that they are inner twist of each other, and neither of them is a base change. Therefore B cannot be of GL_2 -type. Since F is cubic, this means that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$. The same shows that $\mathrm{End}_{\mathbf{Q}}(B') = \mathbf{Z}$. We conclude the proof using Corollaries 5.10 and 5.12. $\square$

Remark 6.2. The argument we used to prove that B is not of GL_2 -type cannot be applied directly to B' as the dimension of the new subspace of cusp forms of level $\mathfrak{N}'$ and weight 2 is 1884288.

6.2. An example over the real cubic subfield of $\mathbf{Q}(\zeta_{43})^+$. Our second example is over the cubic subfield $F = \mathbf{Q}(b)$ of $\mathbf{Q}(\zeta_{43})^+$, here $b^3 + b^2 - 14b + 8 = 0$. We consider the quaternion algebra D ramified at exactly two of the real places of F . We let $\mathcal{O}_D$ be a maximal order, and $X_0^D(1)$ the Shimura curve of level (1). This is a curve of genus 6 defined over $\mathbf{Q}$. Under the Hecke action, we have the decomposition

$$\mathrm{Jac}(X_0^D(1)) \sim E \times A \times A',$$

where E is the base change of the elliptic curve defined over $\mathbf{Q}$ by

$$y^2 + xy + y = x^3 - 39x - 27,$$

with j -invariant $j(E) = 1849 = 43^2$. The abelian surface A has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. It is the Jacobian of the genus 2 curve $C : y^2 + Q(x)y = P(x)$ given by

$$\begin{aligned} P(x) := & (9b^2 - 33b + 20)x^6 + (24b^2 - 88b + 54)x^5 + (42b^2 - 154b + 94)x^4 \\ & + (42b^2 - 154b + 94)x^3 + (21b^2 - 77b + 47)x^2 + (6b^2 - 22b + 13)x \\ & - 15b^2 + 55b - 34; \\ Q(x) := & x^3 + x + 1. \end{aligned}$$

We have $\mathrm{disc}(C) = u(-b^2 + 5b - 3)^{22}$, where $-b^2 + 5b - 3$ generates the unique prime ideal above 43 and $u \in \mathcal{O}_F^\times$, and $\mathrm{cond}(A) = (1)$. In fact, A is a quadratic twist of the base change of the Jacobian of the curve C' given by

$$C' : y^2 - (x^3 + x + 1)y = -x^6 - 2x^5 - 4x^4 - 4x^3 - 2x^2 - x + 1.$$

Proposition 6.3. *There exists an abelian threefold B' defined over $\mathbf{Q}$ such that $\mathrm{End}_{\mathbf{Q}}(B') = \mathbf{Z}$, and $A' = B' \times_{\mathbf{Q}} F$ becomes of GL_2 -type, with RM by the cubic subfield $K = \mathbf{Q}(e)$ of $\mathbf{Q}(\zeta_{19})^+$ given by $e^3 - e^2 - 6e + 7 = 0$. So, the decomposition*

$$\mathrm{Jac}(X_0^D(1)) \sim E \times A \times A'$$

descends to $\mathbf{Q}$.

Proof. The Hecke eigenvalues of the newform corresponding to A are listed in Table 2. One see that it cannot be a base change. Since it belongs to the unique Hecke constituent of dimension 3, it must satisfy the conditions of Theorem 5.4. It follows that its isogeny class descend to an abelian threefold B' with $\mathrm{End}_{\mathbf{Q}}(B') = \mathbf{Z}$. Since E and A also descend to $\mathbf{Q}$, we conclude from Corollary 5.12 that the decomposition of $\mathrm{Jac}(X_0^D(1))$ must descend as well. $\square$

Remark 6.4. From the Hecke eigenvalues listed in Table 2, it is not hard to check that the mod 2 Galois representation attached to A' surjects onto $\mathrm{PSL}_2(\mathbf{F}_8)$. Since this is not a subgroup of S_8 , we see that B' cannot be the Jacobian of a hyperelliptic curve (assuming that it is principally polarised).

TABLE 2. Euler factors for the threefold A' of level (1) over the cubic subfield of $\mathbf{Q}(\zeta_{43})^+$ with RM by the cubic subfield of $\mathbf{Q}(\zeta_{19})^+$.

$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(g)$	Euler factor
2	$\frac{1}{2}(b^2 - 3b)$	$-e^2 - e + 5$	$x^6 - x^5 + 3x^3 - 4x + 8$
2	$b - 3$	$e^2 - 4$	$x^6 - x^5 + 3x^3 - 4x + 8$
2	$\frac{1}{2}(-b^2 - 3b + 2)$	e	$x^6 - x^5 + 3x^3 - 4x + 8$
11	$-2b + 1$	$e^2 + e - 9$	$x^6 + 13x^5 + 83x^4 + 335x^3 + 913x^2 + 1573x + 1331$
11	$-b^2 - b + 11$	$-e - 4$	$x^6 + 13x^5 + 83x^4 + 335x^3 + 913x^2 + 1573x + 1331$
11	$b^2 + 3b - 7$	$-e^2$	$x^6 + 13x^5 + 83x^4 + 335x^3 + 913x^2 + 1573x + 1331$
27	3	0	$x^6 + 81x^4 + 2187x^2 + 19683$
41	$-b^2 - b + 15$	$-2e^2 - 2e + 9$	$x^6 + x^5 + 98x^4 + 113x^3 + 4018x^2 + 1681x + 68921$
41	$b^2 + 3b - 3$	$2e - 1$	$x^6 + x^5 + 98x^4 + 113x^3 + 4018x^2 + 1681x + 68921$
41	$2b - 5$	$2e^2 - 9$	$x^6 + x^5 + 98x^4 + 113x^3 + 4018x^2 + 1681x + 68921$

7. Application to the Gross conjecture for non-solvable Galois number fields

In [Dem09] (see also [Ser09]), the second author proved the following theorem:

Theorem 7.1. *There exists a Galois number field M ramified at 2 only, with Galois group $\text{Gal}(M/\mathbf{Q}) = \text{SL}_2(\mathbf{F}_{2^8})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$.*

Theorem 7.1 settled a conjecture of Gross on the existence of non-solvable Galois number fields which ramify at one prime only for $p = 2$. The number field M was constructed using Galois representations arising from Hilbert modular forms. In this section, we will explain the very interesting connection between Theorem 7.1 and Conjecture 5.1.

Let $F = \mathbf{Q}(\zeta_{32})^+ = \mathbf{Q}(\beta)$, where $\beta := \zeta_{32} + \zeta_{32}^{-1}$ has minimal polynomial $x^8 - 8x^6 + 20x^4 - 16x^2 + 2$. The space $S_2(1)$, of Hilbert newforms of level (1) and weight 2 over F , has dimension 57. The Hecke constituents have dimensions 1, 2, 2, 4, 16 and 32. Let f be one of the newforms in the Hecke constituent of dimension 16; and let $L_f = \mathbf{Q}(\{a_{\mathfrak{p}}(f)\})$ be its coefficient field. Then L_f is a cyclic extension of $K = \mathbf{Q}(\sqrt{5})$. More precisely, L_f is given by the polynomial $x^8 + (-10w + 1)x^7 + (26w - 98)x^6 + (1137w - 257)x^5 + (-2048w + 1263)x^4 + (-32915w + 13419)x^3 + (16985w + 55780)x^2 + (154252w - 20245) - 170835w - 222209$, where $w = \frac{1+\sqrt{5}}{2}$. It is a ray class field of conductor $\mathfrak{c} = (48w - 569)$, which is the product of the primes $(-2w + 1)$, $(w - 10)$, and $(23w - 11)$ above 5, 89 and 661 respectively.

Lemma 7.2. *Let $\text{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$; then there exists $\tau \in \text{Gal}(L_f/K)$ such that*

$$a_{\sigma(\mathfrak{p})}(f) = \tau(a_{\mathfrak{p}}(f)) \text{ for all primes } \mathfrak{p}.$$

Proof. The prime 31 splits completely in F . There are 8 distinct Hecke eigenvalues $a_{\mathfrak{p}}(f)$ for $\mathfrak{p} \mid 31$. So f cannot be a base change from any subfield of F . (In other words, the stabiliser of f under the action of $\text{Gal}(F/\mathbf{Q})$ is trivial.) Furthermore, since f is in the unique constituent of dimension 16, its Hecke orbit must be preserved by the action of $\text{Gal}(F/\mathbf{Q})$. Therefore, letting $\text{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$, we see that there exists $\tau \in \text{Gal}(L_f/K)$ such that

$$a_{\sigma(\mathfrak{p})}(f) = \tau(a_{\mathfrak{p}}(f)) \text{ for all primes } \mathfrak{p}.$$

□

By Theorem 3.1, we can associate a compatible system of λ -adic Galois representations

$$\rho_{f,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}_2(L_{f,\lambda}),$$

where λ runs over all the primes of L_f . The field M in Theorem 7.1 was constructed from the reductions of ρ_{f,λ_1} and ρ_{f,λ_2} modulo the primes λ_1, λ_2 above 2. If we assume that the Eichler-Shimura conjecture holds for F , then there is an abelian variety A_f/F of dimension 16 and conductor (1) such that:

1. $\text{End}_F(A_f) \otimes \mathbf{Q} \simeq L_f$;
2. For all primes λ of K_f , $\rho_{A_f,\lambda} = \rho_{f,\lambda}$, where $\rho_{A_f,\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow \text{GL}_2(L_{f,\lambda})$ is the Galois representation in the λ -adic Tate module of A_f .

This would mean that the number field M arises from the field of 2-torsion of the abelian variety A_f (and its conjugates).

Theorem 7.3. *Assume that the Eichler-Shimura conjecture holds for F , and let A_f be the abelian variety attached to f . Then, there exists an abelian variety $B/\mathbf{Q}$ of dimension 16 such that the followings hold:*

- (i) *The base change $B \times_{\mathbf{Q}} F$ is isogenous to A_f ;*
- (ii) $\text{End}_{\mathbf{Q}}(B) = \mathbf{Z}[\frac{1+\sqrt{5}}{2}]$;
- (iii) *The automorphic representation π_f lifts to an automorphic representation Π on $\text{GSpin}_{17}(\mathbf{A}_{\mathbf{Q}})$, with coefficients in $\mathbf{Q}(\sqrt{5})$, such that*

$$L(B, s) = L(\Pi, s)L(\Pi^{\tau}, s),$$

where $\text{Gal}(\mathbf{Q}(\sqrt{5})/\mathbf{Q}) = \langle \tau \rangle$;

- (iv) *The conductor of B is $d_{F/\mathbf{Q}}^8 = (2^{31})^8 = 2^{248}$. (So, B only has bad reduction at 2.)*

Proof. The discussion above, together with Lemma 7.2, shows that the newform f and the conjecturally attached abelian variety A_f satisfy the conditions of Theorem 5.4. So the isogeny class of A_f descends to $\mathbf{Q}$. $\square$

Remark 7.4. The abelian varieties A_f and B were discussed in [DV13]. But the connection with lifts of Hilbert modular forms wasn't clear to the authors at the time.

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